Homework 4, Math 535

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1a)

$$\int_{|z|=1} \frac{e^z}{z^n} dz = 2\pi i (n-1)! f^{(n)}(0)$$

where $f(z) = e^z$. But the last term is 1 so the answer is

$$\frac{2\pi i}{(n-1)!}.$$

2) For any z let $\gamma = \{\zeta : |\zeta - z| = R\}$ and take R large enough so that any $\zeta \in \gamma$ satisfies $|\zeta| \geq M$. Then on γ, $|\zeta| \leq |z| + R$ and so $|f(\zeta)| \leq (|z| + R)^n$.

Then by the Cauchy inequality

$$|f^{(n+1)}(z)| \leq \frac{(n+1)!}{2\pi} 2\pi R \frac{(|z| + R)^n}{R^{n+2}}.$$

This quantity goes to 0 as $R \to \infty$ and so $f^{(n+1)}(z) = 0$ for all z. This says f is a polynomial.

4) By Cauchy inequality $|f^n(0)| \leq M n! r^{-n}$ where M is the maximum of f on the circle of radius r. Since $|f(z)| \leq \frac{1}{1-|z|}$ we have

$$f^n(0) \leq \frac{n!}{(1-r)r^n}.$$

To find thevalue of r where $\frac{n!}{(1-r)r^n}$ is smallest, we look for the value of r where $(1-r)r^n$ is a maximum. We take derivative with respect to r of this quantity and get $nr^{n-1} - (n+1)r^n$. We set it equal to 0 to find $r = \frac{n}{n+1}$. Plugging into the above we find that

$$|f^n(0)| \leq \frac{n!(n+1)^{n+1}}{n^n}.$$

5) Again by Cauchy estimate

$$|f^n(a)| \leq n! M / r^n$$

where M is the maximum of $|f(z)|$ on a circle of radius r centered at a. If $|f^n(a)| > n! n^n$ then we would have $n^n < M/r^n$ for all small r which is impossible.

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2) Suppose $f(z)$ is analytic in the entire plane with a pole at infinity. Then $g(z) = f(1/z)$ has a pole of order n at 0. Thus

$$g(z) = \frac{h(z)}{z^n}$$

where $h(z)$ is analytic near 0 and $h(0) \neq 0$. Thus h is bounded in a neighborhood of 0. Then $f(z) = g(1/z) = z^n h(1/z)$ and $h(1/z)$ is bounded near $z = \infty$; that is, outside a circle of radius R_0. Thsu for some M we have

$$|f(z)| \leq M|z|^n$$

for some constant M.

We can now apply the Cuachy inequality to show that for any a, for any cirucle of radius $R > R_0 + |a|$ we have

$$|f^{n+1}(a)| \leq (n+1)!M(R+|a|)^n/R^{n+1}$$

and if we let $R \to \infty$ we have $f^{n+1}(a) = 0$ so f is a polynomial.

4) Suppose f is meromorphic in the plane with a pole at infinity. Since poles are isolated it can have only finitely many poles. Suppose these are at $z_1, \ldots, z_j$ with orders n_j. Then $\prod(z - z_i)^{n_i} f(z)$ does not have any poles in the plane and still has a pole at infinity. By the previous problem it is a polynomial which implies that $f(z)$ is rational.

6)

Suppose $f(z)$ has an essential singularity at z_0. Then since the values of f are dense there is a sequence $z_n \to z_0$ such that $f(z_n) \to 0$. Then

$$e^{f(z_n)} \to e^0 = 1$$

so z_0 cannot be a pole of e^f.

If z_0 is a pole of order n, then $g(z) = 1/f(z)$ has a zero of order n at z_0. Then g is open so that every point in a neighborhood of 0 say $|w| < \delta$ is taken on by $g(z)$. Then every point in $|w| > 1/\delta$ is taken on by $f(z)$. In particular values along the imaginary axis going to infinity are taken on. But then $|e^f| = 1$ and so e^f is not a pole.