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Date: 3/6/2023

Modal Model Theory: A Potential Foundation for Mathematics?
Louise Hay Logic Seminar

1 Overview

Overview of this talk:

- Introductory Items/Philosophical Motivation
- A Bit of Modal Logic
- A Bit of Modal Model Theory
- The Expressive Power of Modal Graph Theory
- The Interpretability of Modal Graph Theory
- An Actuality Operator and Full Set-Theoretic Truth
- Meta-Mathematical Issues and Closing Remarks

2 Introductory Items and Philosophical Motivations

First and foremost, it must be mentioned that the content of this talk is based on (and theorems/proofs are taken from) a paper by Joel Hamkins and Wojciech Woloszyn called "Modal Model Theory" which can be found here on Arxiv.

As does Kunen in his book, we begin by briefly concerning ourselves with a bit of the philosophy of mathematics. Among the several schools of thought with respect to the ontology of mathematics—that is, with respect to what *actually exists* mathematically—Platonism enjoys much support among mathematicians implicitly and philosophers explicitly to this day. Crudely, we may understand Platonism in mathematics to be the belief that the objects of mathematics (for example, the sets in set theory) *actually exist* in some abstract sense as do Plato's ideal forms in the realm of forms. Under this view, there is somehow an absolute actual meaning to, e.g., the number three; this is in opposition to, e.g., the formalist's view that the number three is merely the symbol '3', which can be represented in the world by concrete instances of 'three'. Mathematicians occasionally implicitly ascribe to Platonism by regarding what they're working with to exist abstractly; we seem to have no difficulty with the assertion that there is some number three existing outside the symbol we write down just as we have no difficulty with the assertion that there is some function in an abstract sense that is differentiable everywhere (in fact, there are many). Among these implicit ascriptions to Platonism, the set-theoretic universe V is sometimes taken by mathematicians to be an abstract entity that really does exist; Kunen is among these mathematicians, noting in the introduction to his book that throughout there is an underlying assumption of Platonism so that everything being worked with does actually exist in some sense.

However, Platonism does not run unopposed in the philosophy of mathematics as the dominant view through which we should understand what mathematics is (indeed, even Hilbert was a formalist). Among one of the issues philosophers typically bring up about Platonism is an issue regarding the infinite. It is of the view of some that in order for an object of mathematics to exist in any sense, it must be able to be instantiated or represented in the real world; for example, we can see representations of the number three anywhere we look. However, to claim that the infinite exists in a Platonic sense would, under this view, require of the actual universe a representation of the infinite, which may be seen as problematic; we cannot, for example, represent infinity as we can three. A proposed solution to this issue regards a potentialist perspective. That is, infinity does not exist itself in the Platonic sense, but it does potentially exist: we may, in the real universe, represent infinity to an arbitrarily high degree, so, while it is never fully realized in the universe, it is potentially realized in the sense that it is taken to be the limit of the process of realizing its initial segments. (Note: this view may still run into issues if one believes in an absolute limit to, say, the number of particles in the universe.)

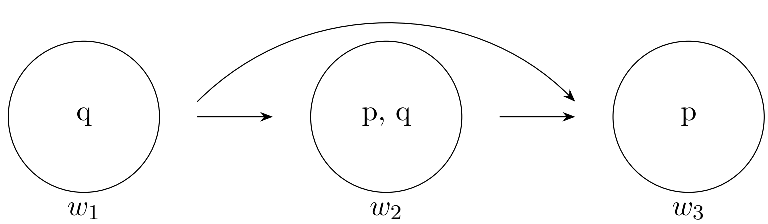
That the set-theoretic universe V exists in a Platonic sense is certainly contestable under similar suspicions to those challenging the Platonic existence of the infinite. In fact, some philosophers of mathematics would, rather than assert the Platonic existence of *the* set-theoretic universe V, assert that the set-theoretic universe can only potentially exist, that it is never fully realized anywhere. Under this view, V is more of a never-ending process of building more and more sets than an actual entity. Among the potentialist systems which describe the set-theoretic universe are the potentialist systems of modal model theory, the topic of this talk. We will focus, in particular, on the potentialist systems of finite and general modal graph theory, and we will focus on the incredible expressive power of these theories, connecting their interpretability with the whole of set-theoretic truth; this will, as a consequence, investigate whether modal graph theory can serve as a foundation of mathematics just as well as set theory does.

