HOMEWORK 3

This problem set is due Friday September 19. You may work on the problem set in groups; however, the final write-up must be yours and reflect your own understanding. In all these exercises assume that k is an algebraically closed field and R is a commutative ring with unit.

Problem 0.1. *Let V and W be two linear spaces of dimension k and m, respectively, in $\mathbb{P}^n$. Prove that if $k+m \geq n$, then $V \cap W$ have non-empty intersection.*

Problem 0.2. *Using problem one, show that given*

$$k \leq \binom{n+d}{d} - 1$$

points in $\mathbb{P}^n$, there exists a non-zero homogeneous polynomial of degree d in $n+1$ variables vanishing on all k points. We say that the points impose independent conditions on polynomials of degree d in $n+1$ variables if the codimension of the vector space of homogeneous polynomials of degree d vanishing on the k points is $\min(k, \binom{n+d}{d})$. Prove the following statements.

1. *$k \leq n+1$ points impose independent conditions on linear polynomials if and only if the points are in general linear position.*
2. *4 points fail to impose independent conditions on degree two polynomials in three variables if and only if they lie on a line. (What is the relation to the problems in week 1?)*
3. *Let $p_1, \dots, p_9$ be the points of intersection of two irreducible cubic polynomials in $\mathbb{P}^2$. Show that any cubic polynomial containing 8 of the points contains the ninth as well.*

Problem 0.3. *Prove the following statements.*

1. *Show that 4 points in $\mathbb{P}^3$ fail to impose independent conditions on quadrics (degree two polynomials) if and only if they lie on a line. Does the same hold for $\mathbb{P}^n$?*
2. *Show that 6 points that lie on a conic in $\mathbb{P}^3$ fail to impose independent conditions on quadrics.*
3. *Show that 8 points that lie on a twisted cubic fail to impose independent conditions on homogeneous polynomials on quadrics. More generally, show that $2n+2$ points that lie on a rational normal curve of degree n in $\mathbb{P}^n$ fail to impose independent conditions on quadrics (degree two polynomials in $n+1$ variables).*

Problem 0.4. *Let $X = \nu_2(\mathbb{P}^2)$, the second Veronese embedding of $\mathbb{P}^2$. Show that the hyperplane sections of X are either rational normal curves of degree 4 or the union of two conics intersecting at a point or a (double) conic. (When do we get a rational normal curve? When do we get a union of two conics/a (double) conic?)*

Problem 0.5. *Find the equations of the projection of the standard twisted cubic $[x_0, x_1] \mapsto [x_0^3, x_0^2x_1, x_0x_1^2, x_1^3]$ from the points $[1,0,0,1]$ and $[0,1,0,0]$. Harder: Show that any projection of a twisted cubic in $\mathbb{P}^3$ from a point outside the twisted cubic is projectively equivalent to one of these two. Challenge: Show that the rational normal quartic in $\mathbb{P}^4$ has smooth projections to $\mathbb{P}^3$ (from points outside the curve) that are not projectively equivalent.*