Algebraic Geometry (Intersection Theory) Seminar

Lecture 1 (January 20, 2009)

We will discuss cycles, rational equivalence, and Theorem 1.4 in Fulton, as well as push-forward of rational equivalence. Notationwise, we will let "scheme" mean an algebraic scheme over a field, "variety" will mean a reduced and irreducible scheme, and a "point" will mean a closed point. Let \ be a scheme and Z © \ a subvariety. Then b@ßB œ b8ßB where 8 is a generic point of Z. For a variety \, let V(\ ) be the field of rational functions.

Let \ be a variety with Z © \ a subvariety of codimension one. Then bZß\ has dimension ", and V(bZß\) œ V(\ ). For any ! Á < − V(\ ) œ V(bZß\) with < œ +Î, for +ß , − bZß\. For any ! Á = − bZß\, j(bZß\ ÎW) □ ∞.

Definition. *Let \ be a variety and Z © \ a subvariety with codimension ". Then there is a well-defined homomorphism* ordZ À V(\ )‡ Ä ™ *such that* (*for* < − V(\ )‡ œ V(bZß\)‡, < œ +Î, *with* +ß , − bZß\):

(i) ordZ (<) œ ordZ (+) □ ordZ (,),

(ii) ordZ (+) œ j(bZß\ Î+), and

(iii) ordZ (,) œ j(bZß\ Î,).

Example. If \ is a veriety which is regular in codimension one, this means for any Z © \ a subvariety with codimension ", bZß\ is a discrete valuation ring.

Cycles and rational equivalence

Definition. *Let \ be a scheme. A* 5*-cycle on \ is a finite formal sum*

∑ 83[Z3] ß 83 − ™

where Z3 © \ *are* 5*-dimensional subvarieties of* \. *Let*

^5 \ œ {∑ 83[Z3] l Z3 *are a* 5*-dimensional subvarieties*}, *so*

^‡ \ œ ⊕5 ! ^5 \.

Any element α œ ∑5 ! {α}5 − ^‡O *is called a cycle.*

For X a scheme, let $X_1, \ldots, X_r$ be irreducible components of X. For each X_i, we have geometric multiplicity $m_i = \ell(\mathcal{O}_{X_i, X})$. Notice X_i has a generic points η_i ($\mathcal{O}_{X_i, X} = \mathcal{O}_{\eta_i}$ is an Artinian ring).

Notation. We define $[X] = \sum_{i=1}^{r} m_i [X_i] \in Z_* X$ (this is a cycle).

We want to give an equivalence relation on cycles. Let X be a scheme and $V \subset X$ a $(k+1)$-dimensional subvariety. For $f \in K(V)^*$, we define

$$[\mathrm{div}(f)] = \sum_{\substack{W \subset V \\ \mathrm{cod}_V W = 1}} \mathrm{ord}_W(f)\,[W].$$

Notice a k-cycle α is rationally equivalent to 0, written $\alpha \sim 0$, if there are a finite number of $k+1$-dimensional subvarieties $V_i \subset X$ and $f_i \in K(V_i)^*$ such that $\alpha = \sum [\mathrm{div}(f_i)]$.

All the cycles equivalent to 0 are written $\mathrm{Rat}_k X$. We can now define the cycle class,

$$A_k X = Z_k X / \mathrm{Rat}_k X, \text{ with}$$

$$A_* X = \bigoplus_{k \geq 0} A_k X = \bigoplus_{k \geq 0} Z_k X / \mathrm{Rat}_k X.$$

Examples. (1) $A_k(X) \cong A_k(X_{\mathrm{red}})$, since X and X_{reduced} have the same subvarieties.

(2) If $X = X_1 \cup X_2 \cup \cdots \cup X_r$, then

$$Z_* X = \bigoplus_{i=1}^{r} Z_* X_i \quad \text{and} \quad A_* X = \bigoplus_{i=1}^{r} A_* X_i.$$

(3) If $\dim X = n$, then $A_n X = Z_n X$, then there are no "$k+1$-dimensional subvarieties", so there is nothing to mod out by.

(4) If X_1 and X_2 are subschemes of X, then

$$A_k(X_1 \cap X_2) \to A_k X_1 \oplus A_k X_2 \to A_k(X_1 \cup X_2) \to 0$$

is an exact sequence.

(5) If **Z** is an irreducible component of ****, then for any cycle class α **−** **E**‡ ****, we define the coefficient of **Z** in α to be the coefficient of [**Z**] in any cycle which represents α (since α **œ** ∑ **8**₃ [**Z**₃]).

(6) If **** **œ** spec **O**, then **E**! **** **œ** **^**! **** **œ** ™[B] (just one point).

(7) If **** **œ** □"₅, then **E**" **** **œ** **^**" **** **œ** ™[B] and **E**! **** **œ** Pic □" **œ** ™[:].

(8) Let **Z** be a **5** □ **"**-dimensional variety. Assume we have **0** À **Z** **Ä** □" a dominant function (<u>**0**(**Z**)</u> **œ** □", or other way to think about it is that **0** maps a generic point of **Z** to a generic point of □", or even **0** **−** **V**(**Z**)‡). Let's look at the fibers **0**□"(**!**) and **0**□"(**∞**) (i.e., if □" **œ** (**B**! À **B**"), we have **!** **œ** (**"** À **!**) and **∞** **œ** (**!** À **"**)). Then **0**□"(**!**) and **0**□"(**∞**) are purely **5**-dimensional subschemes, so [**0**□"(**!**) □ **0**□"(**∞**)] **œ** [div(**0**)].

