A MARKED STRAIGHTEDGE AND COMPASS CONSTRUCTION OF THE REGULAR HEPTAGON

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ABSTRACT. We present a new construction of the regular heptagon using a marked straightedge and compass. The construction is very short and streamlined and has the virtue of producing at the same time three triangles with angle ratios 3:3:1, 2:2:3, and 1:1:5. The vertices of these triangles, on their respective circumscribed circles, are among the vertices of three other regular heptagons. We provide elementary geometric and algebraic proofs of our main construction.

1. INTRODUCTION

The set $\mathcal{C}$ of constructible points with straightedge and compass is the set of points in the Euclidean plane (identified with $\mathbb{C}$) that can be described as follows. The set $\mathcal{C}$ is equal to $\cup_{i\geq 0}\mathcal{C}_r$ with $\mathcal{C}_0 = \{0, 1\}$ and $\mathcal{C}_{r+1}$ consisting of the points in $\mathcal{C}_r$ together with the points that arise as intersections of any two from the following set of curves: lines through points of $\mathcal{C}_r$ and circles centered in $\mathcal{C}_r$ which pass through at least one other point of $\mathcal{C}_r$. The set $\mathcal{C}$ is actually a subfield of $\mathbb{C}$ that is characterized as follows (see e.g. [Cox12, §10.1]):

$$\begin{aligned}\mathcal{C} &= \{\alpha \in \mathbb{C} \mid \exists n \geq 0,\ \mathbb{Q} = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n \ni \alpha,\ [K_{i+1} : K_i] \leq 2,\ \forall 0 \leq i \leq n-1\}\\ &= \{\alpha \in \mathbb{C} \mid \exists a \geq 0, [\mathbb{Q}(\alpha) : \mathbb{Q}] = 2^a\}.\end{aligned}$$

One can define in the same fashion the set of constructible points with respect to an enlarged set of allowable constructions. For example, the set

$$\begin{aligned}\mathcal{C}^* &= \{\alpha \in \mathbb{C} \mid \exists n \geq 0,\ \mathbb{Q} = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n \ni \alpha,\ [K_{i+1} : K_i] \leq 3,\ \forall 0 \leq i \leq n-1\}\\ &= \{\alpha \in \mathbb{C} \mid \exists a, b \geq 0, [\mathbb{Q}(\alpha) : \mathbb{Q}] = 2^a 3^b\}.\end{aligned}$$

arises if the straightedge and compass constructions are enlarged by allowing intersections using conic sections, certain origami folds, or a construction called *neusis* [Cox12, §10.3]. Any of these new tools can be used to construct roots of degree three equations, in particular, to construct cubic roots and to trisect angles.

A marked straightedge is a straightedge with two marks one unit apart. Aside from its normal use as a straightedge, this can be used to construct new points through a procedure called *verging*. In verging, one needs a point P (the pole) and two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ (lines or circles). We are allowed to use the marked straightedge to draw a line that goes through the point P and intersects $\mathcal{C}_1$ and $\mathcal{C}_2$ in a pair of points Q_1 and Q_2 that match the position of the two marks on our straightedge. We say that Q_1 and Q_2 are constructed by verging. Verging of type I, II, and III refers to verging between two lines, a line and a circle, and two circles, respectively. The verging of type I, also called *neusis*, was one of the geometric construction methods

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used in antiquity, in particular by Nicomedes (to construct cubic roots and therefore to double the cube). Verging of type II was used by Archimedes for the trisection of an angle, but it is important to stress that this construction can be achieved only by neusis [Har00, Prop. 30.1]. As already mentioned, the field $\mathcal{C}^*$ of constructible numbers with marked straightedge and compass, where only verging of type I is permissible, has a complete characterization. If verging of type II or type III (or both) is also allowed, then the problem of describing the corresponding field of constructible numbers is still open.

