MATH 531: PROBLEM SET 2

Due Friday, September 12

(1) Atiyah-MacDonald, Ch. 4: 5, 7.

(2) Let $R = k[X,Y]/I$, where $I = (X) \cap (X,Y)^2 = (X^2, XY)$. Denote by x and y the classes of X and Y in R. Show that (y^n) is (x,y)-primary in R, and that $(0) = (x) \cap (y^n)$ is a reduced primary decomposition of (0) in R for any $n \geq 1$.

(3) Let $R = k[X,Y,Z]/(XY - Z^2)$ and $P = (X, Z)$. Set $M = R/P^2$.

(1) Determine $\mathrm{Ass}(M)$.
(2) For each prime in $\mathrm{Ass}(M)$, find the elements in M that it annihilates.
(3) Find a chain of submodules $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ such that $M_i/M_{i-1} \cong R/P_i$ for some prime P_i, for each $i = 1, \ldots, n$ (i.e. a filtration as discussed in class).

(4) Let $R = k[X,Y]/I$, where k is a field and I is the prime ideal $(X^2 - Y^3)$. If x and y denote the classes of X and Y in R, show that $\alpha = x/y$ is integral over R, but $\alpha \notin R$ (so R is not integrally closed).

(5) Show that a unique factorization domain (UFD) is intergally closed. (Recall that a UFD is a domain in which every non-zero, non-unit element, can be expressed as a product of primes – a prime element $r \in R$ is an elment such that the ideal (r) it generates is prime.)

(6) Compute the field of fractions and the integral closure of the ring R in exercise (4).

(7) Let $R = \mathbf{Z}$ and $S = \mathbf{Z}[i]$, the Gaussian integers. Give an example of two different prime ideals of S lying above the same prime ideal of R.