3 A Bit of Modal Logic

Modal logics are systems of logic built out of usual first-order predicate logic with quantifiers by augmenting the system with two modal operators: $\Box$, usually interpreted to signify necessity and $\Diamond$, usually interpreted to signify possibility. Whereas the usual truth-functional connectives' truth-values depend on the truth values of their atomic parts and connectives, the modal operators require some additional structure built into the logic to be able to determine their truth-values.

For this, we introduce Kripke semantics, a system for interpreting truth-values in modal logics based on "possible worlds" and an accessibility relation. In a modal logic, one specifies not only all of what first-order logic requires, but also a collection $\mathcal{U}$ (or universe) of worlds together with a binary operation R and a truth-valuation like for ordinary first-order logic. We take the statement $\Box p$ to be true at a world $w \in \mathcal{U}$ if and only if for every v such that wRv, it is true that p in world v. We take the statement $\Diamond p$ to be true at a world w if and only if there is some world $v \in \mathcal{U}$ such that p holds at v and wRv. Heuristically, it helps to remember that the modal operators are usually interpreted as necessity and possibility and that the relation R is usually interpreted as accessibility: for a statement p to be necessary at a world w, it must be the case that at every world w "sees," p holds; likewise, for p to be possible at a world w, it must be the case that w can "see" a world in which p holds.

For example, we may consider the following diagram of one modal universe, and we may ask whether $\Box p$, $\Diamond p$, $\Box q$, and $\Diamond q$ hold at worlds w_1, w_2, and w_3:



Systems of modal logic, as was mentioned before, augment first-order logic, and therefore all the usual rules of inference may apply to the truth-functional connectives. As for the modal operators, the most basic system of modal logic called **K** assumes the rules N: $\models p \Rightarrow \models \Box p$ and K: $\Box(p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Box q)$. Notice that this system does not impose any conditions on the accessibility relation R. Further important systems for modal model theory (and modal logics in general) can be obtained by imposing conditions on R and other preferred assumptions on the modal operators. For example, the modal logic system **T** is obtained by requiring R to be reflexive on top of **K** (equivalently, $\Box p \Rightarrow p$). In modal model theory, the systems **S4** and **S5**, obtained, respectively, by requiring R to be reflexive and transitive (equivalently $\Box p \Rightarrow \Box\Box p$) and by requiring R to be a reflexive and Euclidean—that is, R is an equivalence relation such that for every $u, v, w \in \mathcal{U}$, $uRv \wedge uRw \Rightarrow vRw$ (equivalently, $\Diamond p \Rightarrow \Box\Diamond p$)—are of utmost importance.

4 A Bit of Modal Model Theory

In modal model theory, we consider structures within a class of similar structures, and we investigate how the modal operators behave among the class. To begin our potentialist system, we define a class $\mathcal{W}$ of our possible worlds, as in Krpike semantics, in a common language $\mathcal{L}$ along with an accessibility relation $M \sqsubseteq N$ which refines the substructure relation (whatever that may be). We define also the truth of the modal operators as one would expect:

$$\Box p \text{ is true at world } M \Leftrightarrow \forall N \in \mathcal{W}(M \sqsubseteq N \Rightarrow p \text{ is true at } N)$$

$$\Diamond p \text{ is true at } M \Leftrightarrow \exists N \in \mathcal{W}(M \sqsubseteq N \wedge p \text{ is true at } N)$$

We may write this more efficiently as

$$M \models \Box p \Leftrightarrow \forall N \in \mathcal{W}(M \sqsubseteq N \Rightarrow N \models p)$$

$$M \models \Diamond p \Leftrightarrow \exists N \in \mathcal{W}(M \sqsubseteq N \wedge N \models p)$$

Note that the modal operators are dual operators—they are definable in terms of one another.

For our purposes, we consider the class $\mathcal{W}$ to be the class of models of a given theory T, denoted $\text{Mod}(T)$, and we take the substructure relation to be with respect to the models of the theory T, so $M \sqsubseteq N$ iff $M \subseteq N$. In this talk, the theory T we will concern ourselves with primarily is graph theory.

