Stat 512: Notes from Professor Ouyang*

Jennifer Pajda-De La O

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Definition 1. Let $(\Omega, \mathscr{B}, \mathbb{P})$ be a probability space. A stochastic process is a family of random variables on $(\Omega, \mathscr{B}, \mathbb{P})$ is $\{X_t\}_{t \in T}$. Here T is an index set. Typically $T = \{0, 1, 2, \ldots\}$, a sequence of random variables or $T = [0, \infty)$, a continuous family of random variables.

Q1: How to specify a stochastic process?

1. How to specify a Random Variable?
   Suppose we have a probability space $(\Omega, \mathscr{B}, \mathbb{P})$ and X is a random variable on $(\Omega, \mathscr{B}, \mathbb{P})$. What do we mean by specifying that X is a standard Gaussian $\mathsf{N}(0, 1)$?

   Look at
   $$\mathbb{P}\{X \in B\} = \int_B \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \mathbb{P} \circ X^{-1}(B).$$
   Here, $X : (\Omega, \mathscr{B}, \mathbb{P}) \to (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ induces a measure $\mathbb{P} \circ X^{-1}$. We need to specify that $\mathbb{P} \circ X^{-1}(B)$ is given by something, i.e. we need to specify the measure that it induces.

2. What do we mean by (X, Y) is a Gaussian with mean 0 and covariance $\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$?
   Here $(X, Y) : (\Omega, \mathscr{B}, \mathbb{P}) \to (\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2))$, where the induced measure is $\mathbb{P} \circ (X, Y)^{-1}$ and
   $$\begin{aligned} \mathbb{P} \circ (X, Y)^{-1} &= \iint_{B_1 \times B_2} \left(\frac{1}{\sqrt{2\pi}} \right)^2 e^{-\cdots} dxdy \\ &= \mathbb{P}\{X \in B_1 \text{ and } Y \in B_2\}. \end{aligned}$$

3. For a stochastic process, to specify the distribution of it, we just give all the finite dimensional distributions. Let μ be the measure on $\mathbb{R}^k$. Then
   $$\mu_{t_1, t_2, \ldots, t_k}(F_1 \times F_2 \times \cdots \times F_k) = \mathbb{P}\{X_{t_1} \in F_1, X_{t_2} \in F_2, \ldots, X_{t_k} \in F_k\},$$
   for all $t_1, t_2, \ldots, t_k \in T$ and $t_1 < t_2 < \cdots < t_k$ (only pick a finite number).

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Q2: Given a family of measures $\mu_{t_1,t_2,\ldots,t_k}$ where $t_1, t_2, \ldots, t_k \in T$, how to know there is indeed a stochastic processes X_t such that

$$\mathbb{P}\left\{X_{t_1} \in F_1, X_{t_2} \in F_2, \ldots, X_{t_k} \in F_k\right\} = \mu_{t_1,\ldots,t_k}\left(F_1 \times F_2 \times \cdots \times F_k\right)?$$

Theorem 1. *(Kolmogorov's Extension Theorem) Suppose we have a family of probability measures which is given by $\mu_{t_1,\ldots,t_k}$, a measure on $\mathbb{R}^k$, $t_1, t_2, \ldots, t_k \in T$, and $t_1 < t_2 < \cdots < t_k$ such that for any $u_1 < u_2 < \cdots < u_l < t_1 < s_1 < t_2 < s_2 < \cdots < t_n < s_n < v_1 < v_2 < \cdots < v_m$ (a sequence of times), then we have the measure*

$$\mu_{u_1 \ldots u_l t_1 s_1 \ldots t_n s_n v_1 \ldots v_m}\left(\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{l \text{ components}} \times \underbrace{F_1}_{\text{for } t_1} \times \underbrace{\mathbb{R}}_{\text{for } s_1} \times F_2 \times \mathbb{R} \times \cdots \times F_n \times \mathbb{R} \times \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{m \text{ components}}\right)$$
$$= \mu_{t_1,\ldots,t_k}\left(F_1 \times \cdots F_k\right) \ \textit{Compatibility Property}$$

then there is a probability space $(\Omega, \mathscr{B}, \mathbb{P})$ and a stochastic process such that

$$\mathbb{P}\left\{X_{t_1} \in F_1, \ldots, X_{t_k} \in F_k\right\} = \mu_{t_1,\ldots,t_k}\left(F_1 \times \cdots F_k\right).$$

Interpretation: If we have a measure μ (distribution), then as long as it satisfies some properties, it has a probability space and a stochastic process.

