Consider the Three-Sector Economy: In an economic system, each of three industries wood, steel and coal depends on the others for raw materials.
Consider the Three-Sector Economy: In an economic system, each of three industries wood, steel and coal depends on the others for raw materials.

Let:

\[ x_w = \text{amount of wood produced (in units of dollars)} \]
\[ x_s = \text{amount of steel produced (in units of dollars)} \]
\[ x_c = \text{amount of coal produced (in units of dollars)} \]
Consider the Three-Sector Economy: In an economic system, each of three industries wood, steel and coal depends on the others for raw materials.

Let:

\( x_w \) = amount of wood produced (in units of dollars)
\( x_s \) = amount of steel produced (in units of dollars)
\( x_c \) = amount of coal produced (in units of dollars)

\( x_w \) = wood needed by wood, steel and coal + outside demand for wood
\( x_s \) = steel needed by wood, steel and coal + outside demand for steel
\( x_c \) = coal needed by wood, steel and coal + outside demand for coal
Consider the Three-Sector Economy: In an economic system, each of three industries wood, steel and coal depends on the others for raw materials.
Let:

\[ x_w = \text{amount of wood produced (in units of dollars)} \]
\[ x_s = \text{amount of steel produced (in units of dollars)} \]
\[ x_c = \text{amount of coal produced (in units of dollars)} \]

\[ x_w = \text{wood needed by wood, steel and coal + outside demand for wood} \]
\[ x_s = \text{steel needed by wood, steel and coal + outside demand for steel} \]
\[ x_c = \text{coal needed by wood, steel and coal + outside demand for coal} \]

Notation: let \( A_{ws} = \text{amount of wood needed for steel, and } d_w = \text{outside demand for wood, \cdots} \).
Matrices
Input-Output Analysis

\[ x_w = (A_{ww} + A_{ws} + A_{wc}) + d_w \]  \hspace{1cm} (1)
\[ x_s = (A_{sw} + A_{ss} + A_{sc}) + d_s \]  \hspace{1cm} (2)
\[ x_c = (A_{cw} + A_{cs} + A_{cc}) + d_c \]  \hspace{1cm} (3)
Matrices
Input-Output Analysis

\[ x_w = (A_{ww} + A_{ws} + A_{wc}) + d_w \]  \hspace{1cm} (1)

\[ x_s = (A_{sw} + A_{ss} + A_{sc}) + d_s \]  \hspace{1cm} (2)

\[ x_c = (A_{cw} + A_{cs} + A_{cc}) + d_c \]  \hspace{1cm} (3)

Useful trick (⇒ Get Noble Prize!) Multiply each \( A_{ij} \) by 1
Matrices
Input-Output Analysis

\[ x_w = (A_{ww} + A_{ws} + A_{wc}) + d_w \]  \hspace{1cm} (1)

\[ x_s = (A_{sw} + A_{ss} + A_{sc}) + d_s \]  \hspace{1cm} (2)

\[ x_c = (A_{cw} + A_{cs} + A_{cc}) + d_c \]  \hspace{1cm} (3)

Useful trick (⇒ Get Noble Prize!) Multiply each \( A_{ij} \) by 1

\[ x_w = \left( \frac{A_{ww}}{x_w} \cdot x_w + \frac{A_{ws}}{x_s} \cdot x_s + \frac{A_{wc}}{x_c} \cdot x_c \right) + d_w \]  \hspace{1cm} (4)

\[ x_s = \left( \frac{A_{sw}}{x_w} \cdot x_w + \frac{A_{ss}}{x_s} \cdot x_s + \frac{A_{sc}}{x_c} \cdot x_c \right) + d_s \]  \hspace{1cm} (5)

\[ x_c = \left( \frac{A_{cw}}{x_w} \cdot x_w + \frac{A_{cs}}{x_s} \cdot x_s + \frac{A_{cc}}{x_c} \cdot x_c \right) + d_c \]  \hspace{1cm} (6)
Matrices
Input-Output Analysis

\[
x_w = (A_{ww} + A_{ws} + A_{wc}) + d_w \tag{1}
\]
\[
x_s = (A_{sw} + A_{ss} + A_{sc}) + d_s \tag{2}
\]
\[
x_c = (A_{cw} + A_{cs} + A_{cc}) + d_c \tag{3}
\]

