Almost Free Abelian Groups

We say that an abelian group is is $\aleph_1$-free if every countable subgroup is free.

We describe the construction of abelian groups of size $\aleph_1$ that are $\aleph_1$-free but not free. Throughout this note all groups are abelian.

Throughout we use the basic fact that a subgroup of a free abelian group is free.

We will need the following algebraic lemma.

Lemma 0.1 *There are free abelian groups $K_1 \subset K_2 \subset \ldots \subset K_n \subset \ldots$, $K = \bigcup K_n$, and $F \supset K$ such that:*

i) F/K_m is free for all m, but F/K is not;

ii) each K_m, F/K, K_{m+1}/K_m has rank $\aleph_0$.

Proof Let G be the subgroup of $\mathbb{Q}$ generated by $\{\frac{1}{2^n} : n \in \omega\}$. Let $\widehat{F}$ be the free abelian group on generators $x_0, x_1, \ldots$ and let $f : \widehat{F} \to G$ be the surjective homomorphism $x_i \mapsto \frac{1}{2^i}$. Let $\widehat{K}$ be the kernel of F. Then $\widehat{K}$ is the free abelian group on generators $\{x_0 - 2x_1, x_1 - 2x_2, \ldots\}$. Then $F/K \cong G$ is not free.

Let $\widehat{K}_m$ be the free abelian group on $\{x_0 - 2x_1, \ldots, x_{m-1} - 2x_m\}$. Then $\widehat{F}/\widehat{K}_m$ is isomorphic to the free abelian group on $x_m, x_{m+1}, \ldots$.

The groups we have constructed satisfy i), but not ii). It is easy to make ii) true by adding in large free factors. Here are the details. Let H be a free abelian group on $\aleph_0$ generators.

Let

$$K_m = \widehat{K}_m \oplus \oplus_{i=0}^{m} H,$$

let

$$K = \bigcup_{m=0}^{\infty} K_m = \widehat{K} \oplus \oplus_{i=0}^{\infty} H$$

and let $F = \widehat{F} \oplus \oplus_{i=0}^{\infty} H \oplus H$. These groups have the desired properties.

Theorem 0.2 *There are $2^{\aleph_1}$ nonisomorphic $\aleph_1$-free groups of cardinality $\aleph_1$.*

Proof We fix a family $\mathcal{F}$ of $2^{\aleph_1}$ stationary subsets of ω_1 such that if $S_1, S_2 \in \mathcal{F}$ are distinct, then $S_1 \triangle S_2$ is stationary. We may assume each $S \in \mathcal{F}$ is a set of limit ordinals. Let $S \in \mathcal{F}$. We construct a sequence of countable free abelian groups

$$G_0 \subset G_1 \subset \dots G_\alpha \subset \dots$$

for $\alpha < \omega_1$. We will do this so that:

$(*)$ if $\beta < \alpha$, then G_α is free over $G_{\beta+1}$.

i) $G_0 = \mathbb{Z}$.

ii) If α is a limit ordinal, let $G_\alpha = \bigcup_{\gamma<\alpha} G_\gamma$. Choose $\gamma_0 < \gamma_1 < \dots$ with $\sup \gamma_n = \alpha$ such that each γ_n is a successor ordinal. By $(*)$ $G_{\gamma_{n+1}}$ is free over G_{γ_n}. Thus G_α is free. Indeed G_α is free over each G_{γ_n}. If $\beta < \alpha$, choose n such that $\beta < \gamma_n$. Then $G_{\gamma(n)}$ is free over $G_{\beta+1}$ and G_α is free over $G_{\gamma(n)}$. Thus G_α is free over $G_{\beta+1}$ and $(*)$ holds.

iii) If $\alpha \notin S$, then $G_{\alpha+1} = G_\alpha \oplus H$, where H is free abelian on $\aleph_0$-generators. Clearly $(*)$ holds.

iv) Suppose $\alpha \in S$. Choose $\gamma_0 < \gamma_1 < \dots$ successor ordinals with $\alpha = \sup \gamma_n$.

Let $F, K, K_0, K_1, \dots$ be as in the Lemma. Since G_{γ_0} and K_0 are both free abelian on $\aleph_0$-generators, we can find an isomorphism $\phi_0 : G_{\gamma_0} \to K_0$. Since $G_{\gamma_{n+1}}$ is free over G_γ of rank $\aleph_0$ and K_{n+1}/K_n is free of rank $\aleph_0$, we can extend ϕ_n to an isomorphism $\phi_{n+1} : G_{\gamma_n} \to K_n$. Then $\phi = \bigcup \phi_n$ is an isomorphism from G_α to K. We then define $G_{\alpha+1} \supset G_\alpha$ such that ϕ extends to an isomorphism from $G_{\alpha+1} \to F$. Thus we have

$(**)$ if $\alpha \in S$, then $G_{\alpha+1}/G_\alpha$ is not free.

On the other hand if $\beta + 1 < \alpha + 1$, there is an n such that $\beta + 1 < \gamma_n$. Then G_{γ_n} is free over $G_{\beta+1}$ and, by construction, $G_{\alpha+1}$ is free over G_{γ_n}. Thus $G_{\alpha+1}$ is free over $G_{\beta+1}$. Thus $(*)$ holds.

This concludes the construction. If $A \subseteq G$ is countable, there is an α such that $A \subseteq G_\alpha$. Since G_α is free abelian so is A. Thus G is $\aleph_1$-free.

Suppose S_1 and S_2 are distinct elements of $\mathcal{F}$. Let G_1 and G_2 be groups we constructed. We claim that $G_1 \not\cong G_2$. Suppose not. Let $f : G_1 \cong G_2$ be an isomorphism. Then $C = \{\alpha : f$ is an isomorphism between $G_{1,\alpha}$ and $G_{2,\alpha}\}$ is closed unbounded.

Since $S_1 \triangle S_2$ is stationary, we can, without loss of generality, find $\alpha \in (C \cap S_1) \setminus S_2$.

Then f is an isomorphism between G_1 and G_2 such that f maps $G_{1,\alpha}$ onto $G_{2,\alpha}$. But $G_1/G_{1,\alpha}$ is not free while $G_2/G_{2,\alpha}$ is free, a contradiction.