Homework 5, Math 535

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1)

Consider the function $f(z) = z^2 + z$. We have $f'(-1/2) = 0$ so the function is not $1-1$ in a neighborhood of $-1/2$. Thus the largest radius circle about 0 in which it could be $1-1$ is $1/2$.

Now if $f(z_1) = f(z_2)$ where $z_1 \neq z_2$ we have $z_1^2 + z_1 = z_2^2 + z_2$ so

$$|z_1 - z_2|^2 = |z_1 - z_2|$$

so $|z_1 + z_2| = 1$. This means that z_1 and $-z_2$ are distance 1 apart and so cannot both be in the circle of radius $1/2$. The same is then true of z_1 and z_2 so on this disc, the map is $1-1$.

3)

We have $\cos(z) = 1 - z^2/2 + z^4/4! \dots$. We are going to use a trigonometric identity which is $\frac{1-\cos(2z)}{2} = \sin^2(z)$. Thus

$$\cos(z) - 1 = -2\sin^2(z/2).$$

Thus $\zeta(z) = i\sqrt{2}\sin(z/2)$.

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2) For z_0 in the upper half plane, the map $S(z) = \frac{z-z_0}{z-\bar{z}_0}$ maps the upper half plane to the unit disc, sending z_0 to 0. Similarly we replace z_0 with $f(z_0)$ and call this map T.

Then

$$g = T \circ f \circ S^{-1}$$

takes the unit disc to itself and $g(0) = 0$. Then Schwarz Lemma says that $|g(\zeta)| \leq |\zeta|$. applying this for $\zeta = S(z)$ we have

$$|T \circ f(z)| \leq |S(z)|.$$

This gives

$$|\frac{f(z) - f(z_0)}{f(z) - \overline{f(z_0)}}| \leq |\frac{z - z_0}{z - \bar{z}_0}|.$$

If we divide by $|z - z_0|$ and multiply by $|f(z) - \bar{f}(z_0)|$ and take limits as $z \to z_0$ we get

$$|f'(z_0)| \leq \lim_{z \to z_0} |\frac{f(z) - \bar{f}(z_0)}{z - z_0} = |\frac{Imf(z_0)}{Imz_0}|.$$

5) Let f be any $1-1$ map between discs D_1 and D_2 with $z_0 \in D_1$ any point. There is a linear transformation g between the unit disc and D_1 sending 0 to z_0 and a linear fractional map h between the unit disc and D_2 taking 0 to $f(z_0)$. Then $F = h^{-1} \circ f \circ g$ is a bijection from the unit disc to itself taking 0 to 0 so $|F(z)| \leq |z|$. But F^{-1} does the same. Thus by Schwarz we have $F(z) = cz$ for some c. Thus F is a linear transformation and so is f.