(1) Let \((R, m)\) be a Noetherian local ring, and suppose that the elements \(a_1, \ldots, a_t\) are part of a system of parameters. If the ideal \(P = (a_1, \ldots, a_t)\) is prime and has height \(t\), show that \(\text{ht} \ P + \text{coht} \ P = \dim R\).

(2) Let \(S = k[[X, Y, Z]]\), with \(k\) a field, and let \(R = S/I\), where \(I = (XY, XZ)\).

(a) Find the minimal prime ideals in \(R\). Find the dimension of \(R\).

(b) Show that \(\{Z, X+Y\}\) is a system of parameters for \(R\), but not an \(R\)-regular sequence.

(c) Show that \(\text{ht} \ (Z) = 0\). (This shows that if \(\{a_1, \ldots, a_r\}\) is a system of parameters, it is not necessarily the case that \(\text{ht} \ (a_1, \ldots, a_i) = i\).)

(3) Let \(R = k[X, Y]/(Y^2 - X^3)\), where \(k\) is a field.

(a) If \(m = (X, Y)\), show that \(R_m\) is not a regular local ring. Compute the height and the depth of its maximal ideal.

(b) If \(\text{char} ~ k \neq 2, 3\) and \(m = (X - 1, Y - 1)\), show that \(R_m\) is a regular local ring and compute its dimension.

(4) Show that any localization of the ring \(R\) in problem 3 at a maximal ideal is Cohen-Macaulay.

For the next two problems you can assume the following fact that will be proved later in the course: if \(R\) is a Noetherian local ring and \(x \in R\) is a non-zero divisor, then \(\text{depth} \ R/xR = \text{depth} \ R - 1\). (In both problems one can replace the power series ring with the localization of the polynomial ring at the maximal ideal generated by the variables.)

(5) Show that the ring \(R = k[[X, Y, Z]]/(XY, XZ)\) in problem 2 is not Cohen-Macaulay.

(6) Show that the ring \(R = k[[X, Y, Z, T]]/(X, Y) \cap (Z, T)\) is not Cohen-Macaulay.