Math 504 Set Theory I
Problem Set 5

Due Wednesday March 5

1) (ZFC) a) Show that $H(\aleph_1) \models$ ZFC−P+ ω is the largest cardinal.

b)[†] Use a) and Gödel's Second Incompleteness Theorem to show that (assuming ZFC is consistent)

$$\text{ZFC} \not\vdash \text{Con}(\text{ZFC} - \text{P}) \rightarrow \text{Con}(\text{ZFC}).$$

2) We say that $(M_\alpha : \alpha \in On)$ is a *smooth hierarchy* if

i) $M_\alpha \subseteq M_\beta$ for all $\alpha < \beta$, and

ii) $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for α a limit ordinal.

Suppose $(M_\alpha : \alpha \in On)$ is a smooth hierarchy and $M = \bigcup_{\alpha \in On} M_\alpha$.

Suppose $X \subseteq M$ and $\phi_1(v, w_1, \ldots, w_n), \ldots \phi_k(v, w_1, \ldots, w_n)$ are formulas. Prove that there is an ordinal α such that $X \subseteq M_\alpha$ and

$$M \models \exists v\ \phi_i(v, \bar{a}) \Rightarrow M_\alpha \models \exists v\ \phi_i(v, \bar{a})$$

for all $a_1, \ldots, a_n \in M_\alpha$ and $i = 1, \ldots, k$.

Convince yourself (but don't hand in) that this is enough to prove the following version of Reflection: for any $a \subseteq M$ and any $\phi(\bar{v})$ there is α such that $a \subseteq M_\alpha$ and

$$M \models \phi(\bar{b}) \Leftrightarrow M_\alpha \models \phi(\bar{b})$$

for all $\bar{b} \in M_\alpha$.

3) a) Prove that $|\text{Def}(X)| \leq \max(|X|, \aleph_0)$.

b) Prove that $|\mathbb{L}_\alpha| = |\alpha|$ for all $\alpha > \omega$.

4) Suppose κ is a weakly inaccessible cardinal. Prove that

$$\mathbb{L} \models \kappa \text{ is a weakly inaccessible cardinal.}$$