The construction of the regular heptagon is equivalent to constructing the roots of the cubic equation

$$z^3 + z^2 - 2z - 1, \tag{1}$$

that is, the real numbers $2\cos(\frac{2\pi}{7})$, $2\cos(\frac{4\pi}{7})$, and $2\cos(\frac{6\pi}{7})$ (the roots of (1) are of the form $\xi+\xi^{-1}$ where ξ is a primitive 7th root of unity, that is a root of $x^6+x^5+x^4+x^3+x^2+x+1$). As these numbers are elements of $\mathcal{C}^*$ but not of $\mathcal{C}$, they can be constructed by neusis but not by straightedge and compass only. As such, explicit constructions of the regular heptagon have been obtained. Most of these constructions rely on solving the relevant cubic equation by trisecting an angle [Gle88, Har00, Hog84, Ple12, Viè70] and are accessible by neusis only; others use origami folds to the same effect [Huz94] or intersections with conic sections, some of these constructions going back to Archimedes [Hea63]. There are also constructions that use tools that are unnecessarily powerful, such as verging of type II [Joh75], or Archimedes's area construction (see [Ric19, Ch. 9]).

While it is theoretically clear that a construction of the regular heptagon is possible (e.g. by neusis), in practice the construction can be long and complicated. For example, the explicit construction detailed in [Har00, pg. 211] lists 28 steps (some of the steps requiring several uses of the instruments). Our goal is to give a very short construction of the regular heptagon by neusis. The construction, which has only 5 steps (if one is willing to count as one step simple constructions such as the median line of a segment or the parallel to a given line through a given point), does not use trisection on an angle or verging of type II, and we employ only elementary tools in the verification of its validity. We present this construction in the following section. The arguments are remarkably short.

2. The geometric construction

We denote by O and I the initial points in the complex plane (corresponding to the numbers 0 and 1).

1. Draw the line k through O and I and construct the points R and R' on k corresponding to the numbers 2 and -2, respectively.
2. Construct the perpendicular bisector of $R'I'$, where I' is the midpoint of $R'O$; this intersects twice the circle centered at O of radius 2. Call the lower intersection point P.
3. Construct the line l parallel to PR which passes through I.
4. Use verging with pole P between the lines k and l to construct four pairs of points. We are only interested in the points on the line k. The origin O is necessarily among them. From right to left, call the remaining three points A, B, and C.
5. Construct perpendiculars to line k at A, B, and C. The intersection points between the circle centered at O with radius 2 and the three perpendiculars, together with the point R are the vertices of a regular heptagon inscribed in the circle of radius 2 centered at the origin.

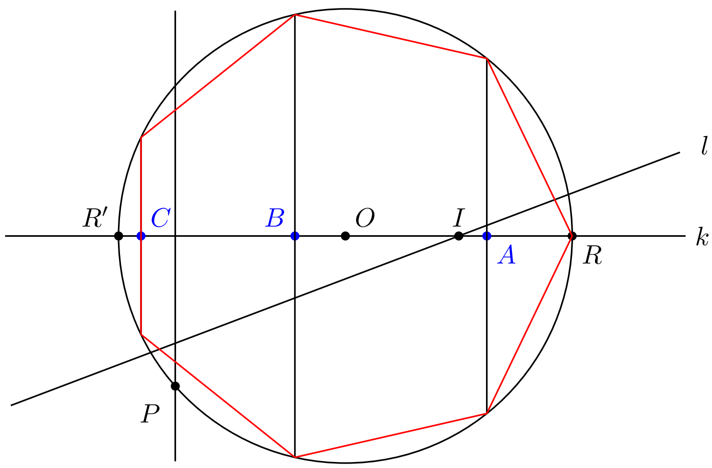

FIGURE 1. A neusis construction of a regular heptagon

3. The Geometric Proof

We present a geometric argument that our construction does indeed produce a regular heptagon. Our goal is to show that the numbers that correspond to the points A, B, and C are precisely the roots of the polynomial $z^3 + z^2 - 2z - 1$.

