MATH 531: PROBLEM SET 1

Due Friday, September 5

All rings are considered to be commutative, with unit, unless otherwise specified.

(1) Atiyah-MacDonald, Ch.1: 5; Ch. 4: 4, 5, 7.

(2) Let \( R \) be a ring. An element \( a \in R \) is nilpotent if there exists an integer \( n > 0 \) such that \( a^n = 0 \). The set of all nilpotent elements of \( R \), denoted \( \text{nil}(R) \), is called the nilradical of \( R \). Show that \( \text{nil}(R) \) is an ideal, and that it is equal to the intersection of the prime ideals of \( R \). Show that \( R/\text{nil}(R) \) has no nonzero nilpotent elements, i.e. it is a reduced ring.

(3) Let \( S \) be a multiplicative system in the ring \( R \). For an ideal \( I \) in \( R \), we define \( S^{-1}I := \{ \frac{x}{s} \mid x \in I, s \in S \} \). Denote by \( h \) the natural homomorphism \( R \to S^{-1}R \).

1. Show that \( S^{-1}I \) is an ideal, which is proper if and only if \( S \cap I = \emptyset \).
2. If \( J \subset S^{-1}R \) is an ideal and \( I = h^{-1}(J) \), then \( I \) is an ideal and \( S^{-1}I = J \).
3. If \( I \subset R \) is an ideal, then \( I \subset h^{-1}(S^{-1}I) \). If \( I \) is prime and disjoint from \( S \), this is an equality.
4. If \( I \) is prime and disjoint from \( S \), then \( S^{-1}I \) is prime in \( S^{-1}R \).
5. Show that there is a one-to-one correspondence between prime ideals \( P \subset R \) disjoint from \( S \) and prime ideals \( Q \subset S^{-1}R \) given by
   \[ P \to S^{-1}P \text{ and } Q \to h^{-1}(Q). \]
   (Hint: use (2), (3) and (4)).

(4) Let \( R \) be a commutative ring. An \( R \) module is free of rank \( n \) if it is isomorphic to \( R^n \). Show the following:

1. If \( R^n \cong R^m \), then \( n = m \).
2. Let \( A = (a_{ij}) \) be an \( n \times m \) matrix over \( R \), of rank \( r \). If \( r < m \), then the column vectors of \( A \) are linearly dependent. Deduce from this an alternative proof of (1).
3. If \( R \) is a local ring, then any minimal set of generators of the module \( R^n \) is basis.

(5) Let \( M \) and \( N \) be finitely generated modules over a local ring \( R \). Show that \( M \otimes_R N = 0 \) if and only if either \( M \) or \( N \) is 0. Show that the result fails to hold if \( R \) is not local.

(6) Show that \( \text{Ass}(M) \) commutes with localization, i.e. if \( S \subset R \) is a multiplicative set, then
\[
\text{Ass}_{S^{-1}R}(S^{-1}M) = \{ P \cdot S^{-1}R \mid P \in \text{Ass}(M), P \cap S = \emptyset \}.
\]