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On O-Minimally Definable Groups and Fields
Louise Hay Logic Seminar

1 Overview

- Background on O-Minimality
- O-Minimally Definable Groups and Fields
- Real and Algebraic Closedness

2 Background on O-Minimality

We begin with some definitions regarding o-minimality. We fix a model $(\mathcal{M}, <, ...)$ of **DLO** without endpoints.

Definition 1. *We say that* $\mathcal{M}$ *is "o-minimal" if every definable set in* $\mathcal{M}$ *is a finite union of points and intervals (including* $\pm\infty$*).*

An important fact about o-minimal structures is that definable sets are very well controlled, even in Cartesian powers. In particular, if $\mathcal{M}$ is o-minimal, then any definable set in $\mathcal{M}^n$ is a finite, disjoint union of special sets. Specifically, we have the so-called "cell decomposition theorem":

Theorem 1 (Cell Decomposition)**.** *For any parameter set* $A \subseteq \mathcal{M}$*: if* $X \subseteq \mathcal{M}^n$ *is* A*-definable, then* X *is a finite, disjoint union of* A*-definable cells.*

(Provide some geometric intuition on cells, noting how they are built up in dimension. Give terminology for "thin" vs. "thick" cells, projections onto a k-dimensional subspace to show X is homeomorphic to $\pi(X)$ for some $k \leq n$.)

We note that there is a well-defined notion of dimension for cells—namely, the number of thick cells used in constructing it. We can characterize this differently, based on a notion of generic points.

Definition 2. *Let* $\bar{a} \in \mathcal{M}^n$*,* $A \subseteq \mathcal{M}$*. We define* $\dim(\bar{a}/A)$ *to be the least cardinality of a subtuple* $\bar{a}'$ *of* $\bar{a}$ *such that* $\bar{a} \in dcl(A \cup \bar{a}')$*. We can define the dimension of types over* A *using the dimension of realizations (assuming we work in a monster model or something).*

We note some quick facts about this notion of dimension:

- if $A \subset B$, then $\dim(\bar{a}/B) \leq \dim(\bar{a}/A)$
- $\dim(\bar{a}/A)$ is the cardinality of any maximally algebraically independent over A subtuple of $\bar{a}$. **(Here, using $acl = dcl$ in o-minimal setting.)**

Definition 3. *Let* $X \subseteq \mathcal{M}^n$ *be* A*-definable for some* $A \subseteq \mathcal{M}$*. We define the dimension of* X *as*

$$\dim(X) = \max(\dim(\bar{a}/A) : \bar{a} \in X).$$

Further, we say that $\bar{a} \in X$ *is a generic point of* X *over* A *if* $\dim(X) = \dim(\bar{a}/A)$*.*

(Remark that definable bijections preserve this dimension of definable sets. Also, dimension of a disjoint union is maximum dimension of the components.)

There is one other general fact about definability in o-minimal structures—regarding definable functions—which we will make use of.

Theorem 2 (Piecewise Continuity)**.** *Let $f : X \to \mathcal{M}$ be a definable function on the definable set $X \subseteq \mathcal{M}^n$ (suppressing parameters). There is a finite, definable decomposition of X into subsets $X_1, ..., X_m$ such that $f \restriction_{X_i}$ is continuous for each i.*

In other words, definable functions on definable sets in the o-minimal setting are (piecewise) very well behaved. Moreover, we can take cell decompositions of domains of definable functions which are compatible with the piecewise behavior of the definable function—that is, we may as well assume the definable sets in the previous theorem are cells.

Our final background result will help our ultimate theorem—that any field definable in an o-minimal structure is either real or algebraically closed.

Definition 4. *We say that a ring R is "formally real" if -1 is not a sum of squares in R.*

Definition 5. *A "real closed field" is a formally real field with no formally real, algebraic extensions.*

The idea here is that all a real closed field misses in falling short of being algebraically closed are imaginary elements, even roots of "negative" elements. The field of real numbers is the paradigm example of a real closed field—in some sense, all one *misses* is $\sqrt{-1} =: i$.

Theorem 3 (Artin-Schreier)**.** *Let F be a formally real field. The following are equivalent:*

1. *F is real closed.*
2. *F has a (nontrivial) finite, algebraically closed extension.*

(Remark that this is different from Ronnie's seminar. There, we characterized F as real closed iff F is not algebraically closed but $F(i)$ is. This is equivalent to the above: if F has a finite, algebraically closed extension, the extension must be degree two since F is formally real—all odd degree polynomials have a root in F.)

3 O-Minimally Definable Groups and Fields

3.1 Groups

Our first main result characterizes o-minimally definable groups as manifold-like structures.