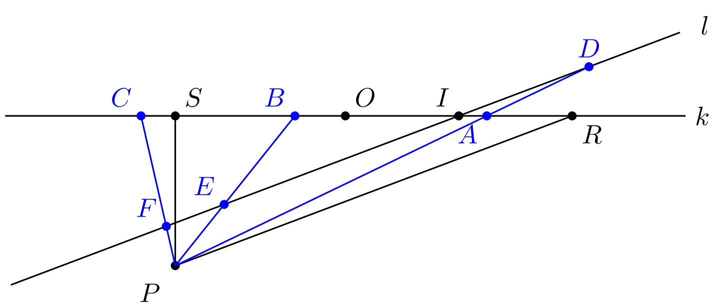

FIGURE 2. The geometric proof

Draw the line PR. Label by S the intersection of the perpendicular bisector of $R'I'$ with k. Draw the lines PA, PB, and PC and label their intersection points with l by D, E, and F, respectively. These are the locations of the second mark on the straightedge from step 4 of the construction; therefore,

$$|AD| = |BE| = |CF| = 1.$$

Let a be the distance from O to A and let x be the distance from P to A. Now, consider the triangles IAD and RAP. Since the lines PR and l are parallel, these triangles are similar. Their similarity gives the equation

$$\frac{|PA|}{|DA|} = \frac{|AR|}{|AI|}.$$

From this, we get the equation

$$\frac{x}{1} = \frac{2-a}{a-1}. \tag{2}$$

The Pythagorean Theorem applied to the right triangle PSA, gives

$$|PS|^2 + |SA|^2 = |PA|^2.$$

Note that $|PS| = \frac{\sqrt{7}}{2}$ (by construction P lies on the circle of radius 2). The previous equation becomes

$$\left(\frac{\sqrt{7}}{2}\right)^2 + \left(a + \frac{3}{2}\right)^2 = x^2 \quad \text{or equivalently,} \quad a^2 + 3a + 4 = x^2.$$

Substituting the value of x from (2) we obtain

$$a^2 + 3a + 4 = \left(\frac{2-a}{a-1}\right)^2.$$

Simplification yields

$$a^4 + a^3 - 2a^2 - a = 0,$$

and since $a \neq 0$, we obtain that a is a root of $z^3 + z^2 - 2z - 1$.

The verification that the numbers b and c corresponding to the negative values of the distances from O to B and C are roots of $z^3 + z^2 - 2z - 1$ follows from a similar argument. For B, the relevant similar triangles are IBE and RBP, and the right triangle is PSB. For C, the relevant similar triangles are ICF and RCP, and the right triangle is PSC.

To conclude, the numbers corresponding to the points A, B, and C are all roots of (1) and, therefore, they are the numbers $2\cos(\frac{2\pi}{7})$, $2\cos(\frac{4\pi}{7})$, and $2\cos(\frac{6\pi}{7})$, respectively. We have indeed constructed a regular heptagon.

4. The Algebraic Proof

Conchoids are plane algebraic curves constructed in the following fashion. Given a point P and a plane algebraic curve $\mathcal{C}$, the conchoid associated with $(P, \mathcal{C})$ is the geometric locus consisting of the points that lie on a variable line passing through P and at distance 1 from the intersection point(s) of that line with the curve $\mathcal{C}$. If $\mathcal{L}$ is a line, the conchoid associated with $(P, \mathcal{L})$ is an algebraic curve of degree 4 called the *Conchoid of Nicomedes* (henceforth simply called conchoid); if $\mathcal{C}$ is a circle, the conchoid associated with $(P, \mathcal{C})$ (henceforth called circle conchoid) is an algebraic curve of degree 6 related to the *Limaçon of Pascal*, which appears in the special case when $P \in \mathcal{C}$ [Bar02].

From this point of view, verging through P between the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ produces the intersection points between $\mathcal{C}_1$ and the conchoid associated with $(P, \mathcal{C}_2)$ (or vice versa). Hence, neusis can be thought of as constructing the intersection points between a conchoid and a line, verging of type II as constructing the intersection points between a conchoid and a circle or between a circle conchoid and a line, and verging of type III as constructing the intersection points between a circle conchoid and a circle. Our particular neusis therefore produces the intersection points between the conchoid associated with (P, l) and the line k.