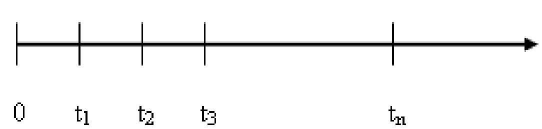
Example 1. Do we have two random variables X_1 and X_2 such that (X_1, X_2) as a random vector is Gaussian? So $\mu_{1,2}$ is a Gaussian measure on $\mathbb{R}^2$. And X_1 is Bernoulli, i.e. μ_1 is a Bernoulli on $\mathbb{R}^1$. No, because all marginals should be Gaussian:

$$\mu_{1,2}\left(F_1 \times \mathbb{R}\right) \neq \mu_1\left(F_1\right).$$

Example 2. (Brownian Motion - BM)

Definition 2. A Brownian motion starting from 0 is a family of random variables $\{B_t\}_{t \geq 0}$ such that

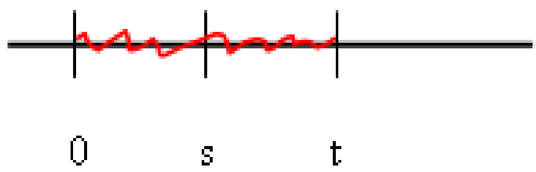
1. $B_0 = 0$
2. It has independent increments. If we have



   then $B_{t_1} - B_{t_0=0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent.

3. $B_t - B_s \sim \mathsf{N}(0, t - s)$

Idea of Brownian Motion:



From 0 to s the displacement $\sim \mathsf{N}(0,s)$. The process keeps moving (as shown above by the red line). The new displacement between s and $t \sim \mathsf{N}(0,t-s)$.
How do we know we have a process? We need to specify the distributions with Kolmogorov's Extension Theorem and make sure we have compatibility. The requirements are 1 - 3 from the definition of Brownian Motion. We have

$$\begin{aligned}
&\mu_{t_1,t_2,\ldots,t_k}\left(F_1 \times F_2 \times \cdots \times F_k\right) \\
&\qquad \overset{\text{want}}{=} \mathbb{P}\left\{B_{t_1} \in F_1, B_{t_2} \in F_2, \ldots, B_{t_k} \in F_k\right\} \\
&\qquad = \int_{F_1} \mathbb{P}\left(t_1, 0, X_1\right) dX_1 \int_{F_2} \mathbb{P}\left(t_2 - t_1, X_1, X_2\right) dX_2 \cdots \int_{F_k} \mathbb{P}\left(t_k - t_{k-1}, X_{k-1}, X_k\right) dX_k.
\end{aligned}$$

Note that the last line of the above is compatible and on $\mathbb{R}^k$.

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Recall: $(\Omega, \mathscr{B}, \mathbb{P})$. A stochastic process is a collection of random variables $\{X_t\}_{t\geq 0}$, $T = \{0, 1, 2, \ldots, n, \ldots\}$ or $T = [0, \infty)$.

Theorem 2. *(Kolmogorov's Extension Theorem) Given a family of probability measures $\mu_{t_1,t_2,\ldots,t_n}$ on $\mathbb{R}^n$, $t_1 < t_2 < \cdots < t_n$ such that*

$$\mu_{s_0 t_1 s_1 t_2 s_2 \cdots s_{n-1} t_n s_n}\left(\mathbb{R} \times F_1 \times \mathbb{R} \times F_2 \times \cdots \times \mathbb{R} \times F_n \times \mathbb{R}\right) = \mu_{t_1 t_2 \cdots t_n}\left(F_1 \times F_2 \times \cdots \times F_n\right),$$

where $\mu_{s_0 t_1 s_1 t_2 s_2 \cdots s_{n-1} t_n s_n}$ is a probability measure on $\mathbb{R}^{2n+1}$. Then there is a probability space $(\Omega, \mathscr{B}, \mathbb{P})$ and a stochastic process $\{X_t\}$ such that

$$\mathbb{P}\left\{X_{t_1} \in F_1, X_{t_2} \in F_2, \ldots, X_{t_n} \in F_n\right\} = \mu_{t_1,\ldots,t_n}\left(F_1 \times F_2 \times \cdots \times F_n\right).$$

Intuition: If we want a distribution to have a process like this, then the given distribution should have compatibility and then we will have the stochastic process.

