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A Consistency Proof of **PA**
Louise Hay Logic Seminar

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2 Preliminaries

2.1 The language $\mathcal{L}_{PA}$

First, we fix our language for the theory **PA**:

$$\mathcal{L}_{PA} = (0, S, +, \cdot, <).$$

2.2 Logical symbols and negation normal form

We will build formulas using the ordinary first-order logical symbols along with equality. However, we will only consider formulas represented in *negation normal form*—that is, formulas in which all negations will be pushed inward "as far as they can be". We will use '$\sim$' as a metasymbol to denote the negation normal form of a formula which follows it.

In particular, our most basic formulas will be the *literals*: atomic formulas and their ordinary negations. The metasymbol '$\sim$' will interact with complex formulas as follows:

$$\sim \varphi \equiv \neg\varphi, \text{ when } \varphi \text{ is a literal;}$$

$$\sim (\neg\varphi) \equiv \varphi;$$

$$\sim (\varphi \wedge \psi) \equiv \sim \varphi \vee \sim \psi \text{ (and similarly for disjunction);}$$

$$\sim \forall x\varphi \equiv \exists x \sim \varphi \text{ (and similarly for existential quantification).}$$

So, for example, the formula

$$\neg\left(\neg\exists x(S(x) = 0) \wedge \forall x\neg(x + y = 0)\right) \vee \neg(z = S(S(0)))$$

is not yet in negation normal form, but its modification to

$$\exists x(S(x) = 0) \vee \exists x(x + y = 0) \vee \neg(z = S(S(0)))$$

is.

2.3 Deductions in PA

Our main object of study in this talk will be well-founded proof trees—what I'll call deductions—in which each node is isolated by a rule. The deductions will be proved in Tait's calculus, the rules of which are given below. We will be proving *sequents* through our deductions—that is, sets of formulas to be interpreted disjunctively. As a bit of notation, '$\vdash \Gamma$' means we have a deduction of some formula in Γ.

We add in axioms for **PA**, in the form of the DEF rule, dictating that the symbols in the language act as they are intended and that equality is really equality. We also add IND, an axiom scheme for induction:

Using these deductions rules and axioms, we may prove, for example, that $\mathbf{PA} \vdash \varphi, \sim \varphi$ for any formula φ:

3 CUT-elimination for PA and its consistency

We have now the technical machinery set up which will do the work of the consistency proof. We need only focus on one rule—namely, the CUT rule—to see how to leverage a consistency proof for **PA**. The key theorem is the following.

Theorem 1. *If* $\textbf{\textit{PA}} \vdash \Gamma$*, where* Γ *is a sequent whose formulas contain no universal quantifiers, then there is a deduction of* Γ *which does not use the CUT rule.*

Let us say that the theory **PA** is consistent if, from the axioms, one cannot derive '$0 = 1$'. Then we have the following main result, in the form of a corollary.

Corollary 1.1. ***PA*** *is consistent. In other words,* $\textbf{\textit{PA}} \nvdash 0 = 1$*.*

Proof. Notice that '$0 = 1$' is a sentence not containing logical operators nor quantifiers. Thus, a deduction of '$0 = 1$' could not have concluded in anything other than an application of the CUT rule. However, '$0 = 1$' does not contain universal quantifiers, so by the previous theorem, there should be a deduction of the sentence without use of the CUT rule. Since this is impossible, there can be no deduction of '$0 = 1$' in our system. □

The consistency of our formalization of **PA**, then, hinges upon so-called "CUT-elimination" for deductions yielding sequents without universal quantifiers. The strategy to prove **Theorem 1** involves four steps.

1. Define an auxiliary system $\mathbf{PA}^\infty$ which is more easily shown to eliminate the CUT rule from deductions.
2. Show that $\mathbf{PA} \vdash \Gamma \implies \mathbf{PA}^\infty \vdash \Gamma$.
3. Show that $\mathbf{PA}^\infty$ has CUT-elimination.
4. Show that if $\mathbf{PA}^\infty \vdash \Gamma$ without CUT, where Γ doesn't contain universal quantifiers, then $\mathbf{PA} \vdash \Gamma$ without CUT.

4 The system $\mathbf{PA}^\infty$ and step four of the CUT-elimination strategy

4.1 The system $\mathbf{PA}^\infty$

The system $\mathbf{PA}^\infty$ keeps much of the same deduction rules as **PA**, but introduces an infinitely branching rule. In particular, it maintains the deduction rules AX, $\wedge$I, $\vee$I, $\exists$I, and CUT as well as the following.

