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A Brief Survey of Hyperreal Numbers
Louise Hay Logic Seminar

1 Overview

- Axioms of a Hyperreal System and the Ultraproduct Construction
- Structure of the Hyperreal Numbers
- Nonstandard Analysis: Differentiation and Integration
- The Hyperreals and ZFC

2 Axioms of a Hyperreal System and the Ultraproduct Construction

Axioms of a general hyperreal number system $(*, \mathbb{R}, \mathbb{R}^*)$:

Axiom 1. $\mathbb{R}$ *is a complete, ordered field.*

Axiom 2. $\mathbb{R}^*$ *is an ordered field extension of* $\mathbb{R}$.

Axiom 3. $\mathbb{R}^*$ *has a positive infinitesimal.*

Axiom 4. *(Function Axiom) For each real function f of n variables, there is a corresponding hyperreal function f^* of n variables, called the natural extension of f. The field operations of $\mathbb{R}^*$ are the natural extensions of the field operations of $\mathbb{R}$.*

Axiom 5. *(Transfer Axiom) Given two formulas S, T with the same variables, if every real solution of S is a solution of T, then every hyperreal solution of S is a solution of T.*

> **Note:** We may alternatively give the Transfer Principle as:
> *Any hyperreal number system $\mathbb{R}^*$ is an elementary extension of the ordered base field $\mathbb{R}$.*

The Ultraproduct Construction:
We fix a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and identify

$$\mathbb{R}^* := \prod_{\mathcal{U}} \mathbb{R}.$$

We identify real elements $r \in \mathbb{R}$ with the $\mathcal{U}$-equivalence class of the sequence $(r, r, r, ...)$. We also extend any function and relation symbols from $\mathbb{R}$ to $\mathbb{R}^*$ as follows:

Definition 1. *We write $a <^* b$ for any $a, b \in \mathbb{R}^*$ iff*

$$\{i : a_i < b_i\} \in \mathcal{U}.$$

Definition 2. *Let f be a real function on n variables. The* natural extension *of f is the function f^* such that*

$$f^*(x) = y \iff \{i : f(x_i) = y_i\} \in \mathcal{U}$$

for any $x = (x_i)_{\mathcal{U}}$ and $y = (y_i)_{\mathcal{U}}$ in $\mathbb{R}^*$.

Note that the above definition is well-defined over $\mathcal{U}$:

Corollary 0.1. *The natural extension f^* of a function f is well-defined.*

Proof. We will show the case when f is a function of one variable. Suppose that $x, y, a, b \in \mathbb{R}^*$ and that $f^*(x) = y$, $x =_{\mathcal{U}} a$, and $y =_{\mathcal{U}} b$. Then:

$$\{i : f(x_i) = y_i\} \in \mathcal{U},\ \{i : x_i = a_i\} \in \mathcal{U},\ \text{and}\ \{i : y_i = b_i\} \in \mathcal{U}$$

and therefore

$$\{i : f(a_i) = b_i\} \supseteq \{i : f(x_i) = y_i\} \cap \{i : x_i = a_i\} \cap \{i : y_i = b_i\}$$

so that $\{i : f(a_i) = b_i\} \in \mathcal{U}$. □

The defining characteristic of the hyperreals as constructed here is that they contain infinite numbers and infinitesimal numbers, so it will be useful to delineate exactly what we mean by these terms.

Definition 3. *We call a hyperreal number x* infinitesimal *iff $|x| <^* r$ for every standard real number $r \in \mathbb{R} \subseteq \mathbb{R}^*$. We call a hyperreal number x* finite *iff $|x| <^* r$ for some real r. We call x* infinite *iff $r <^* |x|$ for every real r.*

Equipped with these definitions, $(*, \mathbb{R}, \mathbb{R}^*)$ defines a hyperreal system satisfying the axioms above:

