Math 533 - Real Analysis HW 1 DUE: Wednesday September 5 2007

Recall:

- For any two non-empty sets X and Y the set Y^X stands for the collection of all functions $f : X \to Y$ (note the order).
- If X is a non-empty set, then $\mathcal{P}(X)$ denotes the collection of all its subsets. $\mathcal{P}(X)$ is naturally identified with $\{0,1\}^X$.
- The cardinality of $\mathbb{N}$ is denoted by the Hebrew $\aleph_0$, the cardinality of the reals $\mathbb{R}$ by $\aleph$
- The latter cardinality (of the reals) is also called the (cardinality of) continuum.
- $\aleph = 2^{\aleph_0}$, or in other words $\mathrm{card}(\{0,1\}^{\mathbb{N}}) = \mathrm{card}(\mathbb{R})$.
- Cantor's "diagonal argument" proves that $\aleph_0 < \aleph$, *i.e* the continuum is uncountable. More generally, for any set X, $\mathrm{card}(X) < \mathrm{card}(\mathcal{P}(X))$.

Questions

(1) For any three non-empty sets X, Y, Z construct a natural bijection between the sets $(Z^Y)^X$ and $Z^{(Y \times X)}$.

(2) Prove that $\mathrm{card}(\mathbb{R}^{\mathbb{N}}) = \mathrm{card}(\mathbb{R}) = \aleph$ (Hint: use problem 1).

(3) Prove that the space $\mathcal{C}(\mathbb{R})$ of all continuous functions $\mathbb{R} \to \mathbb{R}$ has the cardinality of the continuum (Hint: a continuous function is determined by its values at rational points).

(4) Prove that the collection of all open subsets of $\mathbb{R}$ has the cardinality of the continuum (Hint: think first about open intervals)