Push-forward of cycles

Let **0** À **** **Ä** **]** be a proper morphism of two schemes. Then for a unique α **−** **^**‡ ****, we want to understand **0**‡α **−** **^**‡ **]**. Assume α is a **5**-cycle with α **œ** ∑ **8**₃ [**Z**₃] with **Z**₃ a **5**-dimensional subvariety. We want **0**‡[**Z**] **œ** ? **−** **^**‡ **]** to have some (covariant) functorial property relating **** and **]**. If **0** À **** **Ä** **]** and **0** À **Z** **Ä** **[** **œ** **0**(**Z**) (**0**(**Z**) is a subvariety of **]**). We want

(1) dim **[** □ dim **Z** means **0**‡[**Z**] **œ** **!**.

(2) dim **[** **œ** dim **Z** means **V**(**Z**) ä **V**(**[**) has degree [**V**(**[**) À **V**(**Z**)] □ **∞**

We then call **0**‡[**Z**] **œ** [**V**(**Z**) À **V**(**[**)] † [**[**] (so **0**‡α is well-defined).

Theorem. [1.4] *If* **0** À **** **Ä** **]** *is a proper morphism and* α *is a* **5**-*cycle on* **** *which is rationally equivalent to* **!**, *then* **0**‡α *is rationally equivalent to zero on* **]**.

Proof. We write α **œ** [div(**<**)] for **<** **−** **V**(**Z**) for some **5** □ **"**-dimensional subvariety of ****. That is,

α **œ** [div(**<**)] **œ** ∑_{**Z**₃©**Z**, codim**Z** **Z**₃ **œ** **"**} **8**₃[**Z**₃].

For this cycle, we can assume \ œ Z and] œ 0(Z). We can assume \ is a variety and] is a variety and 0 À \ Ä] is a proper surjection. Now we use the below proposition.

Proposition. [1.4] *Let* 0 À \ Ä] *be a proper, surjective morphism and let* < − V(\ )‡. *Then*

> (a) 0‡[div(<)] œ ! *if* dim] □ dim \ .
>
> (b) 0‡[div(<)] œ [div R(<)] *if* dim] œ dim \ ,

where R(<) *is determinant of* V(\ ) Å V(\ ) *as an* V( ])*-linear morphism (*R(V) − V( ])*).*

Proof. *Case 1.* Let] œ spec O for O a field. Let \ œ □"5 with 0 À □"5 Ä spec O. Since V(\ ) œ O(>), and < − V(\ ) œ O(>) (< œ +Î, with +ß , − O[>]), we can assume < − O[>] is an irreducible polynomial (since [div(<)] œ [div(+)] □ [div(,)], so if we prove it for [div(+)] and [div(,)] (and we know + and , are polynomials), then we have shown it for [div(<)]). Furthermore, let T œ (<) be a maximal prime ideal (T − \ œ □"5 is a closed point). In this case, ordT(<) œ " (since b: œ O[>](<)), and ordU(<) œ ! for U © O[>] with U Á T.

Finally, consider the point T∞ − □"5. Then T∞ œ ("/>) § O["/>] (it corresponds to the prime ideal generated by "/> in the ring O["/>]). The way to visualize this is that O[>] is for everything but ∞, and O["/>] is for everything but !. Call degree < œ . . Then < œ ("/>)□. 2("/>), some poly. For example, < œ ># □ " œ ("/>)□#(" □ "/>#). Then on □"5, we have

[div(<)] œ [:] □ .[T∞] and
0‡[div(<)] œ [V(T) À O] □ .[V(T∞) À O] œ . □ . † " œ !,

with V(T) œ O[>]Î<.

Case 2. If 0 À \ Ä] is finite, set with O œ V( ]) and P œ V(\ ) and O ä P. Also, assume all varieties in [div(<)] map to a variety [§] of codimension ". Let E œ b[ß] (the generic point of this image) and 7 œ 7[ß] . Consider the commutative diagram

Spec **P Ä \ ,]** Spec **E Ä **
Æ Æ Æ 0
Spec **O Ä** Spec **E Ò]**

with **E** ä **F** ä **P** and dim **F œ** dim **E œ "**. Finally, **F** has finitely many maximal ideals **7₃** such that **7₃** ∩ **E œ 7**. There is a one-to-one correspondence between □",{**Z₃ § \ l 0**(**Z₃**) **œ [** } and { **7₃ − F l 7₃** ∩ **E œ 7** }þ Also, from the diagram, **F Œ_E O œ P** and **F_7₃ Œ_E O œ P**.

Now assume **< − F**. Then computationally,

div(**<**) **œ** ∑_**Z₃** ord_**Z₃**(**<**) † [**Z₃**], and

0‡[div(**<**)] **œ** ∑_**Z₃** ord_**Z₃**(**<**) † [**V**(**Z₃**) À **V**(**[**)][**[**]

œ ∑_**Z₃** **j**(**F_7₃ Î<**)[**V**(**Z₃**) À **V**(**[**)][**[**].

Now we use two lemmas from the back of the book (A.2.3 and A.2.2), that state if (**7ß E**) **Ä** (**F_7₃ß 7₃**) is a push-forward,

j_E(**F_7₃ Î<**) **œ** [**F_7₃ Î 7₃ À EÎ 7**] † **j**(**F_7₃ Î<**)

œ ∑_**Z₃** ord_**E**(**R**(**<**))[**[**] **œ =** † ord_**E**(**R**(**<**))[**[**].

