Stat 411 – Homework 04 ***Solutions***

1. Problem 6.2.7 in HMC7. The PDF for the $\mathsf{Gamma}(4,\theta)$ distribution is

$$f_\theta(x) = \frac{1}{6\theta^4} x^3 e^{-x/\theta}, \quad x > 0, \quad \theta > 0.$$

   (a) For the Fisher information, we first need second derivative of log-PDF:

$$\frac{\partial^2}{\partial\theta^2} \log f_\theta(x) = \frac{\partial^2}{\partial\theta^2}\Big[\text{const} - 4\log\theta - \frac{x}{\theta}\Big] = \frac{4}{\theta^2} - \frac{2x}{\theta^3}.$$

   If we recall that the expected value of a $\mathsf{Gamma}(\alpha,\beta)$ random variable is $\alpha\beta$ (see middle of p. 158 in HMC7), then

$$I(\theta) = -\mathsf{E}_\theta\Big[\frac{\partial^2}{\partial\theta^2}\log f_\theta(X)\Big] = \frac{\mathsf{E}_\theta(2X)}{\theta^3} - \frac{4}{\theta^2} = 2\cdot\frac{4}{\theta^2} - \frac{4}{\theta^2} = \frac{4}{\theta^2}.$$

   (b) If $X_1,\ldots,X_n \overset{\text{iid}}{\sim} \mathsf{Gamma}(4,\theta)$, then the MLE is found by maximizing the log-likelihood:

$$\ell(\theta) = \log L(\theta) = \text{const} - 4n\log\theta - n\bar{X}/\theta.$$

   Setting the derivative equal to zero and solving for θ gives:

$$\frac{\partial}{\partial\theta}\ell(\theta) = -\frac{4n}{\theta} + \frac{n\bar{X}}{\theta^2} \overset{\text{set}}{=} 0 \iff \hat\theta = \frac{\bar{X}}{4}.$$

   If we recall that the variance of a $\mathsf{Gamma}(\alpha,\beta)$ random variable is $\alpha\beta^2$ (see middle of p. 158 in HMC7), then

$$\mathsf{V}_\theta(\hat\theta) = \frac{\mathsf{V}_\theta(X_1)}{16n} = \frac{4\theta^2}{16n} = \frac{\theta^2}{4n}.$$

   Since $\mathsf{V}_\theta(\hat\theta)$ and the Cramer–Rao lower bound $[nI(\theta)]^{-1}$ are the same, we conclude that $\hat\theta$ is an efficient estimator of θ.

   (c) The asymptotic distribution of $\sqrt{n}(\hat\theta-\theta)$ is $\sqrt{n}(\hat\theta-\theta) \xrightarrow{\text{D}} \mathsf{N}(0,\theta^2/4)$.

2. Problem 6.2.9 in HMC7. First we find the Fisher information for the PDF

$$f_\theta(x) = \frac{3\theta^3}{(x+\theta)^4}, \quad x > 0, \quad \theta > 0.$$

The second derivative of log-PDF is

$$\frac{\partial^2}{\partial\theta^2}\big[\text{const} + 3\log\theta - 4\log(x+\theta)\big] = -\frac{3}{\theta^2} + \frac{4}{(x+\theta)^2}.$$

Then the Fisher information is

$$I(\theta) = -\mathsf{E}_\theta\Big[\frac{\partial^2}{\partial\theta^2}\log f_\theta(X)\Big] = \frac{3}{\theta^2} - 4\int_0^\infty \frac{3\theta^3}{(x+\theta)^6}\,dx = \frac{3}{5\theta^2}.$$

For $X_1, \ldots, X_n \overset{\text{iid}}{\sim} f_\theta(x)$, let $\hat\theta = 2\bar X$. Then $\mathsf{E}_\theta(\hat\theta) = 2\mathsf{E}_\theta(X_1)$ and $\mathsf{V}_\theta(\hat\theta) = 2^2\mathsf{V}_\theta(X_1)/n$. Therefore, we only need to find $\mathsf{E}_\theta(X_1)$ and $\mathsf{V}_\theta(X_1)$. These require some calculus effort, but it's not too bad. I'll use a trick that helps us avoid integration-by-parts.

$$\begin{aligned}
\mathsf{E}_\theta(X_1) &= \int_0^\infty x \cdot \frac{3\theta^3}{(x+\theta)^4}\,dx \\
&= \int_0^\infty [(x+\theta)-\theta]\frac{3\theta^3}{(x+\theta)^4}\,dx \\
&= \int_0^\infty \frac{3\theta^3}{(x+\theta)^3}\,dx - \theta \\
&= \int_\theta^\infty \frac{3\theta^3}{y^3}\,dy - \theta \quad [y = x+\theta] \\
&= \frac{\theta}{2}.
\end{aligned}$$

