MATH 531: PROBLEM SET 6

Due Friday, October 17

(1) Let R be a Noetherian ring, and let X be an indeterminate. Show that
$$\dim R[X, X^{-1}] = \dim R + 1.$$

(2) (a) If $R \subset S$ is an inclusion of affine domains over a field k, then
$$\dim S = \dim R + \dim Q(R) \otimes_R S.$$

(b) If R and S are affine domains over a field k, then
$$\dim R \otimes_k S = \dim R + \dim S.$$

(3) (This problem deals with what can be said about the Hilbert function when the graded ring is not generated in degree 1.) Let $S = k[X_1, \ldots, X_r]$, where X_i is an indeterminate of degree d_i. Let M be a finitely generated graded module over S, and define as usual the Hilbert function of M to be $h_M(n) := \dim_k M_n$. Define the *Hilbert series* of M to be the formal power series in one variable $H_M(t) := \sum_{n \geq 0} h_M(n) t^n$.

(a) Show that $H_M(t)$ is a rational function of t, with poles only at roots of unity (for example by imitating the proof of the fact that the Hilbert function is polynomial-like when all $d_i = 1$). In fact show that we have that $H_M(t) = \frac{P(t)}{\prod_i^r (1 - t^{d_i})}$, where P is a polynomial in t with integral coefficients.

(b) If d is the least common multiple of the d_i, then for each natural number p we have that the function $h_M(dn + p)$ is polynomial-like in n. (This basically means that $h_M(n)$ is not necessarily polynomial-like this time, but it looks like a "polynomial with periodic coefficients".)

(c) Check that the same statements work if S is a graded ring finitely generated over an Artinian ring S_0 by homogeneous elements $x_1, \ldots, x_r$ of strictly positive degrees $d_1, \ldots, d_r$ such that $S/(x_1, \ldots, x_r)$ is Artinian (replacing of course dimension by length).