The conchoid associated to (P, l) in our construction is the algebraic curve described by the equation

$$\left(\left(x + \frac{3}{2}\right) - \sqrt{7}\left(y + \frac{\sqrt{7}}{2}\right) + 1\right)^2 \left(\left(x + \frac{3}{2}\right)^2 + \left(y + \frac{\sqrt{7}}{2}\right)^2\right) = \left(\left(x + \frac{3}{2}\right) - \sqrt{7}\left(y + \frac{\sqrt{7}}{2}\right)\right)^2.$$

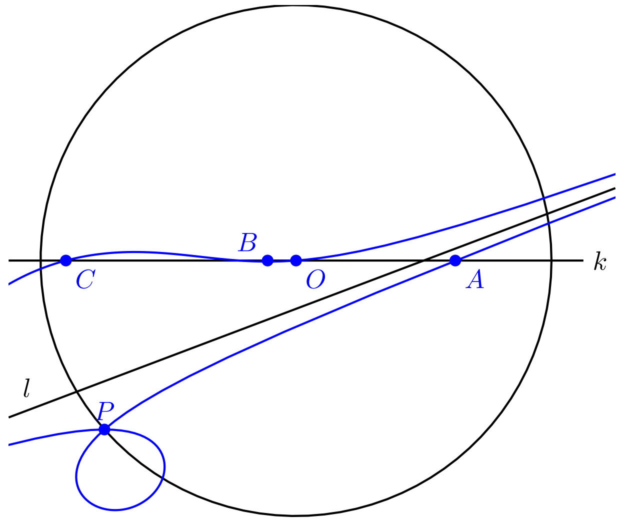

FIGURE 3. The conchoid used to construct the regular heptagon

Therefore, its intersection with the line k is precisely the set of points satisfying the equations

$$y = 0 \quad \text{and} \quad x(x^3 + x^2 - 2x - 1) = 0.$$

In conclusion, the numbers corresponding to the points A,B, and C are roots of the cubic polynomial $z^3 + z^2 - 2z - 1$, and therefore correspond to the real numbers $2\cos(\frac{2\pi}{7})$, $2\cos(\frac{4\pi}{7})$, and $2\cos(\frac{6\pi}{7})$.

For the reader's convenience we briefly indicate how to obtain the equation of a conchoid associated with a point having coordinates (a, b) and a line having equation $cx + dy = e$. The equation for the conchoid associated with the origin and the vertical line $x = \alpha$ can be written as

$$(x - \alpha)^2(x^2 + y^2) = x^2.$$

Here, the number α represents the distance between the pole and the line. With this in mind, to obtain the equation of a conchoid in general position we have to implement two changes of coordinates: one linear to rotate the conchoid until the slope of the line becomes $-c/d$, and a translation to move the pole to (a, b). After simplification, the desired equation is

$$(c(x-a) + d(y-b) + ac + bd - e)^2 \left((x-a)^2 + (y-b)^2\right) = (c(x-a) + d(y-b))^2 .$$

5. Other Heptagons in the same construction

In this section, we draw the reader's attention to an interesting (and rather remarkable) facet of the construction of the regular heptagon presented in this paper: there are several other regular heptagons hidden in the figure as well!

The key to uncovering these extra heptagons is the observation that the triangles ABP, ACP, and BCP are isosceles (see Figure 2). How does this help? Let's assume for the moment that each of the triangles ABP, ACP, and BCP are isosceles, as marked in Figure 4. Label as θ the measure of angle PAB. The angle PBC, exterior to triangle ABP, then, has a measure of 2θ. Finally, since triangle ACP is isosceles, the angles ACP and APC both measure 3θ. Summing the interior angles of triangle ACP, we find that

$$\theta = \frac{\pi}{7}.$$

Thus, the triangles ACP, BCP, and ABP have angle ratios 3:3:1, 2:2:3, and 1:1:5, respectively. If we construct the circumscribed circle for any of these triangles, then we can produce an inscribed regular heptagon which has among its vertices the three vertices of the corresponding triangle. We can construct the remaining vertices of the heptagon by walking the compass around the circumcircle starting with any pair of vertices of the relevant triangle. The three resulting regular heptagons are distinct.

