Math 535 Problem Set 3

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1) Let $z(t) = (1+i)t$ with $0 \le t \le 1$. Then $\int_\gamma x dz = \int_0^1 t(1+i)dt = 1/2(1+i)$.

6) Since $|f(z) - 1| < 1$ if we let $w = f(z)$ then $\log w$ is analytic since w is never a negative real number. Thus we have $\log f(z)$ is an analytic function on the domain Ω. Since

$$d(\log f(z))/dz = f'(z)/f(z)$$

the expression $f'(z)/f(z)dz$ is an exact differential so the integral is 0 along any closed curve in the domain Ω.

7) Express the polynomial

$$P(z) = a_0 + a_1(z-a) + \ldots + a_n(z-a)^n.$$

On the circle C we have $z = a + Re^{it}$ so that $d\bar{z} = -iRe^{-it}dt$. Thus the contour integral is

$$-iR\int_0^{2\pi}(a_0 + a_1Re^{it} + \ldots a_nR^ne^{int})e^{-it}dt.$$

The only nonzero term in the integral is $-2iR^2\pi a_1$. But $a_1 = P'(a)$.

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2) We write

$$\frac{1}{z^2+1} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z^i}\right).$$

The integral of the first term in the parentheses is $2\pi i$; the integral of the second is $-2\pi i$. Therefore the sum is 0.

3) We have $z = \rho e^{it}$ so $dz = i\rho e^{it}dt$ and $|dz| = \rho dt = -i\rho\frac{dz}{z}$. Thus

$$\int_{|z|=\rho}\frac{|dz|}{|z-a|^2} = \int_{|z|=\rho}\frac{-i\rho}{z(z-a)(\bar{z}-\bar{a})}dz = \int_{|z|=\rho}\frac{-i\rho}{(z-a)(\rho^2-\bar{a}z)}.$$

Now suppose $|a| < \rho$. Then the only singularity of the integrand is $z = a$ and the Cauchy integral formula applied to

$$f(z) = \frac{1}{\rho^2 - \bar{a}z}$$

gives that the integral is

$$2\pi i(-i\rho)\frac{1}{\rho^2-|a|^2} = \frac{2\pi\rho}{\rho^2-|a|^2}.$$

If $|a| > \rho$ then the singularity is at $\rho^2/\bar{a}$ and the Cauchy integral formula applied to

$$g(z) = \frac{-\bar{a}}{z - \rho^2/\bar{a}}$$

says that the integral is

$$-2\pi i(-i\rho)\bar{a}\frac{1}{\rho^2/\bar{a} - a} = \frac{2\pi\rho}{|a|^2-\rho^2}.$$