Homework 1, Math 535

January 25, 2008

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2b) $1/z = \bar{z}/|z|^2 = \frac{x-iy}{x^2+y^2}$ so that the real part is $\frac{x}{x^2+y^2}$ and the imaginary part is $\frac{-y}{x^2+y^2}$.

2c)

$$\frac{z-1}{z+1} = \frac{(z-1)(\bar{z}+1)}{|z+1|^2} = \frac{z-\bar{z}+|z|^2-1}{|z+1|^2} = \frac{2iy+x^2+y^2-1}{(x+1)^2+y^2}.$$

The real part is $\frac{x^2+y^2-1}{(x+1)^2+y^2}$ and the imaginary part is $\frac{2y}{(x+1)^2+y^2}$.

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3) Squaring the quantity we get

$$\frac{(a-b)(\bar{a}-\bar{b})}{(1-\bar{a}b)(1-a\bar{b})} = \frac{|a|^2+|b|^2-a\bar{b}-b\bar{a}}{1+|a|^2|b|^2-\bar{a}b-a\bar{b}}.$$

If one of the terms say $|a| = 1$ then the numerator is equal to the denominator.

If both have absolute value 1 then we cannot have $a = b$ for then both numerator and denominator are 0.

4) If we write this in coordinates x, y then we get two linear equations in two unknowns. We get either a single line if the equations are the same or a single point if the lines are different. We see under what conditions we get a single line. This problem is the same as page 17 1) then.

$az + b\bar{z} + c = 0$ is a line iff there exists μ, λ such that

$$z = \mu t + \lambda.$$

This gives

$$a(\mu t + \lambda) + b(\bar{\mu} t + \bar{\lambda}) + c = 0$$

for all t. This gives $a\mu + b\bar{\mu} = 0$ or $\frac{a}{b} = \frac{-\bar{\mu}}{\mu}$. Since the quantity on the right has absolute value 1, this equation can be solved for μ if and only if $|a| = |b| \neq 0$.

Thus we get a line iff $|a| = |b| \neq 0$. If the absolute values are different we get a point. If they are both 0 then we get emptyset unless $c = 0$ in which case we get the entire plane.

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1) Squaring both sides we want to prove

$$|a - b|^2 < |1 - \bar{a}b|^2.$$

Multiplying this out this is equivalent to proving that

$$|a|^2 + |b|^2 < 1 + |a|^2|b|^2$$

or equivalently

$$(1 - |a|^2)(1 - |b|^2) > 0.$$

but this holds because $|a| < 1, |b| < 1$.

4) We look for $z = \lambda a$ to be a solution where λ real. We get

$$|\lambda a - a| + |\lambda a + a| = 2|c|.$$

This gives

$$|a|(\lambda - 1) + |a|(\lambda + 1) = 2|c|$$

or

$$2|a|\lambda = 2|c|.$$

This gives

$$\lambda = \frac{|c|}{|a|}.$$

If there is a solution then

$$2|a| = |z - a - (a + z)| \leq |z - a| + |z + a| = 2|c|$$

so $|a| \leq |c|$.

The set of solutions is an ellipse. The biggest value of $|z|$ is when z is a multiple of a so that

$$z = \frac{|c|}{|a|}.$$

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4)

$$1 + \omega^h + \ldots + \omega^{(n-1)h} = \frac{1 - \omega^{nh}}{1 - \omega^h} = 0$$

since $\omega^n = 1$.

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See page 9 4)

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1) For the points to be diametrically opposite we have

$$x_i = -x_i'$$

for each i. Now if this is true then

$$\bar{z}' = \frac{-x_1 + ix_2}{1 + x_3}$$

so

$$z\bar{z}' = \frac{-x_1^2 - x_2^2}{1 - x_3^2} = -1.$$

Conversely, if $z\bar{z}' = -1$ then $z' = -\frac{z}{|z|^2}$ and $|z| = \frac{1}{|z'|}$. We see that

$$x_3' = \frac{|z'|^2 - 1}{|z'|^2 + 1} = -x_3.$$

Now

$$x_1' = \frac{z' + \bar{z}'}{1 + |z'|^2} = \frac{(-z/|z|^2 - \bar{z}/|z|^2)|z|^2}{1 + |z|^2} = -x_1.$$

The same computation shows $x_2' = -x_2$.