MATH 531: PROBLEM SET 9

Due Friday, November 7

(1) Let $f : R \to S$ be a ring homomorphism.

(a) If M is a flat S-module and S is flat over R, then M is a flat R-module (by restriction of scalars).

(b) If M is a flat R-module, then $S \otimes_R M$ is a flat S-module.

(2) If S is a multiplicative system in a ring R, show that $S^{-1}R$ is a flat R-module.

(3) Show that $\mathrm{Ext}^n_{\mathbb{Z}}(A, B) = 0$ for all abelian groups A and B and every $n \geq 2$.

(4) Show that $\mathrm{Tor}^{\mathbb{Z}}_n(A, B) = 0$ for all abelian groups A and B and every $n \geq 2$.

(5) Show that the following is an infinite free resolution of the module $M = \mathbb{Z}_2$ over $R = \mathbb{Z}_4$ (with the standard module structure):

$$\ldots \xrightarrow{f} \mathbb{Z}_4 \xrightarrow{f} \mathbb{Z}_4 \xrightarrow{f} \mathbb{Z}_4 \xrightarrow{g} \mathbb{Z}_2 \to 0,$$

where $f(x) = 2x \pmod 4$ and $g(x) = x \pmod 2$.

(6) By analogy with the case of projective dimension, prove the characterization of injective dimension via the vanishing of Ext, i.e. that for any R-module N and any $n \geq 0$, the following are equivalent:

(1) $\mathrm{id}_R\, N \leq n$.
(2) $\mathrm{Ext}^i_R(M, N) = 0$, $\forall i > n$ and $\forall$ R-module M.
(3) $\mathrm{Ext}^{n+1}_R(M, N) = 0$, $\forall$ R-module M.
(4) If
$$0 \to N \to E_0 \to \ldots \to E_{n-1} \to Q_n \to 0$$
is an exact sequence with all E_i injective, then Q_n is injective.