We finish this section with a few more definitions of the languages we will be using throughout:

- We denote by $\mathcal{L}$ the language of the ground theory in the potentialist system $\mathcal{W}$; that is, $\mathcal{L}$ is the language of T.
- We denote by $\Diamond\mathcal{L}$ the language $\mathcal{L}$ with modal operators allowed outside the scope of any quantifiers.
- We denote by $\mathcal{L}^{\Diamond}$ the full modal language of $\mathcal{L}$; that is, the modal operators are allowed anywhere so long as the result is a wff.

5 The Expressive Power of Modal Graph Theory

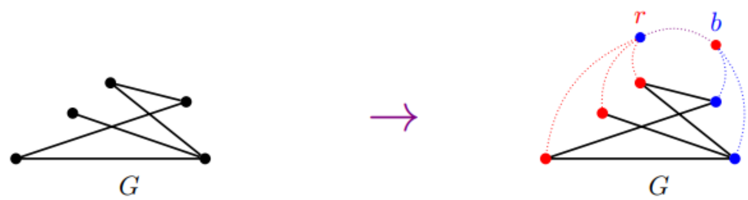
Now, we restrict our attention to $\mathcal{W} =$ the class of all graphs, with language $\mathcal{L}_{\sim}$, where $\sim$ is the irreflexive, symmetric binary edge relation. A graph G is, then, a set of vertices with the edge relation $\sim$.

Now for some theorems:

Theorem 1: In the class of graphs, 2-colorability is expressible in the modal language of graph theory. There is a sentence $p_2 \in \Diamond\mathcal{L}_\sim$ such that for any graph G,

$$G \models p_2 \Leftrightarrow G \text{ is 2-colorable.}$$

Similarly, k-colorability is expressible by a sentence p_k for any finite k.
Proof:
Take the sentence p_2 to be "possibly, there are adjacent nodes r and b, such that every node is adjacent to exactly one of them and adjacent nodes are connected to them oppositely." We may picture the situation as follows:

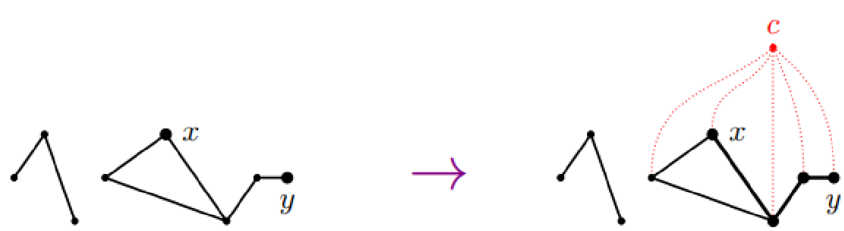


□

Theorem 2: In the class of graphs, connectivity of nodes is expressible in the modal language of graph theory. There is a formula $\varphi(x, y) \in \Diamond\mathcal{L}_\sim$ expressing that vertex x is connected to vertex y.

$$G \models \varphi(x, y) \Leftrightarrow x \text{ is connected to } y \text{ in } G.$$

Similarly, there is a sentence in $\mathcal{L}^\Diamond_\sim$ expressing that the graph as a whole is connected.
Proof:
We may take the formula in the modal language to be "necessarily, any vertex c that is adjacent to x and whose neighbors are closed under adjacency is also adjacent to y. Consider the following case to illustrate the formula:



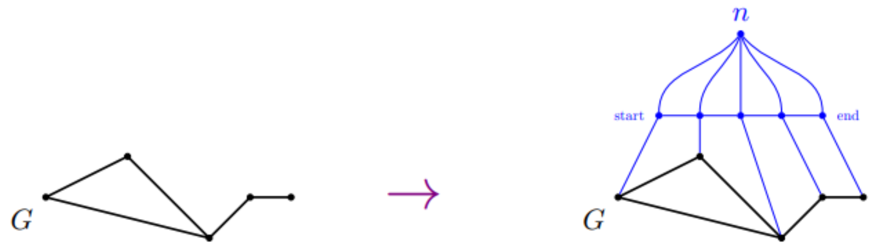
For a nonempty graph to satisfy connectedness as a whole, we may take the formula $\forall x, y \in G(\varphi(x, y))$. □

Note that one may show in the language $\mathcal{L}_\sim$ that 2-colorability and connectedness are *not* expressible (2-colorability is equivalent to infinitely many "no odd-cycle" statements and no-connectedness follows by compactness); thus the modal language $\Diamond\mathcal{L}_\sim$ is strictly stronger than $\mathcal{L}_\sim$.

