**MATH 531: PROBLEM SET 5**

Due Friday, October 10

(1) Let $R \subset S$ be an integral extension of rings, $Q$ a prime ideal in $S$ and $P = Q \cap R$. Prove the following:

(1) $\dim R = \dim S$.
(2) coht $P =$ coht $Q$.
(3) ht $Q \leq$ ht $P$.
(4) If $R$ and $S$ are domains, with $R$ integrally closed, then ht $Q =$ ht $P$.

(2) Let $S = k[[X, Y, Z]]$ be the formal power series ring over a field $k$, and let $R = S/I$, where $I = (XY, XZ)$.

(1) Show that $\dim R = 2$.
(2) Consider the ideal $P = (\overline{Y}, \overline{Z})$ in $R$. Show that it is a prime ideal, with ht $P = 0$ and coht $P = 1$, hence ht $P +$ coht $P < \dim R$.

(3) Prove that if $R$ is any commutative ring, then $\dim R[X] \geq \dim R + 1$. If in addition $R$ is Noetherian, then $\dim R[X] = \dim R + 1$, hence $\dim R[X_1, \ldots, X_n] = \dim R + n$. (Feel free to consult the various commutative algebra books.)

(4) Let $S = \bigoplus_{n \geq 0} S_n$ be a graded ring. Then $S$ is Noetherian if and only if $S_0$ is Noetherian and $S$ is a finitely generated $S_0$-algebra.

(5) Let $M$ be a finitely generated module over the Noetherian ring $R$, and let $P \in \mathrm{Supp}(M)$. Show that $l_{R_P}(M_P) < \infty$ if and only if $P$ is a minimal element in $\mathrm{Ass}(M)$.

(6) Compute in a direct way the Hilbert function of $S = k[X, Y, Z]/(X^2 + Y^2 + Z^2)$, where $k$ is a field. (This is extended by a general procedure in the next exercise.)

(7) Let $F \in k[X_1, \ldots, X_n]$ be a homogeneous polynomial of degree $d$, where $k$ is a field. Compute the Hilbert function of $S = k[X_1, \ldots, X_n]/(F)$.