Useful trick (⇒ Get Noble Prize!) Multiply each \( A_{ij} \) by 1

\[
x_w = \left( \frac{A_{ww}}{x_w} \cdot x_w + \frac{A_{ws}}{x_s} \cdot x_s + \frac{A_{wc}}{x_c} \cdot x_c \right) + d_w \tag{4}
\]
\[
x_s = \left( \frac{A_{sw}}{x_w} \cdot x_w + \frac{A_{ss}}{x_s} \cdot x_s + \frac{A_{sc}}{x_c} \cdot x_c \right) + d_s \tag{5}
\]
\[
x_c = \left( \frac{A_{cw}}{x_w} \cdot x_w + \frac{A_{cs}}{x_s} \cdot x_s + \frac{A_{cc}}{x_c} \cdot x_c \right) + d_c \tag{6}
\]

Let \( \frac{A_{ws}}{x_s} = a_{ws} \) i.e. let \( \frac{A_{ij}}{x_j} = a_{ij} \)
Matrices
Input-Output Analysis

Let \( \frac{A_{ws}}{x_s} = a_{ws} \) i.e. let \( \frac{A_{ij}}{x_j} = a_{ij} \)

\[ x_w = (a_{ww} \cdot x_w + a_{ws} \cdot x_s + a_{wc} \cdot x_c) + d_w \] (7)
\[ x_s = (a_{sw} \cdot x_w + a_{ss} \cdot x_s + a_{sc} \cdot x_c) + d_s \] (8)
\[ x_c = (a_{cw} \cdot x_w + a_{cs} \cdot x_s + a_{cc} \cdot x_c) + d_c \] (9)
Let \( \frac{A_{ws}}{x_s} = a_{ws} \) i.e. let \( \frac{A_{ij}}{x_j} = a_{ij} \)

\[
x_w = (a_{ww} \cdot x_w + a_{ws} \cdot x_s + a_{wc} \cdot x_c) + d_w
\]

(7)

\[
x_s = (a_{sw} \cdot x_w + a_{ss} \cdot x_s + a_{sc} \cdot x_c) + d_s
\]

(8)

\[
x_c = (a_{cw} \cdot x_w + a_{cs} \cdot x_s + a_{cc} \cdot x_c) + d_c
\]

(9)

In matrix form:

\[
\begin{bmatrix}
  x_w \\
  x_s \\
  x_c
\end{bmatrix}
= \begin{bmatrix}
  a_{ww} & a_{ws} & a_{wc} \\
  a_{sw} & a_{ss} & a_{sc} \\
  a_{cw} & a_{cs} & a_{cc}
\end{bmatrix}
\begin{bmatrix}
  x_w \\
  x_s \\
  x_c
\end{bmatrix}
+ \begin{bmatrix}
  d_w \\
  d_s \\
  d_c
\end{bmatrix}
\]
Matrices
Input-Output Analysis

Let \( \frac{A_{ws}}{x_s} = a_{ws} \) i.e. let \( \frac{A_{ij}}{x_j} = a_{ij} \)

\[
\begin{align*}
x_w &= (a_{ww} \cdot x_w + a_{ws} \cdot x_s + a_{wc} \cdot x_c) + d_w \\
x_s &= (a_{sw} \cdot x_w + a_{ss} \cdot x_s + a_{sc} \cdot x_c) + d_s \\
x_c &= (a_{cw} \cdot x_w + a_{cs} \cdot x_s + a_{cc} \cdot x_c) + d_c
\end{align*}
\] (7) (8) (9)

In matrix form:

\[
\begin{bmatrix}
  x_w \\
x_s \\
x_c
\end{bmatrix}
= \begin{bmatrix}
  a_{ww} & a_{ws} & a_{wc} \\
  a_{sw} & a_{ss} & a_{sc} \\
  a_{cw} & a_{cs} & a_{cc}
\end{bmatrix}
\cdot
\begin{bmatrix}
  x_w \\
x_s \\
x_c
\end{bmatrix}
+ \begin{bmatrix}
  d_w \\
d_s \\
d_c
\end{bmatrix}
\]
Let \( \frac{A_{ws}}{x_s} = a_{ws} \) i.e. let \( \frac{A_{ij}}{x_j} = a_{ij} \)

\[
\begin{align*}
x_w &= (a_{ww} \cdot x_w + a_{ws} \cdot x_s + a_{wc} \cdot x_c) + d_w \\
x_s &= (a_{sw} \cdot x_w + a_{ss} \cdot x_s + a_{sc} \cdot x_c) + d_s \\
x_c &= (a_{cw} \cdot x_w + a_{cs} \cdot x_s + a_{cc} \cdot x_c) + d_c
\end{align*}
\]