Theorem 3: In the class of graphs, finiteness is expressible in the modal language of graph theory. There is a sentence $\varphi \in \mathcal{L}^\Diamond_\sim$ such that for any graph G,

$$G \models \varphi \Leftrightarrow G \text{ is finite.}$$

Proof:
A graph G is finite if and only if possibly, there is a point n, whose neighbor graph is connected, with all vertices of degree 2 within that neighbor set, except exactly two vertices of degree 1 in that neighbor set—a starting node and an ending node—and all other nodes of the graph are adjacent to exactly one neighbor of n in a bijective correspondence. Let us clarify with another picture:

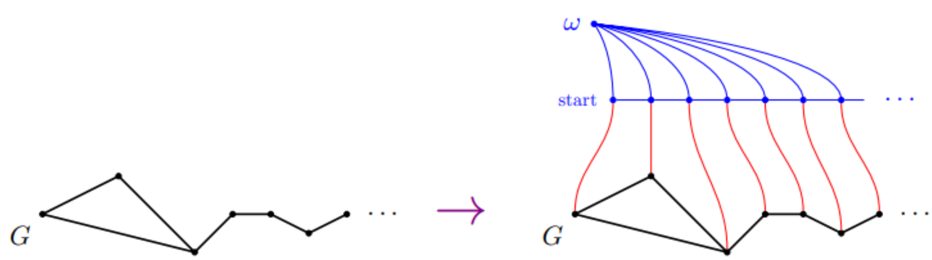


□

Theorem 4: In the class of graphs, countability is expressible in the modal language of graph theory. There is a sentence $\varphi \in \mathcal{L}^{\Diamond}_{\sim}$ such that for any graph G,

$$G \models \varphi \Leftrightarrow G \text{ is countable.}$$

Proof:
A graph G is countable if and only if possibly, there is a point ω, whose neighbor graph is connected, with all vertices of degree 2 except exactly one starting node with degree 1 (amongst the neighbors of ω)—so that these neighbor vertices form an infinite linked chain from the starting node—and furthermore, all other nodes in the graph are adjacent to distinct neighbors of ω:



□

We can likewise establish other cardinality results by matching up vertices of a graph G to distinct subsets of the ω-chain as in the previous proof to obtain

Theorem 5: In the class of graphs, the property of having size at most continuum $\mathfrak{c}$ is expressible in the modal language of graph theory. There is a sentence $\varphi_{\leq\mathfrak{c}} \in \mathcal{L}^{\Diamond}_{\sim}$ such that for any graph G,

$$G \models \varphi_{\leq\mathfrak{c}} \Leftrightarrow |G| \leq \mathfrak{c}.$$

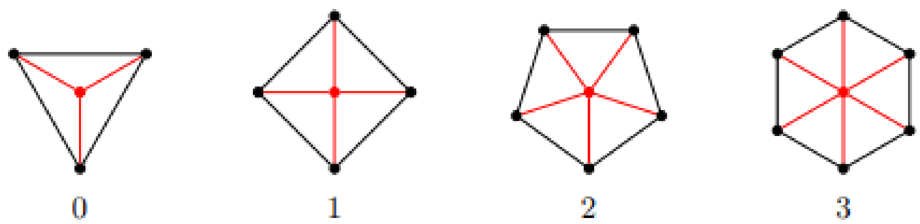
We wrap up this section with

Theorem 6: We can express in the full modal language when a graph G has size exactly $\mathfrak{c}$.
Proof(ish):
A graph G has size continuum if and only if it has size at most continuum and necessarily, if the graph consists of a node a and its neighbors, then necessarily, in any extension having two connected components (the other being a node b and its neighbors), such that the union has size at most continuum, then necessarily, in any extension in which that is exhibited by an association as above of nodes with subsets of ω (as in Theorem 5), then possibly, in a further extension in which that remains true, there is another copy of ω and a new association of the neighbors of a with distinct subsets of it, in such a way that every pattern for that copy of ω is realized by a node adjacent to a. □

