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Date: 5 October 2023

Paradox, Cardinals, Gödel, and Cantor
Louise Hay Logic Seminar

1 Overview

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- Gödel's incompleteness theorem and an argument for inconsistency
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2 Logical paradoxes and our naïve proof methods

Consider the so-called "Liar Sentence": 'This sentence is false.' If we assume the sentence is true, then it expresses its own falsehood, but if we assume it is false, then the sentence expresses something true—hence is true. In mathematics, since at least the turn of the twentieth century, logical paradoxes have cropped up as well, particularly in set theory—the Russell set, the Burali-Forti paradox, etc. Whereas set-theoretic paradoxes are fairly recent in comparison with their semantic counterparts, thinkers have been aware of and have battled with paradoxes such as the Liar Sentence since at least ancient Greece, yet very little can be said of such paradoxes as regards their solutions: do we count the paradoxes as false or perhaps meaningless? do we construct formal systems to avoid them as von Neumann did with the set/class distinction? do we bar ourselves from speaking of them altogether? Let us come at the question of what to make of these semantic and set theoretic paradoxes from an altogether different direction than head-on.

What does it mean in mathematics to *prove a theorem*? In lecture halls and in most academic papers, to prove a theorem is to provide a process by which we establish the truth of some mathematical claim. Usually, this process proceeds by reasoning from previous results we have proven, but it cannot go on backward indefinitely in this way; instead, we hold certain propositions to be true "self-evidently" and begin our preliminary reasoning from these axioms, eventually obtaining more complicated results. Now, why have we cared to ask what is involved in proving mathematical statements to be true? For just this reason: this naïve method of proof—the process of reasoning by which we establish mathematical truths—runs us into a paradoxical corner.

It does not take a stretch of the imagination to think our naïve proof methods could be completely formalized, the axioms codified and the rules of inference encoded; indeed, some might say this is precisely the goal of a logician—to formally uncover the workings of our reasoning, to decide precisely which arguments are *good* (whatever that may mean) and for what reason(s). One might hope that, within this imagined formalization of our long-used naïve proof method, all recursively-definable functions would be representable; in fact, some might insist they must be (for how couldn't they be? after all, they seem to be naïvely recursively definable). But at this point our formalization of the naïve proof method has run into a serious issue: Gödel's first incompleteness theorem. The formalization of our usual proof methods will be unable to prove or refute certain sentences. We may be able to push this uncomfortable truth to the side if it weren't for the paradox at the heart of the issue—namely, such a sentence guaranteed by Gödel's first incompleteness theorem *is* provably true in the naïve sense.

3 Gödel's incompleteness theorem and an argument for inconsistency

Let us examine this alleged paradox more closely. (Following Priest (1979)) Suppose that we have encoded our naïve proof method in a formal system P, and let g be the Gödel code of the Gödel sentence

$$\neg\exists x \text{Prov}(x, g)$$

where here 'Prov(x, y)' is the recursively definable relation which holds exactly when x is the Gödel number of a proof in P of the sentence with Gödel code y. This sentence should be recognizable as the sentence which Gödel's theorem isolates as undecidable in the formal system; however, consider the following "naïve proof" of its truth:

Proof. If there is some Gödel code of the proof of the Gödel sentence, then the sentence expressing this is true:

$$\exists x \text{Prov}(x, g) \rightarrow \ulcorner \exists x \text{Prov}(x, g) \urcorner \text{ is true.}$$

Note that this conditional holds for any sentence in the formal system. Since this conditional holds, we may replace the consequent with 'g is provable', as this is what the Prov relation tells us. Hence:

$$\exists x \text{Prov}(x, g) \rightarrow \text{ g is provable.}$$

Assuming soundness of the formal system, then, we have that g is true, so that

$$\exists x \text{Prov}(x, g) \rightarrow \neg\exists x \text{Prov}(x, g).$$

We may then conclude that $\neg\exists x \text{Prov}(x, g)$. □

What are the crucial ingredients in concocting this paradox that the Gödel sentence is formally undecidable whilst being (naïvely) provably true? For one thing, the assumption of soundness of our formal theory; however, we should not be so quick to give this up, for if we did not believe that our naïve methods of proof yielded truths, then there would be no point to our doing mathematics. For another, we have assumed, in applying Gödel's theorem, that the system formalizing the naïve proof method is consistent. One option we have, then, if one of our goals in logic is to formalize the methods by which we prove things in mathematics classrooms, for example, is to drop this consistency requirement for our formal system. In other words, to solve the paradox, we may be required to recognize our naïve methods as inconsistent and to accept certain paradoxes as both true and false (as our proof of the Gödel sentence could easily be modified to show its truth as well).

