HOMEWORK 2

This problem set is due Friday September 12. You may work on the problem set in groups; however, the final write-up must be yours and reflect your own understanding. In all these exercises assume that $k$ is an algebraically closed field and $R$ is a commutative ring with unit.

Problem 0.1. Consider the following five closed affine sets in $\mathbb{A}^2$. Give a thorough discussion of which among them are isomorphic.

1. $X_1 = \{(x, y) \in \mathbb{A}^2 \mid x = y\}$
2. $X_2 = \{(x, y) \in \mathbb{A}^2 \mid x = y^17\}$
3. $X_3 = \{(x, y) \in \mathbb{A}^2 \mid x^2 = y^2\}$
4. $X_4 = \{(x, y) \in \mathbb{A}^2 \mid x^2 = y^3\}$
5. $X_5 = \{(x, y) \in \mathbb{A}^2 \mid x^2 = y^2 + y^3\}$

Problem 0.2. Prove the following statements:

1. If $X$ is an affine variety, then any non-empty Zariski open subset $U$ of $X$ is dense in $X$.
2. If $X$ is an affine variety, then any non-empty Zariski open subset $U$ of $X$ is irreducible.
3. Let $f : X \to Y$ be a regular, surjective map of closed affine sets. If $X$ is irreducible, then $Y$ is irreducible.

Problem 0.3. Show that any two ordered sets of $n + 2$ points in general position in $\mathbb{P}^n$ are projectively equivalent. Show that two sets of four points in $\mathbb{P}^1$ are projectively equivalent if and only if their cross-ratios are equal. Harder: Characterize when $n + 3$ points in general linear position in $\mathbb{P}^n$ are projectively equivalent.

Problem 0.4. Let $\Gamma$ be a set of points in $\mathbb{P}^n$ of cardinality $d$. Show that $\Gamma$ can be expressed as the zero locus of polynomials of degree at most $d$. Show that if all the points in $\Gamma$ do not lie on a line, then in fact $\Gamma$ can be expressed as the zero locus of polynomials of degree $d - 1$ or less.

Problem 0.5. (1) Show that the Segre image of $\mathbb{P}^1 \times \mathbb{P}^1$ is a quadric hypersurface in $\mathbb{P}^3$.
(2) Let $L, M$ and $N$ be three pairwise skew lines in $\mathbb{P}^3$. Show that union of all the lines in $\mathbb{P}^3$ intersecting $L, M$ and $N$ is isomorphic to the Segre image of $\mathbb{P}^1 \times \mathbb{P}^1$.
(3) How many lines in $\mathbb{P}^3$ intersect the four lines $L_1 = (z_1 = z_2 = 0), L_2 = (z_3 = z_4 = 0), L_3 = (z_1 = z_2, z_3 = z_4$ and $L_4 = (z_1 + 2z_2 = z_3 + z_4, z_1 + 2z_4 = z_2 + z_3)$?

Problem 0.6. Recall that the twisted cubic curve $C$ is the image of the map $\nu : \mathbb{P}^1 \to \mathbb{P}^3$ given by $(x_0 : x_1) \mapsto \begin{pmatrix} x^3_0 \\ x^2_0x_1 \\ x_1 \end{pmatrix}$

1. Show that the homogeneous ideal is generated by $Q_1 : z_0z_2 = z_1^2, Q_2 : z_1z_3 = z_2^2, Q_3 : z_0z_3 = z_1z_2$.
2. Show that an alternative way to describe the twisted cubic is as the rank one locus of the matrix
\[
\begin{pmatrix}
z_0 & z_1 & z_2 \\
\z_1 & z_2 & z_3
\end{pmatrix}
\]
(3) Show that $Q_i \cap Q_j$ for $i \neq j$ is $C$ union a line.
(4) More generally, show that the intersection of $\lambda_1Q_1 + \lambda_2Q_2 + \lambda_3Q_3$ and $\mu_1Q_1 + \mu_2Q_2 + \mu_3Q_3$, where $(\lambda_1 : \lambda_2 : \lambda_3) \neq (\mu_1 : \mu_2 : \mu_3)$ in $\mathbb{P}^2$, is $C$ union a line.