Alternate Definition of Rational Equivalence

Let **Z** ä **\ ,** □" **Ä** □" with **Z** a **5** □ **"**-dimensional variety and **\ ,** □" **Ä **. Then the fibers **0**^□"(**!**) **§ \ ,** {**!**} and **0**^□"(**∞**) **§ \ ,** {**∞**} are purely **5**-dimensional subvarieties of **Z**. Then if **0**^□"(**!**) **§ \ ,** {**!**}, we can define

div(**0**) **œ** [**0**^□"(**!**)] □ [**0**^□"(**∞**)] and
:‡div(**0**) **œ :**‡[**0**^□"(**!**)] □ **:**‡[**0**^□"(**∞**)]

where **:**‡[**0**^□"(**!**)] **œ Z** (**!**) and **:**‡[**0**^□"(**∞**)] **œ Z** (**∞**).

Proposition 1.6. *A cycle* α *in* **^₅ ** *is rationally equivalent to zero if and only if there are* (**5** □ **"**)*-dimensional subvarieties* **Z₃ß þþß Z_>** *of* **\ ,** □" *such that projections from* **Z₃** *to* □" *are dominant with*

α **œ** ∑_{**3œ"**}^{**>**} [**Z₃**(**!**)] □ [**Z₃**(**∞**)].

Proof. We have α œ [div(<)] (< − V([ )‡, [a 5 □ "-dimensional subvariety of \ ). Then consider [Å □" and we have Z ä \ , □" Ä □" and div[<] œ :‡ [div(0)] œ [Z (!)] □ [Z (∞)] (where 0 À Z Ä □").

Lecture 2 (January 27, 2009)

Recall from last time that for \ a scheme, ∑ 8₃[Z₃] is a 5-cycle, and ^5 \ is

^5 \ œ {∑ 8₃[Z₃] l Z₃ are 5-dimensional subvarieties}.

Furthermore, ^‡ \ œ ⊕ 5 ! ^5 \ . For [© X 5 □ "-dimensional, < − V([ ),

[div(<)] œ ∑ cod [Z œ" ord Z (<) [Z].

Furthermore, recall that Rat V \ œ {Eldiv(<)} (< µ !). We defined

E5 \ œ ^5 \ Î Rat5 \

with

E‡ \ œ ⊕ 5 ! E5 \ .

Theorem 1.4. *For* 0 À \ Ä] *a proper morphism, we define*

0\ À ^5 \ Ä ^5]

by 0‡[Z] œ deg(Z Î [ )[[ ] ([œ 0(Z)). *Then for* α µ ! *on* \ , 0‡α µ ! *on*] .

We will motivate this theorem by seeing its application to Bezout's Theorem on plane curves.

Bezout's Theorem on plane curves

For J ß K plane curves in □# with deg(J) œ 7 ß deg(K) œ 8, if J ß K intersect with no common roots, then

∑ 3(T ß J † K) œ 7 † 8,

the intersection multiplicity of J and K at T.

Assume J is irreducible. Then for all Kʷ plane curves, K/Kʷ − V(J), so

∑ 3(T ß J † K) □ ∑ 3(T ß J † Kʷ) œ ∑ ord T (K/Kʷ).

Then for all J″, J/J″ − V(K″),

∑ 3(T ß J † K″) □ ∑ 3(T ß J″ † K″) œ ∑ ord_T (J/J″).

Hence,

∑ 3(T ß J † K) œ ∑ 3(T ß J″ † K″)

Given J a plane curve, < − V(J),

[div(<)] œ ∑ ord_T (<)[T].

We want to show ∑ ord_T (<) œ !.

Take 1 À J Ä □". Then

1‡[div(<)] œ ∑ ord: (<) . [1T] in □",

(with . the degree of J) and notice ord: (<) œ [div(;)] (; − V(□")).\

Alternate Definition of Rational Equivalence

Proposition 1.6. If + µ ! in ^5 \ if and only if

α œ ∑([Z3(!)] □ [Z3(∞)])

Z3 © \ , □" with 03 À Z3 Ä □" dominant with T À \ , □" Ä \, satisfying Z3(!) œ T J □"(!) and Z3(∞) œ T J □"(∞).

Theorem 1.7. *If* 0 À \ Ä] *is a flat morphism (of relative dimension* 8*), define*

0‡ À ^5] Ä ^5□8 \

by 0‡[Z] œ [0□"Z] *(*Z *a variety). Then for* α µ ! *in* ^5], 0‡α µ ! *in* ^5□8 \.

Note. Ramin says: "Keep in mind, for flat morphisms, it essentially means that the dimensions of fibers are constant. This is what it means to be flat, it makes these morphisms nice in the aforementioned sense." [not really a quote, just paraphrase] Specifically, for 0□"(C) a fiber of a flat morphism 0 À \ Ä], the dimension is given by dim \ □ dim].

Lemma 1.7.1. *For all subschemes*]″ ©] ß 0‡[]″] œ [0□"(]″)].

Proposition 1.7. *Consider the fiber square with* 1 *flat and* 0 *proper:*

1″
\″ Ä \
0″ Æ Æ 0

]" Ä]

X2/81" *is flat,* 0" *is proper, and for all* α − ^‡ \ , 0‡"1"‡ œ 1‡0‡α *in* ^‡] ".