4 Cantor's theorem (with a consistent proof)

What is at stake if we were to make this move? Of course, the assumption that the underlying logic of our mental reasonings is classical would have to be replaced with another paraconsistent logic—but we would need to be careful not to end in triviality; we don't want a theory which proves everything, for that would be effectively useless. What principles would we have to forego, then, in replacing this assumption? First and most substantially, the law of non-contradiction—the belief that no contradiction (that is, no sentence and its negation) can be true. What are the downstream effects of such a move? For one thing, we would be required to more carefully consider our methods of proof and the inference rules we are allowed. One worry, then, is how many of our classical results in mathematics would have to be revised since the old methods of proof may be invalid in the new paraconsistent system. For the rest of the talk, we focus on such set theoretic results in an attempt to show the feasibility of the whole enterprise.

Consider the following classical result.

Theorem 1 (Cantor's Theorem). *Suppose that A is any set and $\mathcal{P}(A)$ its powerset. Then $|A| < |\mathcal{P}(A)|$. (That is, for any function $f : A \to \mathcal{P}(A)$, f is not a surjection.)*

Proof. Fix any function $f : A \to \mathcal{P}(A)$, and define the subset $B \subseteq A$ as follows:

$$B := \{x \in A \mid x \notin f(x)\}.$$

Now, suppose **for the purpose of contradiction** that f is surjective; then there is some $y \in A$ such that $f(y) = B$. However, if $y \in B$, then $y \notin f(y) = B$, so y is not a member of B. But then, since $y \notin B = f(y)$, we must have that $y \in B$, so y cannot be a member of B; we have reached a contradiction. Thus, f must not be a surjection. □

It is likely clear where in this proof our new worry might be. If we are to adopt a paraconsistent logic as the formal background to our mathematical reasoning, then we are not allowed reductio ad absurdum arguments without a second thought like Cantor's proof employs (but we are still allowed some of them, as the rules in Weber's appendix (2012) show). Our goal for the moment, then, becomes to see if we may after all salvage Cantor's theorem, which is at the heart of modern set theory.

5 Relevant, inconsistent, naïve set theory with preliminary definitions and lemmas

We begin by redefining what a set is in our new set theory, and since we are working with the naïve methods (wherein we care not so much about inconsistency as we do nontriviality), we define sets very simply and intuitively according to the two axioms:

Axiom 1 (Abstraction). $x \in \{z \mid \varphi(z)\} \leftrightarrow \varphi(x)$

Axiom 2 (Extensionality). $\forall z(z \in x \leftrightarrow z \in y) \leftrightarrow x = y$

From these two axioms and the formal system set up in the appendix to Weber (2012), it can be shown (e.g. as in Weber 2010, 2012) that unrestricted comprehension follows from the Abstraction axiom, and since this comprehension schema is unrestricted, the rest of ZF follows easily.

Next, we set some definitions.

Definition 1. *The "Russell set" is the set $R := \{x \mid x \notin x\}$.*

From this definition, we may see (via the system of inconsistent reasoning) that $(R \in R) \wedge (R \notin R)$ holds; this is the paradox of Russell's set, and it is both true and false in our new system. Using the Russell set, we define the "Routley reduct" of any set.

Definition 2. *Fix any set X. The "Routley reduct of X" is the set $\mathcal{R}(X) := \{x \in X \mid R \in R\}$.*

The key property of the Routley reduct of any set X is that for any $x \in X$, we have both $x \in \mathcal{R}(X)$ and $x \notin \mathcal{R}(X)$. This follows easily from the Abstraction axiom and the fact that $R \in R$ and $R \notin R$ both hold in the system. It is obvious that the Routley reduct of any set has some inconsistent properties, as we have just noted, but the crucial property we'll need for our modified proof of Cantor's theorem is the following.

Lemma 1. *Fix any set X. Then $\mathcal{R}(X) \neq \mathcal{R}(X)$. (Also, $\mathcal{R}(X) = \mathcal{R}(X)$.)*

Proof. Let $x \in X$. Then we have $x \in \mathcal{R}(X)$, as was just noted. On the other hand, we have $x \notin \mathcal{R}(X)$. For ease of notation, let us denote by $\mathcal{R}_1(X)$ the version of $\mathcal{R}(X)$ which contains x and let us denote by $\mathcal{R}_2(X)$ the set $\mathcal{R}(X)$ when viewing it as not containing x. Then the following fails:

$$\forall z(z \in \mathcal{R}_1(X) \leftrightarrow z \in \mathcal{R}_2(X))$$

so that by Extensionality $\mathcal{R}_1(X) \neq \mathcal{R}_2(X)$. Dropping our notation, we have $\mathcal{R}(X) \neq \mathcal{R}(X)$. □

We will also make use of the following similar result.