It should be highlighted here that the defining axioms dictating the behavior of the symbols in $\mathcal{L}_{\mathbf{PA}}$ are absent and are instead replaced either by TRUE by itself or by TRUE along with the ω rule. For example:

4.2 Step four of the CUT-elimination strategy

We proceed with the demonstration of step four of the CUT-elimination proof of **PA** since steps two and three are a bit more involved.

Proposition 1. *Let* Γ *be a sequent whose formulas do not contain universal quantifiers. If* $\boldsymbol{PA}^\infty \vdash \Gamma$ *without CUT, then* $\boldsymbol{PA} \vdash \Gamma$ *without CUT.*

Proof. If we have a CUT-free deduction of a universal-free sequent Γ in $\mathbf{PA}^\infty$, then the deduction could not have applied the ω rule anywhere. If it had, then there would be universal quantifiers left over in Γ (since they were not CUT out). Thus, we may replace the deduction in $\mathbf{PA}^\infty$ with one in **PA** by replacing every instance of the TRUE rule with the corresponding actual deduction in **PA**. □

We end this section by remarking that it was crucial in this proof that Γ not contain any universally-quantified formulas, for this made the proof of Γ in $\mathbf{PA}^\infty$ nearly a proof in **PA** to begin with.

5 Steps two and three of the CUT-elimination strategy

5.1 Step two of the CUT-elimination strategy

We begin with a definition of the *rank* of a formula, which is somehow meant to measure complexity of a formula. For any formula φ, we define $rk(\varphi)$ by induction on φ as follows.

- If φ is a literal, then $rk(\varphi) = 0$.
- If φ is of the form $\psi_1 \wedge \psi_2$ or of the form $\psi_1 \vee \psi_2$, then $rk(\varphi) = \max\{rk(\psi_1), rk(\psi_2)\} + 1$.
- If φ is of the form $\forall x\psi$ or of the form $\exists x\psi$, then $rk(\varphi) = rk(\psi) + 1$.

Further, we introduce the notation $T \vdash_r \Gamma$ if there is a deduction in T with conclusion Γ such that all applications of the CUT rule in the deduction apply to formulas of rank less than r. In this notation, to write $T \vdash_0 \Gamma$ is to say that a theory T has a CUT-free deduction of Γ. Now we are ready for the main proposition of step two of the CUT-elimination proof for **PA**.

Proposition 2. *If* $\boldsymbol{PA} \vdash \Gamma$*, where* Γ *is a sequent all of whose free variables are among* $x_1, ..., x_n$*, then there is some* $r \in \mathbb{N}$ *such that for any* $m_1, ..., m_n \in \mathbb{N}$ *we have that* $\boldsymbol{PA}^\infty \vdash_r \Gamma(m_1, ..., m_n)$*.*

Put differently, this proposition tells us that if **PA** proves some sequent, then so does $\mathbf{PA}^\infty$—and, in fact, the applications of the CUT rule in $\mathbf{PA}^\infty$ can be uniformly bounded no matter which natural numbers are chosen to instantiate the free variables of Γ.

Proof of Proposition 2. We proceed by induction on the deduction of Γ in **PA**. First, if the deduction concludes with any of the rules AX, $\wedge$I, $\vee$I, $\exists$I, or CUT, then we repeat the step and induction takes care of this case. For example:

Next, for deductions whose final rule is DEF, we proceed as follows:

Then, for deductions ending in an application of ∀I, we proceed as follows:

And finally, for deductions concluding in an application of IND, we proceed as follows:

Note that in none of the modifications did we introduce an application of the CUT rule. Therefore, in the new deduction in $\mathbf{PA}^{\infty}$, all applications of the CUT rule are applications straight from the original deduction. Thus, since the deduction in **PA** is finite (since it is a finite proof tree dealing with finite formulas), the CUT-rank was bounded in **PA**, hence it is bounded in $\mathbf{PA}^{\infty}$. □

5.2 Step three of the CUT-elimination strategy

The final proposition required in the consistency proof for **PA** is the CUT-elimination property for $\mathbf{PA}^{\infty}$.

Proposition 3. *If* $\boldsymbol{PA}^{\infty} \vdash_{r+1} \Gamma$*, then* $\boldsymbol{PA}^{\infty} \vdash_{r} \Gamma$.

To prove this proposition requires the use of several lemmas which we will state and mostly leave unproven. The use of these lemmas will be to systematically replace instances of the CUT rule in any given deduction in $\mathbf{PA}^\infty$ to an instance of CUT over lower rank formulas. To do this, we need to know how to reduce the CUT rank in deductions on formulas of various complexities. We consider the simplest example case first, then state (all) and prove (some of) the lemmas, and finish by showing how these lemmas provide Proposition 3.