Note. Ramin says "What is a fiber square? Think about a product structure. What is a product? The product of two sets \ and] is the Cartesian pairs (BßC) such that you have two projections onto B and onto C. The above is the situation when B and C don't have any maps onto any other thing. Now, if you did have a third map, then you'd want the collection of pairs (BßC) which actually map into the same thing in D. Set theoretically, for a diagram

^" Ä]
Æ Æ0
\ Ä ^,

the fiber product is ^" © \ ,] given by ^" œ {(BßC) l 1(B) œ 0(C)}. Suppose you have spec V Ä Spec W and Spec X Ä Spec W and you want to construct something ‡ Ä Spec V and ‡ Ä Spec X (diagram). First thing you do is reverse the arrows so you get W Ä Vß W Ä X, and then the ring you construct is V Ä V Œ W X and X Ä V Œ W X. Then you consider the spectrum of these rings and that is the fiber product. Problem is, scheme-theoretically you have to do this locally, so you have to make sure the data glues together." ☐

Proof. For \ and] varieties with α œ [\ ], assume surjective. Then

0(\ ) œ] and 0‡[\ ] œ .[ ]].

We want 0‡"[\ "] œ .[ ] "]. Take \ œ Spec(P) and] œ Spec(O) with OßP fields, and let] " œ Spec(E) with E local Artinian, and E" œ Spec(F) with F œ E Œ O P. Then we are done by Lemma A.1.3. ☐

We are now ready to prove the main theorem.

Theorem 1.7. *If* 0 À \ Ä] *is a flat morphism (of relative dimension* 8*), define*

0‡ À ^5] Ä ^5☐8 \

by 0‡[Z] œ [0☐"Z] *(*Z *a variety). Then for* α µ ! *in* ^5] , 0‡α µ ! *in* ^5☐8 \

Proof. Let α µ ! in ^O] . Then α œ [Z (!)] ☐ [Z (∞)] with

Z (!) œ ; : ☐"(!), for

: À Z8 Ä □" and ; À] , □" Ä] with (0 , ") À \ , □" Ä] , □". Take [œ (0 , ")□"Z a subscheme. Then 2 À [Ä □" and : À \ , □" Ä \ so that we can construct a fiber square

\ , □"(0 Ò ")] , □"
: Æ Æ 0
\ Ò 0] .

So

0‡α œ 0‡[; : □"(!)] □ [; : □"(∞)] œ 0‡;([: □"(!)] □ [: □"(∞)])

and then by the previous proposition, this equals

:‡(0 , ")‡([: □"(!)] □ [: □"(∞)])

œ :‡(0 , ")□"([: □"(!)] □ [: □"(∞)]) œ :‡([2□"(!)] □ [2□"(∞)])

and so by Theorem 1.4 it suffices to show [[ ] œ ∑ 73[[3] for [œ ⋃ [3. Need 23 œ 2 l[3 with [2□"(T)] œ ∑ 73[23□"(T)] for T œ !ß ∞.

1.8 An Exact Sequence

See Proposition 1.8 and Example 1.9.3.

Lecture 3 (February 3, 2009)

Recall that if \ is a variety and Z § \ is of codimension ", then we defined a= − bZß\ (regular functions in the local ring), then j(bZß\ Î(=)) □ ∞.

Furthermore, recall that a 5-cycle is a finite formal sum ∑Z § \ ßdim Z œ5 8<[Z].

Divisors

Definition. *Let \ be a variety (8-dimensional). Then a **Weil divisor** is an (8□")-cycle on \ . Furthermore, ^8□" \ œ abelian group of Weil divisors.*

Definition. *A C**artier divisor** on \ is defined by {(Yαß0α)} œ H, where Yα © \ is open, and \ œ ⋃α Yα, and 0α are non-zero functions in V(Yα) œ V(\ ) that satisfy (1) 0α Î ! on Yα, (2) 0αÎ0" is a unit on Yα ∩ Y" (a unit is nowhere-vanishing, regular).*

Definition. *The 0α's are called the local equation of H.*

Definition. *If* **H** *is a Cartier divisor of* ****, *and* **Z © ** *is a subvariety of codimension* **"ß** *then we can define an order function on the divisor,*

$$\mathrm{ord}_{\mathbf{Z}}\, \mathbf{H} \;\; ³ \;\; \mathrm{ord}_{\mathbf{Z}}\, (\mathbf{0}_{\alpha}).$$

This is well-defined (independent of our choice of local equations) because they differ by a unit: ord($\mathbf{0}_\alpha$) **œ** ord(**?0**")*, where* **?** *is a unit in* b$_{\mathbf{Y}_\alpha \cap \mathbf{Y}"ß\backslash}$.

Definition. *The associated Weil divisor to* **H** *is* [**H**] **œ** $\sum_{\mathbf{Z} © \backslash ß\, \mathrm{codim}\, "}$ ord **H**[**Z**].

Note. There are only finitely many **Z © ** of order$_{\mathbf{Z}}$ **H Á !**.

Definition. *Let* Div(****) **œ** *group of Cartier divisors. Let* **H œ** ($\mathbf{Y}_\alpha$ß$\mathbf{0}_\alpha$)ß **I œ** ($\mathbf{Y}_\alpha$ß$\mathbf{1}_\alpha$). *Then we define* **H □ I œ** ($\mathbf{Y}_\alpha$ß$\mathbf{0}_\alpha\mathbf{1}_\alpha$).