6 The Interpretability of Modal Graph Theory

We begin first with a toy example:

Theorem 7: True arithmetic is interpretable in finite modal graph theory. There is a translation $\varphi \to \varphi^*$ from arithmetic sentences φ to sentences of modal graph theory φ^*, such that $\langle \mathbb{N}, +, \cdot, 0, 1, \leq \rangle \models \varphi \Leftrightarrow \varphi^*$ holds in every finite graph in the class of finite graphs.
Proof(ish):
We define the natural numbers isomorphically in the class of graphs as



and we define addition, multiplication, and the order-relation naturally. □

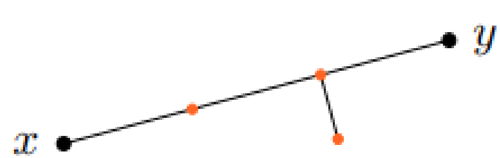
As the title of the talk suggests, we have greater ambitions about the interpretive power of modal graph theory than to use it for arithmetic, and the next theorem shows the first instance of this nontrivial interpretive power. Recall that the a set is called "hereditarily finite" if it is composed of a finite number of finite sets. Similarly, we may define "hereditarily countable sets"; let H_{ω_1} denote the set of all hereditarily countable sets.

Theorem 8: Hereditarily countable set theory is interpretable in countable modal graph theory. We shall represent hereditarily countable sets with countable graphs and vertices and define a translation $\varphi \to \varphi^*$ of set-theoretic assertions φ to modal graph-theoretic assertions φ^* such that

$$\langle H_{\omega_1}, \in \rangle \models \varphi(a_0, ..., a_n) \Leftrightarrow \Gamma_0 \oplus ... \oplus \Gamma_n \models \varphi^*(\hat{a_0}, ..., \hat{a_n}),$$

where $(\Gamma_i, \hat{a_i})$ is a countable graph and vertex representing the set a_i, and $\Gamma \oplus \Lambda$ is the disjoint sum of graphs Γ and Λ.
Proof:
Every hereditarily countable set a is a member of some countable transitive set T since, e.g., $a \in \mathrm{TrCl}(\{a\})$. The structure $\langle T, \in \rangle$ is a countable set with the well-founded relation $\in$, so we can therefore find an isomorphism between the set with relations given by $\in$ and a countable transitive set using the Mostowski collapse. The idea of the rest of the proof, then, is to uniquely "code" this collapsed set, keeping track of where the set a goes, and then to show how to interpret $a \in b$ and $a = b$ for b any other hereditarily countable set. To code the set a with its respective collapse (as above), we define a graph consisting of:

- a node t (standing in for the set T which contains a) whose neighbors are related under the well-founded extensional relation $x \leq_t y$ defined to hold between two neighbors x, y of t whenever they are in the following configuration:



- a copy of ω as in the proof of countability that is in bijection with the neighbors of t and the nodes used in the $\leq_t$ relation (this is to ensure that t has countably many nodes below it, since the set T is countable)
- a vertex $\hat{x}$ pointing toward t and one of its neighbors x in the $\leq_t$ relation (the $\hat{x}$ element represents our analog of the set a, and the node x is its image in the Mostowski collapse of the graph under the $\leq_t$ relation—keeping track of where a goes in the collapsed version of T)

One should check that the $\leq_t$ relation really is well-founded and extensional (we omit the details here). After we have coded the set a into this graph with pointed node (Γ, x), one can show that one code (Γ, x) codes an element of another code (Λ, y) if (Γ, x) is isomorphic to the code of some $\leq_t$ predecessor of y in Λ; in this way, we are able to express of two coded "sets" $x \in y$ and $x = y$. Extending the interpretation through connectives then works just as it would in the proof of the interpretability of arithmetic. □

It turns out that this method for interpreting certain hereditary sets generalizes quite extensively. Before we see the generalizations to larger cardinals, we need to define a certain restrictions our cardinals need to meet. We call a cardinal κ "stably representable" in modal graph theory if there is a property φ expressible in modal graph theory with the following properties:

- there is a graph G with a vertex v satisfying $\varphi(v)$;
- if $\varphi(v)$ holds in a graph G of a vertex v, then v has exactly κ many neighbors in G;
- the truth of $\varphi(v)$ in G depends only on the induced subgraph consisting of v and its neighbors and the neighbors of the parameters;
- if $\varphi(v)$ holds in G and in an extension graph H, then v has the same neighbors in G as in H.

