MATH 531: PROBLEM SET 1

Due Friday, September 5

All rings are considered to be commutative, with unit, unless otherwise specified.

(1) Atiyah-MacDonald, Ch.1: 5; Ch. 4: 4, 5, 7.

(2) Let R be a ring. An element $a \in R$ is *nilpotent* if there exists an integer $n > 0$ such that $a^n = 0$. The set of all nilpotent elements of R, denoted $\mathrm{nil}(R)$, is called the *nilradical* of R. Show that $\mathrm{nil}(R)$ is an ideal, and that it is equal to the intersection of the prime ideals of R. Show that $R/\mathrm{nil}(R)$ has no nonzero nilpotent elements, i.e. it is a *reduced* ring.

(3) Let S be a multiplicative system in the ring R. For an ideal I in R, we define $S^{-1}I := \{\frac{x}{s} \mid x \in I,\ s \in S\}$. Denote by h the natural homomorphism $R \to S^{-1}R$.

(1) Show that $S^{-1}I$ is an ideal, which is proper if and only if $S \cap I = \emptyset$.
(2) If $J \subset S^{-1}R$ is an ideal and $I = h^{-1}(J)$, then I is an ideal and $S^{-1}I = J$.
(3) If $I \subset R$ is an ideal, then $I \subset h^{-1}(S^{-1}I)$. If I is prime and disjoint from S, this is an equality.
(4) If I is prime and disjoint from S, then $S^{-1}I$ is prime in $S^{-1}R$.
(5) Show that there is a one-to-one correspondence between prime ideals $P \subset R$ disjoint from S and prime ideals $Q \subset S^{-1}R$ given by
$$P \to S^{-1}P \text{ and } Q \to h^{-1}(Q).$$
(Hint: use (2), (3) and (4).)

(4) Let R be a commutative ring. An R module is *free of rank* n if it is isomorphic to R^n. Show the following:

(1) If $R^n \cong R^m$, then $n = m$.
(2) Let $A = (a_{ij})$ be an $n \times m$ matrix over R, of rank r. If $r < m$, then the column vectors of A are linearly dependent. Deduce from this an alternative proof of (1).
(3) If R is a local ring, then any minimal set of generators of the module R^n is basis.

(5) Let M and N be finitely generated modules over a local ring R. Show that $M \otimes_R N = 0$ if and only if either M or N is 0. Show that the result fails to hold if R is not local.

(6) Show that $\mathrm{Ass}(M)$ commutes with localization, i.e. if $S \subset R$ is a multiplicative set, then
$$\mathrm{Ass}_{S^{-1}R}(S^{-1}M) = \{P \cdot S^{-1}R \mid P \in \mathrm{Ass}(M),\ P \cap S = \emptyset\}.$$