Theorem 1. *The above hyperreal number system satifies axioms 1-5.*

Proof. Axiom 1 follows from theorems of Dedekind, Hilbert, among others.
Axiom 4 follows from the fact that the definition of function extensions are well-defined as checked in Corollary 0.1.
Axiom 5 is a consequence of Łoś's Theorem.
To show Axiom 2, we need to show that $<^*$ satisfies transitivity and the sum, product, and trichotomy laws. That $<^*$ in $\mathbb{R}^*$ satisfies transitivity and the sum and product laws is a consequence of Axiom 5, so we need to check that $<^*$ satisfies trichotomy—that is, for any $x, y \in \mathbb{R}^*$, we must have either $x <^* y$, $x =^* y$, or $y <^* x$. This follows from the fact that only one of the following sets can be in $\mathcal{U}$:

$$\{i : x_i < y_i\},\ \{i : x_i = y_i\},\ \{i : y_i < x_i\}.$$

Finally, Axiom 3 follows from considering the hyperreal number given by the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. Let the hyperreal number given by this sequence be called x. Note that for every real number $r \in \mathbb{R}^*$, we must have that $r < \frac{1}{N}$ for some $N \in \mathbb{N}$ large enough. This means that the set of n for which $\frac{1}{n} < r$ is cofinite, hence in $\mathcal{U}$. Thus, $|x| = x <^* r$ for every real r. □

3 Structure of the Hyperreal Numbers

We are now ready to examine the structure of the hyperreals. We begin first with some definitions which shed light on how the hyperreals look locally.

Definition 4. *We call two hyperreal numbers x and y* infinitely close*, written $x \approx y$, iff $x - y$ is infinitesimal.*

Definition 5. *For any hyperreal number x, we define the* monad of *x to be the set*

$$monad(x) := \{y \in \mathbb{R}^* : x \approx y\}.$$

Similarly, we define the galaxy of *x to be the set*

$$galaxy(x) := \{y \in \mathbb{R}^* : x - y \text{ is finite}\}.$$

We see, then, that the hyperreal line "looks like" the real line, but at each point one may "zoom in" to see the monad of any point. However, it turns out that each monad of a finite hyperreal number contains exactly one standard real number.

Theorem 2. *(Standard Part Principle) Every finite $x \in \mathbb{R}^*$ is infinitely close to a unique real number $r \in \mathbb{R}$. That is, every finite monad contains a unique real number.*

Proof. Suppose $x \in \mathbb{R}^*$ is finite.
Existence: Let $X := \{s \in \mathbb{R} : s < x\}$. The set X is nonempty since x is finite, so there is some real r with $|x| <^* r$; in fact, r is an upper bound for the set X. By Axiom 1, then, X has a least upper bound ($\mathbb{R}$ is complete). Call the (real) least upper bound of X t. Then we need to show that $x - t \approx 0$, i.e., that $|x - t| < r$ for every real r. For any positive r, we have that

$$x <^* t + r \implies (x - t) <^* r \text{ and } t - r <^* x \implies -r <^* (x - t)$$

so that $|x - t| <^* r$ for every real r. It follows that $x - t \approx 0$, so $x \approx t$, i.e., $t \in monad(x)$.
Uniqueness: Suppose that $r, s \in \mathbb{R}$ are in $monad(x)$. Then $r \approx x$ and $x \approx s$, so $r \approx s$. That is, $|r - s| < t$ for every positive real number t, i.e. $|r - s| = 0$, so $r = s$. □

For any finite hyperreal number x, we will write $st(x)$ to denote the *standard part of x*—that is, the unique real number in $monad(x)$. It is easy to show the following corollary.

Corollary 2.1. *Let $x, y \in \mathbb{R}^*$ be finite. Then:*

1. *$x \approx y$ iff $st(x) = st(y)$*
2. *$x \approx st(x)$*
3. *for all $r \in \mathbb{R}$, $st(r) = r$*
4. *if $x \leq^* y$, then $st(x) \leq st(y)$*

We have already seen an example of an infinitesimal number in $\mathbb{R}^*$. One can easily construct a positive infinite hyperreal number using a similar sequence of reals. In fact, it turns out that every infinitesimal and every infinite number is constructed similarly.

