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$\mathbf{x_w}$ = wood needed by wood, steel and coal + outside demand for wood
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Notation: let $\mathbf{A_{ws}}$ = amount of wood needed for steel, and $\mathbf{d_w}$ = outside demand for wood, $\cdots$.

$$\mathbf{x_w} = (\mathbf{A_{ww}} + \mathbf{A_{ws}} + \mathbf{A_{wc}}) + \mathbf{d_w} \tag{1}$$

$$\mathbf{x_s} = (\mathbf{A_{sw}} + \mathbf{A_{ss}} + \mathbf{A_{sc}}) + \mathbf{d_s} \tag{2}$$

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$$\mathbf{x_w} = (\frac{\mathbf{A_{ww}}}{\mathbf{x_w}} \cdot \mathbf{x_w} + \frac{\mathbf{A_{ws}}}{\mathbf{x_s}} \cdot \mathbf{x_s} + \frac{\mathbf{A_{wc}}}{\mathbf{x_c}} \cdot \mathbf{x_c}) + \mathbf{d_w} \quad (4)$$
$$\mathbf{x_s} = (\frac{\mathbf{A_{sw}}}{\mathbf{x_w}} \cdot \mathbf{x_w} + \frac{\mathbf{A_{ss}}}{\mathbf{x_s}} \cdot \mathbf{x_s} + \frac{\mathbf{A_{sc}}}{\mathbf{x_c}} \cdot \mathbf{x_c}) + \mathbf{d_s} \quad (5)$$
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In matrix form:

$$\begin{bmatrix} x_w \\ x_s \\ x_c \end{bmatrix} = \begin{bmatrix} a_{ww} & a_{ws} & a_{wc} \\ a_{sw} & a_{ss} & a_{sc} \\ a_{cw} & a_{cs} & a_{cc} \end{bmatrix} \cdot \begin{bmatrix} x_w \\ x_s \\ x_c \end{bmatrix} + \begin{bmatrix} d_w \\ d_s \\ d_c \end{bmatrix}$$

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$$\mathbf{X} = \mathbf{AX} + \mathbf{D} \tag{10}$$

$$\mathbf{X} - \mathbf{AX} = \mathbf{D} \tag{11}$$

$$\mathbf{I} \cdot \mathbf{X} - \mathbf{AX} = \mathbf{D} \tag{12}$$

$$(\mathbf{I} - \mathbf{A}) \cdot \mathbf{X} = \mathbf{D} \tag{13}$$

$$(\mathbf{I} - \mathbf{A})^{-1} \cdot (\mathbf{I} - \mathbf{A}) \cdot \mathbf{X} = (\mathbf{I} - \mathbf{A})^{-1} \cdot \mathbf{D} \tag{14}$$

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Three-Sector Economy In an economic system, each of three industries depends on the others for raw materials.

- To make \$1 of processed wood requires:
  - \$.30 wood,
  - \$.20 steel
  - \$.10 coal.
- To make \$1 of processed steel requires:
  - \$.00 wood,
  - \$.30 steel
  - \$.20 coal.
- To make \$1 of processed coal requires:
  - \$.10 wood,
  - \$.20 steel
  - \$.05 coal.

To allow for \$1 consumption in wood, \$4 consumption in steel, and \$2 consumption in coal, what levels of production for wood, steel, and coal are required?

$$\mathbf{A} = \begin{bmatrix} 0.30 & 0 & 0.10 \\ 0.20 & 0.30 & 0.20 \\ 0.10 & 0.20 & 0.05 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \quad (17)$$

$$\mathbf{B} = (\mathbf{I} - \mathbf{A}) = \begin{bmatrix} 0.70 & 0 & -0.10 \\ -0.20 & 0.70 & -0.20 \\ -0.10 & -0.20 & 0.95 \end{bmatrix} \quad (18)$$

$$\mathbf{B}^{-1} = (\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1.4654 & 0.0469 & 0.1641 \\ 0.4924 & 1.5358 & 0.3751 \\ 0.2579 & 0.3283 & 1.1489 \end{bmatrix} \quad (19)$$

$$\mathbf{X} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{D} = \mathbf{B}^{-1}\mathbf{D} = \begin{bmatrix} 1.4654 & 0.0469 & 0.1641 \\ 0.4924 & 1.5358 & 0.3751 \\ 0.2579 & 0.3283 & 1.1489 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \tag{20}$$

$$\mathbf{X} = \begin{bmatrix} 1.98 \\ 7.39 \\ 3.87 \end{bmatrix} \tag{21}$$