In matrix form:

\[
\begin{bmatrix}
x_w \\
x_s \\
x_c
\end{bmatrix} =
\begin{bmatrix}
a_{ww} & a_{ws} & a_{wc} \\
a_{sw} & a_{ss} & a_{sc} \\
a_{cw} & a_{cs} & a_{cc}
\end{bmatrix}
\begin{bmatrix}
x_w \\
x_s \\
x_c
\end{bmatrix} +
\begin{bmatrix}
d_w \\
d_s \\
d_c
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} +
\begin{bmatrix}
d_1 \\
d_2 \\
d_3
\end{bmatrix}
\]

\[
X = A \cdot X + D
\]
\[ X = A \cdot X + D, \]
Matrices
Input-Output Analysis

- \( X = A \cdot X + D \),
- \( A \) has elements \( \frac{A_{ws}}{x_s} = a_{ws} \) or \( \frac{A_{ij}}{x_j} = a_{ij} \)
Matrices
Input-Output Analysis

\[
X = A \cdot X + D,
\]

- \( A \) has elements \( \frac{A_{ws}}{x_s} = a_{ws} \) or \( \frac{A_{ij}}{x_j} = a_{ij} \)
- the ratio \( \frac{A_{ij}}{x_j} = a_{ij} \) typically does not change as the demand and production levels change. Once \( A \) is known, it can be reused for different demand and production levels.
Matrices
Input-Output Analysis

- \( X = A \cdot X + D, \)
- \( A \) has elements \( \frac{A_{ws}}{x_s} = a_{ws} \) or \( \frac{A_{ij}}{x_j} = a_{ij} \)
- the ratio \( \frac{A_{ij}}{x_j} = a_{ij} \) typically does not change as the demand and production levels change. Once \( A \) is known, it can be reused for different demand and production levels.
- \( a_{ws} \) is the amount of output from wood industry needed by the steel industry for each dollar of output produced by the steel industry.
\[ X = A \cdot X + D, \]

- \( A \) has elements \( \frac{A_{ws}}{x_s} = a_{ws} \) or \( \frac{A_{ij}}{x_j} = a_{ij} \)

- The ratio \( \frac{A_{ij}}{x_j} = a_{ij} \) typically does not change as the demand and production levels change. Once \( A \) is known, it can be reused for different demand and production levels.

- \( a_{ws} \) is the amount of output from wood industry needed by the steel industry for each dollar of output produced by the steel industry.

- \( a_{ij} \) is the amount of output from industry \( i \) needed by industry \( j \) for each dollar of output produced by industry \( j \).
\[ \mathbf{X} = \mathbf{A} \cdot \mathbf{X} + \mathbf{D}, \]

- \( \mathbf{A} \) has elements \( \frac{A_{ws}}{x_s} = a_{ws} \) or \( \frac{A_{ij}}{x_j} = a_{ij} \)

- the ratio \( \frac{A_{ij}}{x_j} = a_{ij} \) typically does not change as the demand and production levels change. Once \( \mathbf{A} \) is known, it can be reused for different demand and production levels.

- \( a_{ws} \) is the amount of output from wood industry needed by the steel industry for each dollar of output produced by the steel industry.

- \( a_{ij} \) is the amount of output from industry \( i \) needed by industry \( j \) for each dollar of output produced by industry \( j \).

- \( \mathbf{A} \) is called the Input-Output matrix. This is usually known.
Matrices
Input-Output Analysis

- \( \mathbf{X} = \mathbf{A} \cdot \mathbf{X} + \mathbf{D} \),
- \( \mathbf{A} \) has elements \( \frac{A_{ws}}{x_s} = a_{ws} \) or \( \frac{A_{ij}}{x_j} = a_{ij} \)
- the ratio \( \frac{A_{ij}}{x_j} = a_{ij} \) typically does not change as the demand and production levels change. Once \( \mathbf{A} \) is known, it can be reused for different demand and production levels.
- \( a_{ws} \) is the amount of output from wood industry needed by the steel industry for each dollar of output produced by the steel industry.
- \( a_{ij} \) is the amount of output from industry \( i \) needed by industry \( j \) for each dollar of output produced by industry \( j \).
- \( \mathbf{A} \) is called the Input-Output matrix. This is usually known.
- \( \mathbf{D} \) is called the demand matrix. This is usually known.
\[ \mathbf{X} = \mathbf{A} \cdot \mathbf{X} + \mathbf{D}, \]

