MATH 531: PROBLEM SET 10

Due Monday, November 17

(1) Show that an R-module N is injective $\iff$ $\mathrm{Ext}^n_R(R/I, N) = 0$ for all $n > 0$ and all ideals $I \subset R$ $\iff$ $\mathrm{Ext}^1_R(R/I, N) = 0$ for all ideals $I \subset R$. Deduce that

$$\mathrm{gd}(R) = \sup \{\mathrm{id}_R(R/I) \mid I \text{ ideal in } R\}.$$

(2) Let $R \to S$ be a ring homomorphism, and M an R-module. Prove that the following are equivalent:

(a) $\mathrm{Tor}^R_1(M, N) = 0$ for all S-modules N.

(b) $\mathrm{Tor}^R_1(M, S) = 0$ and $M \otimes_R S$ is a flat S-module.

You don't have to write down anything for problems (3) and (4); they are important theoretical statements that I will not do in class.

(3) Read the proof of the fact that minimal free resolutions are unique up to isomorphism, in [Eisenbud] Appendix A.3.6 and Thm 20.2.

(4) Read about the tensor product of two complexes in [Eisenbud] §17.3 or any book on homological algebra, and then understand why for the Koszul complex associated to $x_1, \dots, x_n \in R$ we have

$$K_\bullet(x_1, \dots, x_n) \cong K_\bullet(x_1) \otimes \dots \otimes K_\bullet(x_n).$$

(This can also be found in §17.3 in [Eisenbud].)

(5) Show that if $I = (x_1, \dots, x_n) \subset R$, and M is an R-module, then $H_0(\underline{x}, M) \cong M/IM$ and $H_n(\underline{x}, M) \cong \{m \in M \mid I \cdot m = 0\}$.