We also induce a homomorphism Div(****) **Ä ^**$_{\mathbf{8□"}}$ **** with **H È** [**H**].

Definition. *For any* **0 − V**(****)‡*, we define a Principle Cartier Divisors* div(**0**) *by all local equations* **œ 0**.

Definition. *Two Cartier divisors* **H** *and* **H**^{w} *are linearly equivalent if there exists an* **0 − V**(****)‡ *such that* **H**^{w} **œ H □** *div*(**0**).

Definition. Pic(****) **œ** Div(****)**Î µ**, *where* **µ** *is the above linear equivalence.*

Hence, **H**^{w} **œ H □** div(**0**) **Ö** [**H**] □ [**H**^{w}] **µ !** as (**8 □ "**)-cycle. This induces Pic(****) **Ä E**$_{\mathbf{8□"}}$ ****.

Definition. *The support of a Cartier divisor* **HÎ** *denoted* supp(**H**)*, or* |**H**| *is the unit of all subvarieties* **^ © ** *such that if* **H œ** ($\mathbf{Y}_\alpha$ß$\mathbf{0}_\alpha$) *then* $\mathbf{0}_\alpha$ *is not a unit of* b$_{\backslash ß ^}$. *Or* $\mathbf{0}_\alpha$ *somewhere on some* **^ © ** *(that is,* $\mathbf{0}_\alpha$ *has a zero or a pole on* **^ © ***).*

Example. *If* **\\ ©** □$^\$$ is defined by **D**$^\#$ **œ BC**, then **B œ D œ !** is a Weil divisor, but not a Cartier divisor. This is because there is no way to have local equations about the cycles define **B œ D œ !** exclusively. If **D œ ! Ö B œ D œ !** or **C œ D œ !**. No matter what local equation you choose, you will always "get another line."

§2.1 - Line bundles as pseudo-divisors

Definition. *A pseudo-divisor is a triple* (**Pß^ß=**) *where* **P œ** *line bundle,* **^ œ** *closed subset of* **** *("support"), and s* **œ** *nowhere vanishing section on* **\\ □ ^**.

We say (Pß^ß=) œ (P"ß^"ß=") if (i) ^ œ ^"ß (ii) b5 À P Ä P" s.t. 5l\□^ sends = È =". In other words, they are equal if on the closed subsets we have isomorphic line bundles. (Recall isomorphism of line bundles!)

Definition. *For* \ *an algebraic scheme,* E *a Cartier divisor, a divisor* H *determines a line bundle* b(H) *by the sheaf of sections on the* b\ *-subsheaf generated by* "Î0₃ *on* Y₃ *(with* \₃ œ ⋃₃ Y₃*).*

Definition. *A cartier divisor* H *is* effective *if the* 0₃ œ =H lY₃ *for* =H − b\ *, where* =H *is the canonical section (i.e., if* H *has a canonical section that is regular, it is effective).*

Definition. *A canonical divisor* H *determines a pseudo-divisor* (bB(H)ß|H|ß=H)*. If* (Pß^ß=) *if* |H| œ ^ *and* b\(H) z P*.*

Lemma. *If* \ *is a variety, any pseudo-divisor* (Pß^ß=) *is represented by a cartier divisor* H *such that*

(1) *If* ^ Á \ *, the operation is unique.*

(2) *If* ^ œ \ *, the operation is unique up to linear equivalence.*

Definition. *If* H *is a pseudo-divisor on an* 8*-dimensional variety with support* |H|*, then the* Weil divisor class *is* [H] − E8□"(|H|)*.*

Definition. *If* E œ (Pß^ß=) *and* F œ (P"ß^"ß=") *are pseudo-divisors, then*

E □ F œ (P Œ P"ß^ □ ^"ß= Œ ="), *and*

□E œ (P□"ß^ß"Î=),

where the tensor product is taken over the trivial bundle, (P Œ P□"ß^ß= Œ =□").

Definition. *Let* 0 À \" Ä \ *with* H œ (Pß^ß=)Î\ *. Then* 0‡H œ (0‡Pß0□"(D)ß0‡=).

Notice

0‡(H □ H") œ 0‡(P Œ P"ß^ □ ^"ß= Œ =") œ (0‡(P Œ P")ß0□"(^ □ ^")ß0‡(= Œ ="))

œ (0‡(P) Œ 0‡(P")ß0□"(^) □ 0□"(^")ß0‡(=) Œ 0‡(=")).

Intersecting with divisors

Definition. *If* **H** *is a pseudo-divisor on a scheme* **** *and* **Z** *is a* **5**-*dimensional sub- variety of* **** , *define* **H**[**Z**] *or* **H** † **Z** *in* **E**5□"(|**H**| ∩ **Z**) *to be*

4‡**H**, *a pseudo-divisor on* **Z** *with support* |**H**| ∩ **Z** .

Then **H** † **Z** *is the Weil divisor class of* 4‡**H** œ [4‡**H**].

Note: If **H** is a Cartier divisor, and **Z** Ȿ̂ |**H**|, then **H** retracts (pullback) to a Cartier divisor by 4‡**H**.

Definition. *Let* α œ ∑ **8**Z [**Z**] *be a* **5**-*cycle. Then the support of* α, |α| *is the union of subvarieties with non-zero coefficients* **8**Z .