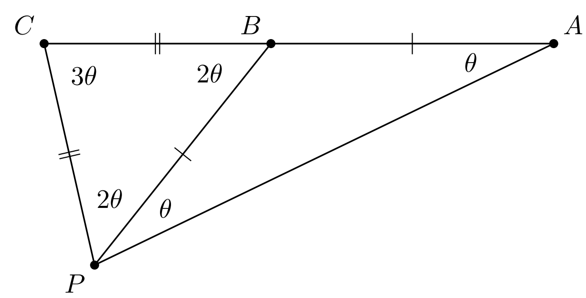

FIGURE 4. The 3:3:1, 2:2:3, and 1:1:5 triangles

Yet another regular heptagon, this time inscribed in the circle centered at O of radius 2, is constructed as follows. Extend the lines PA, PB, PC to intersect the circle of radius 2, as in Figure 5. ; call the intersection points A', B', and C'. Since P lies on the same circle, the arcs between $A'B'$ and $B'C'$ measure $\frac{2\pi}{7}$ and $\frac{4\pi}{7}$, respectively, and therefore they are among the vertices of a regular heptagon. To construct the remaining vertices, simply walk the compass around the circle of radius 2 starting at A' and B'.

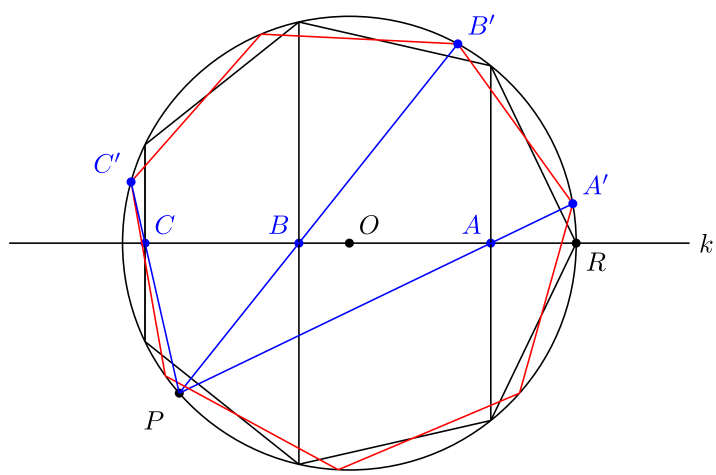

FIGURE 5. Another regular heptagon

It remains to show that the triangles ABP, ACP, and BCP are isosceles. Recall our notation for $a = |OA|$, $b = -|OB|$, and $c = -|OC|$, each being a root of $z^3 + z^2 - 2z - 1$.

For triangle ABP, we have

$$|AB|^2 = (a-b)^2 = a^2 - 2ab + b^2$$

and

$$|BP|^2 = \left(b + \frac{3}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2 = b^2 + 3b + 4.$$

Since $a = 2\cos(\frac{2\pi}{7})$ and $b = 2\cos(\frac{4\pi}{7})$, we can use the trigonometric identity $\cos(2\theta) = 2\cos^2(\theta) - 1$ to write $b = a^2 - 2$. Substituting this into the equations for $|AB|^2$ and $|BP|^2$ gives

$$|AB|^2 = a^2 - 2a(a^2-2) + (a^2-2)^2 = a^4 - 2a^3 - 3a^2 + 4a + 4$$

and

$$|BP|^2 = (a^2-2)^2 + 3(a^2-2) + 4 = a^4 - a^2 + 2$$

Finally, subtracting $|AB|^2$ from $|BP|^2$ yields

$$(a^4 - a^2 + 2) - (a^4 - 2a^3 - 3a^2 + 4a + 4) = 2(a^3 + a^2 - 2a - 1) = 0$$

showing that $|AB| = |BP|$.

The verification that the triangles ACP and BCP are isosceles follows the same general principle. We leave the details to the reader.

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