Definition 3. A Gaussian process is a stochastic process $\{X_t\}$ such that for any $0 \leq t_1 < t_2 < t_3 < \cdots < t_n$, $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ is a Gaussian vector.

For Gaussian vectors, to specify the distribution, we only need to specify the mean and the covariance. For simplicity, we only look at Gaussian processes with mean 0, that is, $\mathbf{E}\, X_t = 0, \forall t \in T$. This is a centered Gaussian because the mean is 0. We only look at the

mean because X_t is Gaussian, $\mathbf{E}\, X_t = m(t)$. Then take $Y_t = X_t - m(t)$ and now we have a centered Gaussian. The covariances are still the same.
For a centered Gaussian Process, to specify the distribution of $(X_{t_1}, \ldots, X_{t_n})$ for any $t_1 < t_2 < \cdots < t_n$, we only need to specify the covariance of $(X_{t_1}, \ldots, X_{t_n})$, that is

$$a_{t_i t_j} = \mathbf{E}\, X_{t_i} X_{t_j}, 1 \leq i, j \leq n.$$

If you look at $\left(a_{t_i t_j}\right)_{i,j=1}^n$, this should be

1. symmetric
2. non-negative definite

These properties guarantee there is a Gaussian process, and that the Gaussian process has this covariance. This is further reduced to specify a function $C(s,t)$, where C is the covariance. We want $\mathbf{E}\, X_t X_s = C(s,t)$.

1. Symmetric condition: we want $C(s,t) = C(t,s)$
2. We want $\sum_{i,j=1}^n b_i C(t_i, t_j) b_j \geq 0$ for all $b_1, \ldots, b_n \in \mathbb{R}$ and all choices of $t_1, \ldots, t_n$.

Example 3. Why do we have compatibility/consistency? Take

$$\mu_{t_1 t_2 t_3 t_4 t_5}\left(F_1 \times F_2 \times F_3 \times F_4 \times F_5\right) = \mathbb{P}\left\{B_{t_1} \in F_1, B_{t_2} \in F_2, B_{t_3} \in F_3, B_{t_4} \in F_4, B_{t_5} \in F_5\right\},$$

where $B_{t_1}, \ldots, B_{t_5}$ is a centered Gaussian with covariance

$$\begin{bmatrix} C(t_1,t_1) & C(t_1,t_2) & \cdots & C(t_1,t_5) \\ \vdots & \vdots & \ddots & \vdots \\ C(t_5,t_1) & C(t_5,t_2) & \cdots & C(t_5,t_5) \end{bmatrix}.$$

Also,

$$\mu_{t_2, t_4}\left(F_2 \times F_4\right) = \mathbb{P}\left\{\bar{B}_{t_2} \in F_2, \bar{B}_{t_4} \in F_4\right\},$$

where $\left(\bar{B}_{t_2}, \bar{B}_{t_4}\right)$ is a centered Gaussian with covariance

$$\begin{bmatrix} C(t_2,t_2) & C(t_2,t_4) \\ C(t_4,t_2) & C(t_4,t_4) \end{bmatrix}.$$

Consistency is to pick some of F_i to be $\mathbb{R}$, i.e., is

$$\mathbb{P}\left\{B_{t_1} \in \mathbb{R}, B_{t_2} \in F_2, B_{t_3} \in \mathbb{R}, B_{t_4} \in F_4, B_{t_5} \in \mathbb{R}\right\} = \mathbb{P}\left\{\bar{B}_{t_2} \in F_2, \bar{B}_{t_4} \in F_4\right\}?$$

We have that

$$\mathbb{P}\left\{B_{t_1} \in \mathbb{R}, B_{t_2} \in F_2, B_{t_3} \in \mathbb{R}, B_{t_4} \in F_4, B_{t_5} \in \mathbb{R}\right\} = \mathbb{P}\left\{B_{t_2} \in F_2, B_{t_4} \in F_4\right\}$$

because when we have $\in \mathbb{R}$, it is no longer a restriction. It is still a Gaussian vector with appropriate covariance (delete appropriate rows and columns), which is the same as the other covariance. So they have the same distribution, so we have consistency.