Definition. *Each* **H** † [**Z**] *is a class in* **E**5□"(|**H**| ∩ |α|). *We define the intersection class*

H † α œ ∑ **8**Z **H** † [**Z**].

Proposition. If **H** is a pseudo-divisor on **** and α is a **5**-cycle, then

(1) **H**(α □ αʷ) œ **H** † α □ **H** † αʷ,

(2) (**H** □ **H**ʷ) † α œ **H** † α □ **H**ʷš

(3) If **0** À **** Ä ****ʷ is a morphism, then there is

1‡ À **0**□"(|**H**| ∩ |α|) Ä |**H**| ∩ **0**(|α|)

with 1‡(**0**‡**H** † α) œ **H** † **0**‡(α).

(4) Same as above but other direction.

(5) If **H** is a pseudo-divisor with b\\ (**H**) is trivial, then **H** † α œ !

Lecture 4 (February 10, 2009)

Theorem 2.4 Let **H**ß **H**ʷ be Cartier divisors on an **8**-dimensional variety **** . Then

H † [**H**ʷ] œ **H**ʷ † [**H**] in **E**8□#(|**H**| ∩ |**H**ʷ|),

where [**H**ʷ] œ ∑ codim **Z**3œ" **8**3[**Z**3] − ^8□"(****).

Proof. Assume **H** and **H**ʷ are effective Cartier divisors that intersect properly (that is, there is no codimension " subvariety of **** in their intersection, |**H**| ∩ |**H**ʷ|). Recall the following fact:

Let **E** be a local domain of dimension **#**. Take **+ß +ʷ − E**. Define **/E**(**+ß EÎ+ʷE**) as follows:

$$\mathbf{!}\ \mathbf{Ä}\ \text{ker}\ \mathbf{Ä}\ \mathbf{EÎ+^{w}E}\ \mathbf{\dot{Ä}}\ \mathbf{EÎ+^{w}E}\ \mathbf{Ä}\ \text{coker}\ \mathbf{Ä}\ \mathbf{!}$$

Then let **/E**(**+ß EÎ+ʷE**) **œ jE**(coker) □ **jE**(ker) (length of cokernel minus length of kernel).

Next, we will need the lemmas:

Lemma A.27. $e_{\mathbf{E}}$(**+ß EÎ+ʷE**) **œ** $\sum_{\substack{\text{ht}\,©\,\mathbf{œ"}\\ \mathbf{:}\,-\,\text{Spec}\,\mathbf{E}}}$ **jE**(**E©Î+ʷE©**) † **jEÎ©**(**EÎ©** □ **+E**).

Lemma A.28. $e_{\mathbf{E}}$(**+ß EÎ+ʷE**) **œ /**(**+ʷß EÎ+E**).

Note: For the above lemma, we want **H** † [**Hʷ**] **œ** ∑ **7₃**[**A₃**] to be **Hʷ** † [**H**] **œ** ∑ **7₃**[**A₃**].

<u>Case 1</u> Assume **H** and **Hʷ** are effective that intersect properly. Calculate the coefficient of **[** in **H** † [**Hʷ**], with dim **E œ #**. Then all primes © with ht **"** correspond to a subvariety **Z** of codimension **"**. Hence,

$$[\mathbf{H^{w}}]\ \mathbf{œ} \sum_{\substack{\text{ht}\,©\,\mathbf{œ"}\\ \mathbf{:}\,-\,\text{Spec}\,\mathbf{E}}} \mathbf{8_3}[\mathbf{Z_3}]\ □\ \mathbf{þþþ}$$

Here, **8₃ œ** ord$_{\mathbf{Z_3}}$(**+ʷ**). Since b$_{\mathbf{Z_3ß\backslash}}$ **œ E©₃ß** we further have **8₃ œ j**$_{\mathbf{E_{©_3}}}$(**E©₃Î+ʷE©₃**). To continue, we first must compute

$$\mathbf{H}\dagger[\mathbf{H^{w}}]\ \mathbf{œ}\sum_{©_3-\mathbf{E}}\mathbf{8_3H}\dagger[\mathbf{Z_3}],$$

with

$$\mathbf{H}\dagger[\mathbf{Z_3}]\ \mathbf{œ}\sum_{\substack{\mathbf{[_3\,§\,Z_3}\\ \text{codim}\ \mathbf{[_3\,œ\,"}}}\mathbf{7_3}[\mathbf{[_3}].$$

Well, the coefficient of **[** in **H** † [**Z₃**] is

$$\sum\nolimits_{©_3}\mathbf{j_{EÎ©_3}}(\mathbf{EÎ+^{w}E_{©_3}})\dagger\mathbf{j_{EÎ©}}(\mathbf{EÎ}(©\ □\ (\mathbf{+})))\ \mathbf{œ}\ \mathbf{/_E}(\mathbf{+ß\ EÎ+^{w}E})$$

by the lemma.

<u>Case 2</u> Assume **H** and **Hʷ** are effective. We wil reduce to the case of proper intersection.