Theorem 4 (Pillay 1988)**.** *Let G be an $\emptyset$-definable group in an o-minimal structure $\mathcal{M}$ with $\dim(G) = n$. Then there are a* ***large*** *$\emptyset$-definable subset $V \subset G$ and a topology T on G such that:*

1. *G with T is a topological group;*
2. *V is a finite, disjoint union of $\emptyset$-definable sets $U_1, ..., U_r$ s.t. each U_i is T-open in G and there is a definable homeomorphism between U_i and some open subset U_i' of $\mathcal{M}^n$.*

(Remark about "large" in definition: a large subset Y of a definable set X is one such that $\dim(X \setminus Y) < \dim(X)$; equivalently, Y contains every generic [over the parameters of X and Y] point of X. Note also that being large is a definable condition.)

To see why this makes such groups "look like" manifolds, note the following fact.

Fact 1. *Let X be a large definable subset of G. Finitely-many translates of X cover G.*

(Remark that this is reasonable, given dimension considerations.)

In other words, each point in G is included in some (translate of) of the U_i's, open sets which are homeomorphic to open sets of $\mathcal{M}^n$. In the case where $\mathcal{M} = \mathbb{R}$, this is just what it is for the group to be a manifold.

Proof (sketch) of Theorem 4.

- Take cell decomposition of G by which $U_1, ..., U_r$ are the n-dimensional cells, and set $V_0 := U_1 \cup \cdots \cup U_r$. Note, then, that V_0 is by definition large in G. Moreover, each of these cells (and therefore V_0 itself) is definably homeomorphic to an open set in $\mathcal{M}^n$, and we identify them.
- By Piecewise Continuity, we can refine the U_i further such that inversion is continuous on the pieces of the refinement of U_i. Then the pieces of U_i either map continuously into U_j or not at all.
- Note that every generic point of G is in one of the U_i's. Denote by U_i^j the smallest subset of U_i containing all generic points of G which map to U_j under inversion–and we know the generics map to some U_j since if $\bar{a}$ is generic, so is $\bar{a}^{-1}$. Setting, $V_1 := \bigcup_{i,j} U_i^j$, we see that V_1 is large in V_0 (hence in G) and that inversion is continuous $V_1 \to V_0$.
- Note that we can do a similar sort of move to find a large (in V_0), definable subset $Y_1 \subset V_0 \times V_0$ on which multiplication is continuous $Y_1 \to V_0$.
- Next, we overlay the continuity conditions on V_1 and Y_1 generically so that we obtain a large subset of $G \times G$ for which multiplication is continuous and projections onto either coordinate yield elements of V_1. **(Draw picture.)**
- In the overlaying process, we impose further conditions on the subset leftover to ensure that, for any element $(\bar{a}, \bar{b})$ in the "good" set, the product $\bar{a}\bar{b}$ ends up back in V_1. **(Three dimensionality of the picture.)**
- This process yields a definable partition of G into subsets $W_1, ..., W_k$ for which $V := \bigcup W_i$ is large in G and s.t. inversion is a continuous map $V \to V$ and multiplication is continuous $V \times V \to V$. It remains to define the new topology T such that inversion and multiplication turn into T-continuous operations on all of G; this requires that "good" translates of open sets remain open.
- Define the topology T as follows:

  $$Z \subset G \text{ is } T\text{-open iff for every } g \in G,\ g \cdot Z \cap V \text{ is open.}$$

  This topology treats translates of V as charts in the sense of manifolds. Due to the following fact, we can think of the transition maps between these translates as continuous—so inversion and multiplication will extend continuously to all of G:

  $$\text{For } Z \subset V \text{ and } g \in G,\ g \cdot Z \text{ is } T\text{-open iff } Z \text{ is open in } V.$$

□

3.2 Fields

The previous argument may be adapted to reconcile both field operations with the large subset V of the field F. The topology T defined in this case is with respect to additive translates, but essentially everything works the same.

4 Real and Algebraic Closedness

Finally, we wish to prove the following:

Theorem 5. *If F is an infinite field definable in the o-minimal structure $\mathcal{M}$, then F is either real closed or algebraically closed.*

To do so, we invoke the following lemmas.

Lemma 1. *Let the infinite field F be definable in $\mathcal{M}$, and let K be a proper, finite field extension of F. Then K is definable in $\mathcal{M}$ and $\dim(K) > \dim(F)$.*

Proof. The idea here is to define K using F^k, where $[K : F] = k$—i.e., where k is the F-vector space dimension of K. Clearly, then, $\dim(K) = k \cdot \dim(F) > \dim(F)$ since $k > 1$ as the extension is proper. □

Lemma 2. *(In the relevant setting) If $\dim(F) > 1$, then F is algebraically closed.*

(Remark that the second lemma above depends upon the topological and definable structure of the field F.)

Now for the proof of the final theorem.

Proof of Theorem 5. If F is not algebraically closed, then it has a proper, finite, algebraic extension—say K. By the first lemma above, K is definable in $\mathcal{M}$ and has larger dimension than F, so it is algebraically closed. But then F has a proper, finite extension to an algebraically closed field, so it is real closed by the Artin-Schreier Theorem (**Note that F is definable as an *ordered* field, so it is formally real**). □