- \( \mathbf{A} \) has elements \( \frac{A_{ws}}{x_s} = a_{ws} \) or \( \frac{A_{ij}}{x_j} = a_{ij} \)
- The ratio \( \frac{A_{ij}}{x_j} = a_{ij} \) typically does not change as the demand and production levels change. Once \( \mathbf{A} \) is known, it can be reused for different demand and production levels.
- \( a_{ws} \) is the amount of output from wood industry needed by the steel industry for each dollar of output produced by the steel industry.
- \( a_{ij} \) is the amount of output from industry \( i \) needed by industry \( j \) for each dollar of output produced by industry \( j \).
- \( \mathbf{A} \) is called the Input-Output matrix. This is usually known.
- \( \mathbf{D} \) is called the demand matrix. This is usually known.
- \( \mathbf{X} \) is the production matrix. It is the unknown that must be solved for. It gives the amount that each industry must produce to satisfy all needs.
Matrices
Input-Output Analysis

- \( X = A \cdot X + D \),
- \( A \) has elements \( \frac{A_{ws}}{x_s} = a_{ws} \) or \( \frac{A_{ij}}{x_j} = a_{ij} \)
- the ratio \( \frac{A_{ij}}{x_j} = a_{ij} \) typically does not change as the demand and production levels change. Once \( A \) is known, it can be reused for different demand and production levels.
- \( a_{ws} \) is the amount of output from wood industry needed by the steel industry for each dollar of output produced by the steel industry.
- \( a_{ij} \) is the amount of output from industry \( i \) needed by industry \( j \) for each dollar of output produced by industry \( j \).
- \( A \) is called the Input-Output matrix. This is usually known.
- \( D \) is called the demand matrix. This is usually known.
- \( X \) is the production matrix. It is the unknown that must be solved for. It gives the amount that each industry must produce to satisfy all needs.
- must solve \( X = A \cdot X + D \) for the production level \( X \).
Must solve $X = A \cdot X + D$ for the production level $X$

\[
X = AX + D \quad (10)
\]
\[
X - AX = D \quad (11)
\]
\[
l \cdot X - AX = D \quad (12)
\]
\[
(I - A) \cdot X = D \quad (13)
\]
\[
(I - A)^{-1} \cdot (I - A) \cdot X = (I - A)^{-1} \cdot D \quad (14)
\]
\[
l \cdot X = (I - A)^{-1} \cdot D \quad (15)
\]
\[
X = (I - A)^{-1} \cdot D \quad (16)
\]
Three-Sector Economy In an economic system, each of three industries depends on the others for raw materials.

- To make $1 of processed wood requires:
  - $.30 wood,
  - $.20 steel
  - $.10 coal.
- To make $1 of processed steel requires:
  - $.00 wood,
  - $.30 steel
  - $.20 coal.
- To make $1 of processed coal requires:
  - $.10 wood,
  - $.20 steel
  - $.05 coal.

To allow for $1 consumption in wood, $4 consumption in steel, and $2 consumption in coal, what levels of production for wood, steel, and coal are required?
Matrices

Input-Output Analysis: solution for example

\[
A = \begin{bmatrix}
0.30 & 0 & 0.10 \\
0.20 & 0.30 & 0.20 \\
0.10 & 0.20 & 0.05 \\
\end{bmatrix}
, \quad
D = \begin{bmatrix}
1 \\
4 \\
2 \\
\end{bmatrix}
\quad (17)
\]

\[
B = (I - A) = \begin{bmatrix}
0.70 & 0 & -0.10 \\
-0.20 & 0.70 & -0.20 \\
-0.10 & -0.20 & 0.95 \\
\end{bmatrix}
\quad (18)
\]

\[
B^{-1} = (I - A)^{-1} = \begin{bmatrix}
1.4654 & 0.0469 & 0.1641 \\
0.4924 & 1.5358 & 0.3751 \\
0.2579 & 0.3283 & 1.1489 \\
\end{bmatrix}
\quad (19)
\]
Matrices
Input-Output Analysis: solution for example

\[ X = (I - A)^{-1}D = B^{-1}D = \begin{bmatrix}
1.4654 & 0.0469 & 0.1641 \\
0.4924 & 1.5358 & 0.3751 \\
0.2579 & 0.3283 & 1.1489
\end{bmatrix} \begin{bmatrix}
1 \\
4 \\
2
\end{bmatrix} \]

\( X = \begin{bmatrix}
1.98 \\
7.39 \\
3.87
\end{bmatrix} \)