Suppose we are trying to reduce the CUT rank over a conjunction—that is, suppose we have a deduction such that the last step is an application of the CUT rule on a rank r formula of the form $\varphi \wedge \psi$:

In the right side sub-deduction, some step must introduce the formula $\sim \varphi \vee \sim \psi$. The relevant case in which this occurs is the one in which an application of $\vee$I is responsible:

To reduce the rank of the CUT used in the last step of the deduction of Γ, we may replace the CUT over $\varphi \wedge \psi$ with one over, say, φ by itself. To do this, we require two items: first, a way to systematically eliminate the introduction of the disjunction $\sim \varphi \vee \sim \psi$; second, a way to deduce Γ, φ from $\Gamma, \varphi \wedge \psi$.

The lemmas provide exactly these items.

Lemma 1 ($\wedge$ Inversion)**.** *If $\boldsymbol{PA}^\infty \vdash_r \Gamma, \varphi_1 \wedge \varphi_2$, then $\boldsymbol{PA}^\infty \vdash_r \Gamma, \varphi_i$ for $i = 1, 2$.*

Proof. We induct on the deduction of $\Gamma, \varphi_1 \wedge \varphi_2$. □

Lemma 2 ($\forall$ Inversion)**.** *If $\boldsymbol{PA}^\infty \vdash_r \Gamma, \forall x \varphi(x)$, then for any $n \in \mathbb{N}$ $\boldsymbol{PA}^\infty \vdash_r \Gamma, \varphi(n)$.*

Lemma 3 ($\perp$ Inversion)**.** *If $\boldsymbol{PA}^\infty \vdash_r \Gamma, \eta$, where η is a false literal, then $\boldsymbol{PA}^\infty \vdash_r \Gamma$.*

Lemma 4 ($\vee$ Elimination)**.** *If $\boldsymbol{PA}^\infty \vdash_r \Gamma, \sim \varphi_1 \vee \sim \varphi_2$, $\boldsymbol{PA}^\infty \vdash_r \Delta, \varphi_1$, and $\boldsymbol{PA}^\infty \vdash_r \Sigma, \varphi_2$ with $rk(\varphi_1 \wedge \varphi_2) \leq r$, then $\boldsymbol{PA}^\infty \vdash_r \Gamma, \Delta, \Sigma$.*

Proof. Note that the only interesting case occurs when $rk(\varphi_1 \wedge \varphi_2) = r$. Induct on the deduction of $\Gamma, \sim \varphi_1 \vee \sim \varphi_2$. □

Lemma 5 ($\exists$ Elimination)**.** *If $\boldsymbol{PA}^\infty \vdash_r \Gamma, \exists x \sim \varphi$ and for each $n \in \mathbb{N}$ $\boldsymbol{PA}^\infty \vdash_r \Sigma, \varphi(n)$, then $\boldsymbol{PA}^\infty \vdash_r \Gamma, \Sigma$.*

We are then in position to sketch the proof of Proposition 3.

Proof (sketch) of Proposition 3. We induct on the rank $r+1$ deduction of Γ. If the last rule is anything other than a CUT of rank r, we're done by the inductive hypothesis. Else, if the deduction ends with

we split into cases depending on the complexity of φ. If φ is a conjunction, we proceed as outlined above, applying CUT to one of the conjuncts with $\wedge$ Inversion and $\vee$ Elimination. We conclude similarly if the rank r CUT applies at the end over a quantified formula or over a false literal. □

Corollary 3.1. $\boldsymbol{PA}^\infty \vdash_r \Gamma \implies \boldsymbol{PA}^\infty \vdash_0 \Gamma$. *In other words,* $\boldsymbol{PA}^\infty$ *admits CUT-elimination on deductions of bounded CUT rank.*

Of course, this completes the CUT-elimination strategy for **PA** over deductions of existential sequents.

6 Gödel's second incompleteness theorem

The main result of this talk—the consistency of **PA**—should strike logicians as suspect. We know, from Gödel's second incompleteness theorem, that no theory, **PA** included, ought to be able to prove its own consistency. So exactly what is going on in this proof such that the consistency is shown?

The answer must be: we have not worked out a proof which can be formalized in **PA** itself. Indeed this is the case. The key observation is that we have used the system $\mathbf{PA}^\infty$ as an intermediary within the proof, but this system has a rule which cannot be formalized within **PA**—namely, the ω rule. By means of infinitary reasoning, then, we have concluded the consistency of **PA**, but we have importantly not done so within **PA**. Thus, no contradiction lurks in the background of our work here.