Theorem 3. *Let $a_1, a_2, a_3, ...$ be a sequence of real numbers and let $r \in \mathbb{R}$.*

1. *$(a_1, a_2, a_3, ...)_{\mathfrak{U}} \approx r$ for every nonprincipal ultrafilter over $\mathbb{N}$ iff $\lim_{n\to\infty} a_n = r$*
2. *$(a_1, a_2, a_3, ...)_{\mathfrak{U}}$ is positive infinite for every nonprincipal ultrafilter over $\mathbb{N}$ iff $\lim_{n\to\infty} a_n = \infty$*

Proof. We prove the first statement. First, let's assume that $\lim_{n\to\infty} a_n = r$. Then, for any positive $s \in \mathbb{R}$, the set $\{n : |a_n - r| < s\}$ is cofinite and is therefore in the ultrafilter. Hence, $(a_n)_{\mathfrak{U}} \approx r$.
Conversely, suppose that $\lim_{n\to\infty} a_n \neq r$. Then there is some real s such that the set $\{n : |a_n - r| \geq s\}$ is infinite. Using Zorn's Lemma, we may construct a free ultrafilter over this set, yielding a hyperreal number system satisfying the desired axioms and for which $(a_n)_{\mathfrak{U}} \not\approx r$. □

4 Nonstandard Analysis: Differentiation and Integration

Here, we take a moment to examine how calculus looks different in the hyperreal number system. We define both the derivative of a function f and the definite integral over a region.

Definition 6. *For a real function f defined at a real value a, we define the* derivative of f at a *to be*

$$f'(a) := st\left(\frac{f(a+\varepsilon)-f(a)}{\varepsilon}\right)$$

for every nonzero infinitesimal $\varepsilon \in \mathbb{R}^*$, if such a number exists.

Using the definition, we can calculate derivatives of usual functions and get results we're used to—for example, $f(x) = x^2 \implies f'(x) = 2x$.
Integration can be defined as one would expect via the intuition from calculus courses.

Definition 7. *Let f be a real function, $a < b$ be real numbers, and $dx \in \mathbb{R}^*$ be a positive infinitesimal. The* Riemann integral of f from a to b *is*

$$\int_a^b f(x)dx := st\left(\sum_a^b f(x)\Delta x\right),$$

where the summation on the right side of the equality is the typical (say) midpoint rectangle summation of n rectangles such that $a + n\Delta x \approx b$.

5 The Hyperreals and ZFC

Consider the hyperreal number defined by the following sequence of reals:

$$1, \frac{1}{2}, 3, 1, \frac{1}{3}, 6, 1, \frac{1}{4}, 9, ...$$

The class, according to the chosen nonprincipal ultrafilter of $\mathbb{N}$, that this sequence gets assigned to will depend upon which sequence of indices modulo 3 is in the ultrafilter. Thus, we see that choice of ultrafilter dictates structure of the produced hyperreal number system. In this way, the hyperreal number system $\mathbb{R}^*$ that we've constructed here is not unique—even up to isomorphism.
What's more, the existence of a hyperreal number system along with a transfer principle as we've required entails the existence of a nonprincipal ultrafilter over $\mathbb{N}$. Thus, the existence of a hyperreal number system is strong enough to yield some sort of axiom of choice since the construction of a nonprincipal ultrafilter is independent of ZF.

Theorem 4. *If N is some positive, infinite hypernatural number in $\mathbb{R}^*$, then the sets $X \in \mathbb{N}$ such that $N \in X^*$ define a nonprincipal ultrafilter on $\mathbb{N}$.*

Proof. Let $\mathfrak{U}$ be the collection of such sets $X \in \mathbb{N}$. Note that:

- for $X \in \mathfrak{U}$ with $X \subseteq Y$, $N \in X^* \subseteq Y^* \implies Y \in \mathfrak{U}$
- for $X, Y \in]mathcalU$, $N \in X^* \cap Y^* \implies X \cap Y \in \mathfrak{U}$
- $\emptyset \notin \mathfrak{U}$ since $N \notin \emptyset$
- for $X \subseteq \mathbb{N}$, every natural number is either in X or $\mathbb{N} \setminus X$, so this tranfers to X^*

□

Thus, the existence of the hyperreals requires some version of choice strong enough for ultrafilters over $\mathbb{N}$. However, as one final salvaging point for $\mathbb{R}^*$, it has been shown (by Kanovei and Shelah) that there is a definable in ZFC hyperreal number system satisfying the axioms we have demarcated here. That is, in ZFC, there is a definable, unique (up to isomorphism) hyperreal number system, so the hyperreals are (essentially) on the same ontological footing as the reals are.