Lemma 2 (Lemma 2.1 in Weber, 2012)**.** *Let X be any set such that $X \neq X$. Then $Y \neq X$ for any set Y.*

Proof. We proceed by cases. First, if $Y = X$, then by substitution $Y \neq X$, so $Y \neq X$. Conversely, if $Y \neq X$, then $Y \neq X$. □

Finally, some definitions regarding functions and cardinality. (Functions are defined analogous to how they are defined in classical mathematics.)

Definition 3. *A function $f : X \to Y$ is "injective" iff $f(x) = f(y) \vdash x = y$.*

Note that, since contraposition of the deduction relation '$\vdash$' does not hold in general in our system, this is not the same as saying $x \neq y \vdash f(x) \neq f(y)$.

Definition 4. *A function $f : X \to Y$ is "surjective" iff $\forall y(y \in Y \to \exists x(x \in X \wedge f(x) = y))$.*

Ordinals are defined analogously to how they are classically defined (but the set axioms give unusual results such as *On* itself being an ordinal). Cardinals are defined in more details than we have time to get into, but the essential notion plays the same role: the cardinality of a set X is the least ordinal κ with which the set is in bijective correspondence. (The extra details are due to the fact that we are using a relevance logic, so we need to be careful with our conditionals and deductions; see Weber 2012.)

6 Cantor's theorem, revisited (with an inconsistent proof)

We are now in a position to provide a new proof of Cantor's theorem within the relevant formal system using the previous results and definitions.

Theorem 2 (Cantor, 1892)**.** *Suppose that X is any nonempty set and $\mathcal{P}(X)$ its powerset. Then $|X| < |\mathcal{P}(X)|$.*

Proof (Weber, 2012). First, we see that there is an injection $f : X \to \mathcal{P}(X)$: define the function f such that $f(x) = \{z \in X \mid z = x\}$. This is a well-defined function and it is injective since $\{x\} = \{y\}$ proves that $x = y$ by Extensionality. Since X injects into $\mathcal{P}(X)$, $|X| \leq |\mathcal{P}(X)|$.
Next, we wish to establish that no surjection exists from X onto its powerset. To this end, let $f : X \to \mathcal{P}(X)$ be any function and consider the Routley reduct of X

$$\mathcal{R}(X) = \{x \in X \mid R \notin R\}.$$

Clearly, $\mathcal{R}(X) \in \mathcal{P}(X)$. By Lemma 1, $\mathcal{R}(X) \neq \mathcal{R}(X)$, so for any $z \in X$, $f(z) \neq \mathcal{R}(X)$ by Lemma 2 (as $f(z)$ is some set). Thus, there is some subset of X which f does not meet; hence, f is not surjective. □

In our new, inconsistent system we have salvaged Cantor's theorem, so there is hope for the salvation of further set theoretic results. Indeed, Weber (2010 and 2012) goes on to show very many classical results in set theory in the same formal system.

7 Further results in Weber's relevant set theory

Theorem 3 (Cantor-Schröder-Bernstein)**.** *For any two sets X and Y,*

$$|X| \leq |Y|,\ |Y| \leq |X| \vdash |X| = |Y|.$$

Theorem 4. $|V| < |\mathcal{P}(V)|$.

Corollary 4.1. $|V| = |\mathcal{P}(V)|$ *and* $|\mathcal{P}(V)| < |V|$.

Corollary 4.2. $|V| \neq |\mathcal{P}(V)|$, *and so* $|\mathcal{P}(V)| \neq |\mathcal{P}(V)|$.

Theorem 5 (Wellorder of V)**.** *There is a contra-injection $\Omega : V \to On$.*

Corollary 5.1 (Zermelo, 1904)**.** *Every set can be well-ordered.*

Theorem 6 (Choice)**.** *There is a choice function on every nonempty set X.*

Corollary 6.1 (Global Choice)**.** *There is a choice function on V.*

Theorem 7. $|V| = |On|$.

Theorem 8 (König's Lemma)**.** *Let cardinals $\kappa_i < \lambda_i$ for all i in some index set I. Then*

$$\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i.$$

Theorem 9. $\aleph_{On} = On$.

Theorem 10. *GCH both holds and fails at On.*

Theorem 11. *The continuum hypothesis is false.*

Theorem 12 (Large cardinals)**.** *There exists regular, inaccessible, Mahlo, and measurable cardinals.*

8 Bibliography

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