Homework 2, Math 535

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4) Suppose $|f(z)|$ is a constant. Then $|f(z)|^2 = u^2 + v^2$ is a constant. Differentiating we get

$$2uu_x + 2vv_x = 0$$

and

$$2uu_y + 2vv_y = 0.$$

Applying the Cauchy Riemann equations, we get

$$-uv_x + vu_x = 0.$$

If $u = 0, v = 0$ at any point, then they are identically 0. Otherwise assume they are never both zero at the same point. The fact that the above system of two linear equations in the two unknowns u, v is homogeneous, says that the determinant must be identically 0. That is to say $u_x^2 + v_x^2 = 0$. But this means that u_x and v_x are identically 0 and so therefore are u_y and v_y. This says that u, v are constants and so is f.

5)

$$\lim_{z \to z_0} \frac{\overline{f}(\overline{z}) - \overline{f}(\overline{z}_0)}{z - z_0} = \lim_{z \to z_0} \frac{f(\overline{z}) - f(\overline{z}_0)}{\overline{z} - \overline{z}_0} = \overline{f}'(\overline{z}_0).$$

This says that if $f'(z_0)$ exists then $\bar{f}(\bar{z})$ has a derivative at $\bar{z}_0$. The other direction is the same.

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4) Let $R(z)$ the rational function. Let

$$S(z) = \frac{1}{\bar{R}(\frac{1}{\bar{z}})}.$$

Since $z\bar{z} = 1$ on the unit circle and $R\bar{R} = 1$ on the unit circle, we have that $R = S$ on the unit circle. Now $R - S$ is again a rational function which is identically 0 on the unit circle and since a rational function has only finitely many zeroes, $R - S$ is identically 0 everywhere. (Notice that S is a rational function because it is formed by taking the conjugate of the argment z and then again conjugate at the end.) Now if R has a zero at z then $S = R$ has a pole at $\frac{1}{\bar{z}}$ and if it has a pole at z then it has a zero at $\frac{1}{\bar{z}}$.

5) Now we let $S(z) = \bar{R}(\frac{1}{\bar{z}})$. Since R is real on the circle we now have $S = R$ on the unit circle and therefore everywhere. This says that if R has a zero at z it has a zero at $\frac{1}{\bar{z}}$ and similarly with poles.

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2) Given ϵ let N_1 such that for $n \geq N_1$, $|z_n - A| < \epsilon/3$. Pick $N_2 > N_1$ large enough so that

$$\frac{|z_1| + \ldots |z_{N_1}|}{N_2} < \epsilon/3$$

and $N_1|A|/N_2 < \epsilon/3$. Then for $n \geq N_2$ we have

$$|\sum_{j=1}^{n} z_j/n - A| \leq \sum_{j=1}^{N_1} |z_j|/n + \sum_{j=N_1+1}^{n} (|z_j - A|/n + N_1|A|/n < 3\epsilon/3 = \epsilon.$$

4) The sequence $\{nz^n\}$ converges for $|z| < 1$, diverges for $|z| \geq 1$ and converges uniformly on any set of the form $|z| \leq \rho$ where $\rho < 1$.

5) Differentiating

$$f_n(x) = \frac{x}{n(1 + nx^2)}$$

and setting it equal to 0 we get $f_n'(x) = 0$ when $x = \frac{1}{\sqrt{n}}$ This is a maximum of the function and plugging in we see the maximum is

$$\frac{1}{2n^{3/2}}.$$

Since

$$\sum \frac{1}{n^{3/2}}$$

is a convergent series, our series converges uniformly on the whole real line by the Weierstrass M test.

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3a) Since

$$\lim (n^p)^{1/n} = 1,$$

the radius of convergence is $R = 1$

4) The coefficient of z^n in $a_n z^{2n}$ is $a_{n/2}$. Since $\limsup |a_{n/2}|^{1/n} = (\limsup |a_{n/2}|^{1/(n/2)})^{1/2}$ we have $R' = R^{1/2}$.

For the second part we have $R' = R^2$.

5) $f'(z) = \sum n a_n z^{n-1}$ so $zf'(z) = \sum n a_n z^n$. Differentiating again and multiplying by z we get

$$z(f'(z) + zf''(z)) = \sum n^2 a_n z^n.$$

Finally differentiating again and then multiplying by z we get

$$z(f'(z) + zf''(z) + z^2(f''(z) + zf'''(z) + f''(z)) = \sum n^3 a_n z^n.$$

8) We need

$$|\frac{z}{1+z}| < 1.$$

We need $|z| < |1+z|$. Squaring we get

$$z + \bar{z} + 1 > 0$$

or $x > -1/2$.