Math 504 Set Theory I
Problem Set 9

Due Wednesday May 6

Do three of the following problems.

1) Suppose $\mathcal{M}$ is a countable transitive model of ZFC, such that

$$\mathcal{M} \models \kappa \text{ and } \lambda \text{ are infinite cardinals}$$

Let $\mathbb{P} = \mathrm{Fn}(\kappa, \lambda)$ and let $G \subset \mathbb{P}$ be an $\mathcal{M}$-generic filter.

a) Prove that $\mathbb{P}$ has λ^+-cc. Conclude that all cardinal $\mu > \lambda$ remain cardinals in $\mathcal{M}[G]$.

b) For $\alpha < \lambda$ show $D_\alpha = \{p : \exists n \in \omega \; p(n) = \alpha\}$ is dense and conclude that $\mathcal{M}[G] \models \lambda$ is countable.

2.5) Suppose $\mathcal{M}$ is a countable transitive model of ZFC with $2^{\aleph_0} = \kappa > \aleph_1$. Let $\mathbb{P} = \mathrm{Fn}(\omega_1, 2, \omega_1)^{\mathcal{M}}$. Let $G \subset \mathbb{P}$ be an $\mathcal{M}$-generic filter.

a) Prove that $\mathcal{M}[G] \models$ CH. [Hint: For $r : \omega \to 2$ with $r \in \mathcal{M}$, consider $D_r = \{p \in \mathbb{P} : \exists \alpha \forall n \in \omega \; p(\alpha + n) = r(n)\}$.]
This give a proof that Con(ZFC)$\to$ Con(ZFC+CH)*that avoids* $\mathbb{L}$. *Even more is true.* $\mathcal{M}[G] \models \diamondsuit$. (see Kunen VII 8.3)

b) Prove that all cardinals λ of $\mathcal{M}$ with $\aleph_2^{\mathcal{M}} \leq \lambda \leq \kappa$ are collapsed to $\aleph_1$ in $\mathcal{M}[G]$.

4) Let $\mathcal{M}$ be a countable transitive model of ZFC. Let $\mathbb{P} = Fn(\omega, 2)$ and let $G \subset \mathbb{P}$ be $\mathcal{M}$-generic and $g = \bigcup_{p \in G} p$.

Recall that $2^{<\omega}$ is the set of all finite sequences of zeros and ones. We say that $T \subseteq 2^{<\omega}$ is a *tree* if whenever $\sigma \in T$ and $\tau \subseteq \sigma$, then $\tau \in T$. If T is a tree then $[T] = \{f : \omega \to 2 : \forall n \; f|n \in T\}$ is the set of paths through T.

We say that a tree T is *nowhere dense* if for all $\sigma \in 2^{<\omega}$ there is $\tau \supset \sigma$ with $\tau \notin T$.

Prove that if $T_0, T_1, \ldots \in \mathcal{M}$ are nowhere dense trees, then $g \notin \bigcup_{n \in \omega} [T_n]$.

This is a proof that a Cohen real avoids all meager sets in $\mathcal{M}$ *and shows some relationship between forcing and the Baire Category Theorem.*

5) Let $\mathcal{M}$ be a countable transitive model of ZFC. Let $\mathbb{P} = \mathrm{Fn}(\omega, 2)$ and let $G \subset \mathbb{P}$ be $\mathcal{M}$-generic.

Let

$$G_0 = \{p \in \mathbb{P} : \exists q \in G \ p(n) = q(2n)\}$$

and

$$G_1 = \{p \in \mathbb{P} : \exists q \in G \ p(n) = q(2n+1)\}.$$

If $g = \bigcup_{p \in G} p$ and $g_i = \bigcup_{p \in G_i} p$, then $g_0(n) = g(2n)$ and $g_1(n) = g(2n+1)$.

a) Prove that G_0 is an $\mathcal{M}$-generic filter. [A similar argument would show that G_1 is $\mathcal{M}$-generic.]

b)† Prove that G_1 is an $\mathcal{M}[G_0]$-generic filter. [Hint: Suppose $D \in \mathcal{M}[G_0]$ is a dense subset of $\mathbb{P}$. Let $\tau \in \mathcal{M}^{\mathbb{P}}$ such that $\tau_{G_0} = D$. Let $p \Vdash$ " τ is dense". Show that $\{q \in \mathbb{P}$: either the "even part" of q is incompatible with p or the "even part" of q forces that the "odd part" of q is in $\tau\}$ is dense.]

c) Prove that $\mathcal{M}[G] = \mathcal{M}[G_0][G_1]$.

This shows that adding one Cohen real is the same as adding two independent *Cohen reals.*