Similarly, we can get

$$\begin{aligned}
\mathsf{E}_\theta(X_1^2) &= \int_0^\infty x^2 \cdot \frac{3\theta^3}{(x+\theta)^4}\,dx \\
&= \int_0^\infty [(x+\theta)^2 - 2\theta x - \theta^2]\frac{3\theta^3}{(x+\theta)^4}\,dx \\
&= \int_0^\infty (x+\theta)^2\frac{3\theta^3}{(x+\theta)^4}\,dx - 2\theta\mathsf{E}_\theta(X_1) - \theta^2.
\end{aligned}$$

Essentially the same work as above evaluates the remaining integral, and it simplifies to $\mathsf{E}_\theta(X_1^2) = \theta^2$. Therefore, $\hat\theta$ is unbiased and

$$\mathsf{V}_\theta(\hat\theta) = \frac{4}{n}\bigl[\mathsf{E}_\theta(X_1^2) - \mathsf{E}_\theta(X_1)^2\bigr] = \frac{4}{n}\cdot\frac{3\theta^2}{4} = \frac{3\theta^2}{n}.$$

Consequently, the efficiency of $\hat\theta$ is

$$\mathsf{eff}_\theta(\hat\theta) = \frac{[nI(\theta)]^{-1}}{\mathsf{V}_\theta(\hat\theta)} = \frac{5\theta^2/3n}{3\theta^2/n} = \frac{5}{9} < 1.$$

3. The importance of Theorem 6.2.2 in HMC7 is that it gives an approximation the sampling distribution of the MLE when n is sufficiently large. That is, for large n, $\hat\theta_n$ is approximately normal with mean θ and variance $[nI(\theta)]^{-1}$. Since the goal of statistical inference is to give an estimate of an unknown quantity θ, it helps to have a summary of the precision of that estimate to go along with the estimate itself. This precision summary is encoded by the sampling distribution of $\hat\theta_n$. For example, one may use the asymptotically approximate sampling distribution to construct a confidence interval.

4. (Graduate only) Problem 6.2.10 in HMC7. Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathsf{N}(0, \theta)$, where $\theta > 0$ denotes the variance. Let $Y = c\sum_{i=1}^n |X_i|$. The goal is to find c such that Y is an

unbiased estimator of $\sqrt{\theta}$. By linearity, it is clear that

$$\mathsf{E}_\theta(Y) = c \sum_{i=1}^n \mathsf{E}_\theta(|X_i|) = cn\mathsf{E}_\theta(|X_1|).$$

To calculate $\mathsf{E}_\theta(|X_1|)$, take advantage of the symmetry of the normal PDF:

$$\mathsf{E}_\theta(|X_1|) = \int_{-\infty}^{\infty} |x| \cdot \frac{1}{\sqrt{2\pi\theta}} e^{-x^2/2\theta}\, dx = 2 \int_0^\infty x \cdot \frac{1}{\sqrt{2\pi\theta}} e^{-x^2/2\theta}\, dx.$$

For the integral, use substitution (i.e., let $u = x^2/\theta$) so that

$$\mathsf{E}_\theta(|X_1|) = \frac{2\sqrt{\theta}}{\sqrt{2\pi}} \int_0^\infty e^{-u}\, du = \sqrt{\frac{2}{\pi}} \cdot \sqrt{\theta}.$$

Therefore, $\mathsf{E}_\theta(Y) = cn\sqrt{2/\pi}\sqrt{\theta}$ so taking $c = n^{-1}\sqrt{\pi/2}$ makes Y and unbiased estimator of $\sqrt{\theta}$. Since the X_i's are independent, so too are the $|X_i|$'s, and so

$$\mathsf{V}_\theta(Y) = c^2 \sum_{i=1}^n \mathsf{V}_\theta(|X_i|) = cn\mathsf{V}_\theta(|X_1|).$$

To find $\mathsf{V}_\theta(|X_1|)$, we need $\mathsf{E}_\theta(|X_1|^2)$ and $\mathsf{E}_\theta(|X_1|)$. The latter we already have, and the former is the same as $\mathsf{E}_\theta(X_1^2)$ which we know equals θ. Therefore,

$$\mathsf{V}_\theta(|X_1|) = \mathsf{E}_\theta(|X_1|^2) - \mathsf{E}_\theta(|X_1|)^2 = \theta - 2\theta/\pi = \frac{\theta(\pi-2)}{\pi},$$

and so

$$\mathsf{V}_\theta(Y) = c^2 n \frac{\theta(\pi-2)}{\pi} = \frac{1}{n} \cdot \frac{\pi}{2} \cdot \frac{\theta(\pi-2)}{\pi} = \frac{\theta(\pi-2)}{2n}.$$

Recall the Fisher information calculation from class: $I(\theta) = 1/2\theta^2$. Consequently, the Cramer–Rao lower bound for estimating $\sqrt{\theta}$ is

$$\text{LB} = \frac{[\frac{1}{2}\theta^{-1/2}]^2}{nI(\theta)]} = \frac{\theta}{2n},$$

and the efficiency of Y is

$$\mathsf{eff}_{\sqrt{\theta}}(Y) = \frac{\text{LB}}{\mathsf{V}_\theta(Y)} = \frac{\theta/2n}{\theta(\pi-2)/2n} = \frac{1}{\pi-2} < 1.$$