Homework 6, Math 535

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4) On any simply connected domain not containing the origin $f(z) \neq 0$ which means that $log(z)$ is analytic. But $z^\alpha = e^{\alpha log(z)}$ is analytic as well as $z^z = e^{zlogz}$.

5) Any closed curve γ in the domain Ω that winds around 1 also winds around -1. Now we start with some point $z_0 \in \gamma$ and choose some value of $\theta_1 = arg(1-z)$ so that $(1-z_0) = r_1 e^{i\theta_1}$. Take a branch of

$$\sqrt{1-z_0} = r_1^{1/2} e^{i\theta_1/2}.$$

Now as we travel around γ; as we come back to z_0 we add 2π to the argument of $(1-z)$. The same thing happens to the argument of $(1+z)$ and so the argument of the product changes by 4π. Now when we take square roots we divide the argument by 2 which means that as we move around γ and come back to z_0 the argument of $\sqrt{1-z^2}$ changes by 2π.

Now we compute

$$\int_\gamma \frac{1}{\sqrt{1-z^2}} dz.$$

By Cauchy's theorem we can assume γ is a very large circle and so acts as a small disc about infinity. this suggests a change of variables $z = 1/w$ on the circle. From $dz = -dw/w^2$ we get the integral is

$$-\int_{|w|=\delta} \frac{dw}{w\sqrt{w^2-1}}.$$

Now $\sqrt{w^2-1}$ is analytic in a neighborhood of 0 and so we can compute the integral by the Cauchy integral formula which gives the integral to be $-2\pi i \sqrt{(-1)} = 2\pi$.

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1) Let $f(z) = 6z^3$ and $g(z) = z^7 - 2z^5 + 6z^3 - z + 1$ Then $|f(z) - g(z)| = |z^7 - 2z^5 - z + 1| \leq |z^7| + 2|z^5| + |z| + 1 = 5 < |f(z)| = |6z^6| = 6$ on the circle $|z| = 1$. Thus they have the same number of zeroes in side the circle and that is 3.

2) Let $f(z) = z^4$ and $g(z) = z^4 - 6z + 3$. On the circle of radius 2, we have $|f(z) - g(z)| \leq 6|z| + 3 = 15 < |f(z)| = 16$. Thus $g(z)$ has the same number of zeroes as $f(z)$ inside which is 4. Inside the circle of radius 1 we let $f(z) = -6z$. Now $|f(z) - g(z)| = |z^4 + 3| \leq |z^4| + 3$ which on the circle of radius 1 is 4 which is less than $f(z)$ on the circle of radius 1. Thus g has 1 zero inside the circle of radius 1 and so g has 3 zeroes in the annulus.