Lemma. *If* H *and* H" *are Cartier divisors on* \ *and* 1 À Ĩ Ä \ *is a proper birational morphism, with*

$$1^{‡}\mathbf{H}\ œ\ \mathbf{F}\ „\ \mathbf{G}ß\ 1^{‡}\mathbf{H}"\ œ\ \mathbf{F}"\ „\ \mathbf{G}"ß\ |\mathbf{F}| \square |\mathbf{G}|\ §\ 1^{\square"}(|\mathbf{H}|)ß$$
$$|\mathbf{F}"| \square |\mathbf{G}"|\ §\ 1^{\square"}(|\mathbf{H}"|).$$

We have four pairs on Ĩ, (FßF")ß (FßG")ß (GßF")ß *and* (GßG"). *If the theorem holds true for each pair, then it is true in general.*

Definition. *Let* HßH" *be effective divisors on* \. *Define*

$$\&(\mathbf{H}ß\mathbf{H}")\ œ\ \max\{\mathrm{ord}_{Z}\mathbf{H} † \mathrm{ord}_{Z}\mathbf{H}"\ \mathbf{I}\ \mathrm{codim}_{\backslash} Z\ œ\ "\}.$$

Note that HßH" intersect properly if and only if &(HßH") œ !. If &(HßH") □ !, then we blow up \ along |H| ∩ |H"|. We get

$$1\ À\ Ĩ\ Ä\ \backslash$$

with exception divisor Iß so $1^{‡}$H œ G □ I and $1^{‡}$H" œ G" □ I. Then

(a) G ∩ G" œ g

(b) if &(HßH") □ !, then &(GßI) □ &(HßH") and &(G"ßI) □ &(HßH").

<u>Case 3</u> Assume H" is effective. Let 1 be the denominator of the ideal sheaf of H. Then we can have a blow-up

$$1\ À\ Ĩ\ Ä\ \mathbf{F}\mathbf{I}_{1}\backslash\ Ä\ \backslash \text{ with } 1^{‡}\mathbf{H}\ œ\ \mathbf{G} \square \mathbf{I}$$

with I an exceptional divisor. Then we can just consider the pair (Gß$1^{‡}$H") and (Iß$1^{‡}$H") and apply these to case 2.

<u>Case 4</u> Let H and H" be arbitrary divisors. Let 1 be the denominator of the ideal sheaf of H. We can have

$$1\ À\ Ĩ\ Ä\ \mathbf{F}\mathbf{I}_{1}\backslash\ Ä\ \backslash \text{ with } 1^{‡}\mathbf{H}\ œ\ \mathbf{G} \square \mathbf{I}.$$

Consider the pairs (Gß$1^{‡}$H") and (Iß$1^{‡}$H"). We can apply case 3 to this and we're done! □ (of theorem)

Corollary 2.4.1. *Let* H *be a pseudodivisor on* \. *Let* α − $^{\wedge}_{5}$\ *and* α µ !. *Then* H † α œ ! *in* $E_{5\square"}$(|H|)ß $E_{5\square"}$(|H| ∩ |α|).

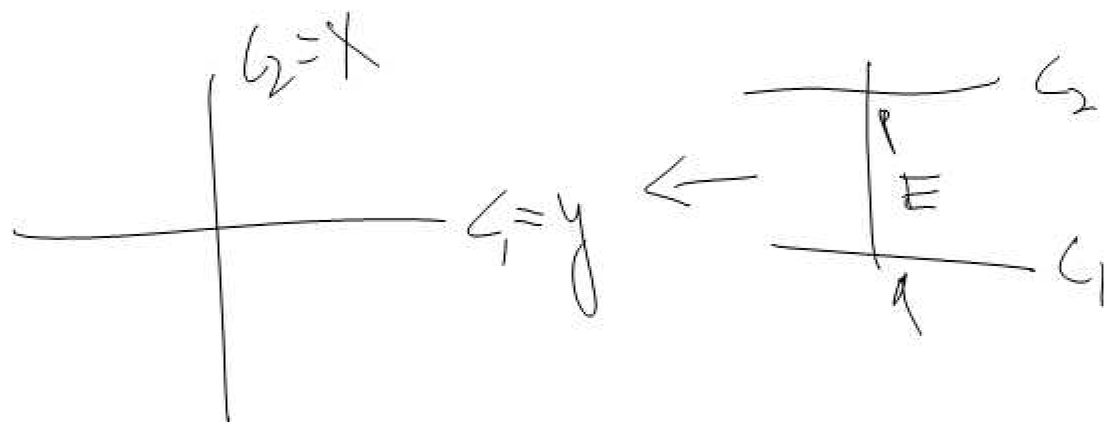
Corollary 2.4.2. *Let* H *and* H" *be pseudo-divisors on* \. *Then*

$$\mathbf{H} † (\mathbf{H}"α)\ œ\ \mathbf{H}"(\mathbf{H} † α)$$

for any α − $^{\wedge}_{5}$(\).

As cycles, H † [H"] Á H" † [H], but as classes, they are equal.

Example 2.4.1. Consider □#. Consider the situation.



First, we have

H œ 1‡G‥ œ G‥ □ I (principal), and

H‴ œ 1‡G# œ G# □ I (principal),

where by principal we mean b ĩ z b ĩ (H) and b ĩ z b ĩ (H‴) respectively. We have

H † [H‴] œ H † ([G#] □ [I]) œ H † [G#] □ H † [I],

where : is just a point (: − ^!(|H| ∩ |G#|)) and E!(|H|)E!(:). Notice H † [G#] − ^!(|H| ∩ |G#|) œ ^!(I:I). W ecan also think of H † [G#] − E!(|G#|) and in that case : œ !, because G# œ □". Thus, H † [G#] œ :. On the other hand, H † [I] œ !. Why? Because | I | § |H| in ^!(I). Hence,

H † [G#] □ H † [I] œ : − ^!(I).