With this stability condition on cardinals, we see the true interpretive power of modal graph theory:

Theorem 9: If a cardinal κ is stably representable in modal graph theory, then $\langle H_{\kappa^+}, \in \rangle$ is interpretable in modal graph theory.

Theorem 10: If κ is stably representable in modal graph theory, then so is κ^+, 2^κ, $\aleph_\kappa$, $\beth_\kappa$, the next $\beth$-fixed point above κ, the first $\beth$-hyper-fixed point above κ, and so on. Here, the first $\beth$-hyper-fixed point above κ is the cardinal λ that is the λ^{th} $\beth$-fixed point above κ.

Corollary 10.1: Set-theoretic truth in V_θ, where θ is the first $\beth$-hyper-fixed point, is interpretable in modal graph theory. Moreover, we may proceed arbitrarily high along the $\beth$-hyper-fixed points.

If we interpret generously, then Corollary 10.1 tells us that set-theoretic truth can be interpreted indefinitely into the set-theoretic cumulative hierarchy. It does not tell us that *every* set-theoretic truth is interpretable in terms of modal graph theory, but it does entail that most non-set-theoretic mathematics can take place within modal graph theory.

7 An Actuality Operator and Full Set-Theoretic Truth

With the addition of a new "actuality" operator, it has been shown that modal graph theory actually can interpret full set-theoretic truth. This actuality operator @ allows one to refer to the "actual world" (the ground model) and more generally to the various worlds that are in effect referenced during the course of interpreting a modal statement. As an example of how one may use the actuality operator, consider the meaning of the following statement in modal graph theory:

$$\Diamond \exists x \forall y (x \sim y \Leftrightarrow (@y \wedge @\forall z(\neg y \sim z)))$$

This statement asserts that it is possible of any graph G that there is a node which is adjacent to every and only the isolated points *of G itself.* The use of the actuality operator is not restricted to the ground model of the potentialist system either; one may refer to the "actual" world at any level of modal operators (and then one need specify at which world each modal and actuality operator is occurring).

8 Meta-Mathematical Issues and Closing Remarks

The issue with the actuality operator is that it likely is not expressible in terms of the language $\mathcal{L}_{\sim}^{\Diamond}$; the ground world to which @ refers might change from formula to formula, so there is no way to pin down the operator with the modal operators. It has not been proven yet, but it is conjectured by the authors of the paper that any modal model theory with actuality is not fully a modal language, but an augmented modal language. Moreover, since set-theoretic truth is not referentiable within set theory, it must be noted that the theorem stating that set-theoretic truth is interpretable in modal graph theory is not itself a theorem of, say, ZFC. Instead, we may interpret the theorem as a theorem-scheme of ZFC such that every set-theoretic truth Σ_i may be interpreted as one would in the proof of the interpretability of set-theoretic truths as a whole; in other words, each instantiation of the theorem-scheme has exactly the same proof.

Further still, it must be noted that for any interesting system of modal model theory, the underlying class of worlds, $\mathrm{Mod}(T)$, is just that—a proper class, and the recursive truth-definition the authors give for theorems of modal model theory is not a set-like recursion. Therefore, with what has been used in this talk, we cannot legitimately define the modal model theory within ZFC. The authors leave the question of whether one may define a model's satisfaction of statements in modal graph theory open.

Is it fair to say, then, that modal graph theory may serve as a foundation for all mathematics? For most mathematical practical purposes, sure. Its status as the end-all be-all foundation is left open by the authors, but it is conjectured unlikely that the underlying meta-mathematical issues are able to be settled under ZFC alone.