Example 4. Brownian Motion $\{B_t\}_{t\geq 0}$ is called a Brownian Motion starting from 0 if

1. $B_0 = 0$
2. Independent Increments (see diagram from last time)
3. $B_t - B_s \sim \mathsf{N}(0, t-s)$

Question: Do we have a process with these properties?

Answer: Check the covariance function for $t > s$.

$$
\begin{aligned}
C(t,s) &= \mathbf{E}\, B_t B_s \\
&= \mathbf{E}(B_t - B_s + B_s) B_s \\
&= \mathbf{E}(B_t - B_s) B_s + \mathbf{E}\, B_s^2 \\
&= \mathbf{E}(B_t - B_s)\, \mathbf{E}\, B_s + s, \text{ since } B_t - B_s \text{ is independent of } B_s \\
&\qquad \text{and since } \mathbf{E}\, B_s = 0, \mathbf{E}\, B_s^2 = \mathbf{V}\, B_s^2 = s - 0 = s \\
&= s \wedge t,
\end{aligned}
$$

which is symmetric. Why is it non-negative definite? We need to show $\sum_{i,j=1}^n b_i C(t_i, t_j) b_j \geq 0$ for all $b_1, \ldots, b_n \in \mathbb{R}$ and for all $t_1, \ldots, t_n$.

$$
\begin{aligned}
\sum_{i,j=1}^n b_i C(t_i, t_j) b_j &= \sum_{i,j=1}^n b_i \left(t_i \wedge t_j\right) b_j \\
&= \sum_{i,j=1}^n \left[b_i b_j \int_0^\infty 1_{[0,t_i]}(u) 1_{[0,t_j]}(u) du \right] \\
&= \int_0^\infty \left[\sum_{i,j=1}^n \left(b_i 1_{[0,t_i]}(u)\right) \left(b_j 1_{[0,t_j]}(u)\right) \right] du \\
&= \int_0^\infty \left(\sum_{i=1}^n b_i 1_{[0,t_i]}(u) \right)^2 du \\
&\geq 0.
\end{aligned}
$$

So we do have a process, and we call this process Brownian Motion.

Question: Why is Brownian Motion continuous in some sense?

Definition 4. Let X_t, Y_t be two stochastic processes. We say X_t and Y_t is a modification of each other if $\mathbb{P}\left[X_t = Y_t\right] = 1$ for every $t \in T$ almost surely, i.e. it is the same process in the sense that the distribution is the same.

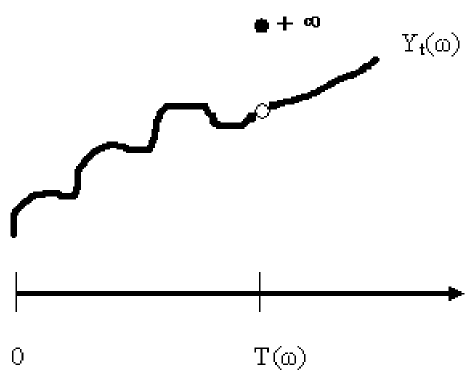
Theorem 3. *A Brownian Motion always has a continuous modification, i.e. fix* $\omega \in \Omega$, *then* $X_t(\omega)$ *becomes a function of* t. *This is the sample path of the process. If the sample path is continuous almost surely, then the process is continuous.*

Theorem 4. *(Kolmogorov's Continuity) If X_t is a stochastic process, suppose $\mathbf{E}|X_t - X_s|^\alpha \leq C|t-s|^{1+\beta}$ for some $\alpha, \beta > 0$. Then X_t has a continuous modification.*

Example 5. Suppose X_t is a stochastic process (we don't know if it is continuous). Let T be a random variable of the continuous type. That is, $\mathbb{P}[T = t] = 0$. Let

$$Y_t(\omega) = \begin{cases} X_t(\omega), & \text{not at } T(\omega) \\ \infty, & \text{at } T(\omega) \end{cases}.$$

We look at $Y_t(\omega)$ and the sample path. An example of this is shown below.



Then look at $T(\omega)$.

$$\mathbb{P}[X_t = Y_t] = 1 - \mathbb{P}[T(\omega) = t] = 1.$$