Notice that since I œ □", the other one turns out to be ;. Hence, they are non-equal as cycles, but are as classes. □

Definition. Let H‥ß þþþß H8 be pseudodivisors on \ . For any α − ^5 \ , define

H‥ † þþþ † H8 † α œ H‥ † (H# † þþþ † H8 † α) − E5□8(|H‥| ∩ þþþ ∩ |H8| ∩ |α|).

More generally, for any homogeneous polynomial of degree . ,

T (X‥ß þþþß X8)

with integer coefficients. Then

T (H‥ß þþþß H8) † α − E5□‥(|H‥| □ þþþ □ |H8| ∩ |α|).

If 8 œ 5ß] œ |H¨| ∩ þþþ ∩ |H8| ∩ |α| is complete, define the intersection # to be

(H¨ † þþþ † H8 † α)\ œ̃ ∫D T(H¨ß þþþß H8) † α.

Note. The "integral" means if α œ ∑83[T3] with \ over 5 then

∫\ α œ ∑ 83[5(:) † 5].

If Z is a pure 5-dimensional subscheme of \ , then

T(H¨ß þþþß H8) † Z œ̃ T(H¨ß þþþß H8)[Z].

Chern classes of line bundles

Let P be a line bundle on a scheme \ . We can define a homomorphism

G3(P) ∩ _ À ^5 \ Ä E5□¨ \ with α œ ∑ 83[Z3] È G3(P) ∩ α

as follows. For any 5-dimensional subvariety Z ß P lZ œ bZ (G) where G is a cartier divisor on Vþ Then

G3(P) ∩ [Z] œ [G] − E5□¨(\ ).

Note that if P œ b\ (H) for a pseudo-divisor on \ , then

G3(P) ∩ α œ H † α.

Proposition 2.5 (a) *If* α µ ! *on* \ *, then* G3(P) ∩ α œ !. *So we have*

G3(P) ∩ __ À E5(\ ) Ä E5□¨(\ ).

(b) *If* Pß P" *are line bundles on* \ , α − ^5 \ *, then*

G¨(P) ∩ (G¨(P") ∩ α) œ G¨(P") ∩ (G¨(P) ∩ α).

(c) *If* 0 À \ " Ä \ *is a proper morphism,* P *is a line bundle on* \ , α *is a* 5-*cycle on* \ ", *then*

0‡(G¨(0‡P) ∩ α) œ G¨(P) ∩ 0‡α.

(d) *If* 0 À \ " Ä \ *is flat of relative dimension* 8, *and* P *is a line bundle on* \ *, with* α *a* 5-*cycle on* \ *, then*

G¨(0‡P) ∩ 0‡α œ 0‡(G¨(P) ∩ α).

(e) *If* Pß P" *are line bundles with* α − ^5 \ ß *then*

G¨(P Œ P") œ

Definition. *If* P¨ß þþþß P8 *are line bundles on* \ ß α − E5(\ )*, and we have a homogeneous polynomial* T(X¨ß þþþß X8) *of degree* . *, then*

T (G¨ (P¨)ß þþþß G¨ (P8)) ∩ α − E5□. (\ ).

In particular,

G¨ (P)˙ ∩ − E5□. (\ ).

Definition. *Let* H *be an effective divisor on* \ *and let* 3 À H Ä \ *be the inclusion. Define the Gysin homomorphism*

3‡ À ^5 \ Ä E5□¨ H

α È 3‡(α) œ H † α.

Proposition 2.6. (a) *If* α µ ! *on* \ *, then* 3‡α œ !. *So we have*

3‡ À E5 \ Ä E5□¨ H.

(b) *If* α − *Z*5 \ *, then*

3‡3‡α œ G¨ (bB (H)) ∩ α.

(c) *If* α − ^5 Hß *, then*

3‡3‡α œ G¨ (R) ∩ α,

where R œ 3‡ b\ (H).

(d) *If* \ *is purely* 8*-dimensional, then*

3‡[\ ] œ [H].

(e) *If* P *is a line bundle on* \ *, then*

3‡(G¨ (P) ∩ α) œ G¨ (P) ∩ 3‡(α)

for any α − ^5 \ þ J ?<>2/<ß

C¨ (P) ∩ (G¨ (Pw) ∩ α) œ G¨ (Pw) ∩ (G¨ (P) ∩ α).

Lecture 5 (February 16, 2009)

(1) T *is projective if and only if* T *is proper.*

(2) b I (")ß proj E[B!ß þþþß B/].

T (I) Ä \ , with α − E‡(\ ).

So =8(I) ∩ α ³ __ ∩ T‡α.

(1) 0 À \w Ä \ ß 0‡(=8(0‡ I) ∩ α) œ =8(I) ∩ 0‡α.

(a) (1) =8(I) œ ! if 3 □ !.

(2) =!(I) ∩ α œ α.

\ (b \ (H)) œ deg((" □ H)(" □ "/# O))„.

g\ œ H"\ œ b\ (□O\ )" œ b\ (□O\ ) œ ...

\ (I) œ deg(" □ H □ "/# H#)(" □ "/# O □ "/"# (O# □ G#)) œ

deg(□ "/# HO □ "/# H# □ "/"# (O# □ G#)) œ deg("/# H(H □ O)) □

"/"# (O# □ G#).