

Algebraic Geometry (Intersection Theory) Seminar

Lecture 1 (January 20, 2009)

We will discuss cycles, rational equivalence, and Theorem 1.4 in Fulton, as well as push-forward of rational equivalence. Notationwise, we will let "scheme" mean an algebraic scheme over a field, "variety" will mean a reduced and irreducible scheme, and a "point" will mean a closed point. Let X be a scheme and $Z \subset X$ a subvariety. Then $\mathcal{O}_{Z, \mathfrak{p}} \subset \mathcal{O}_{X, \mathfrak{p}}$ where \mathfrak{p} is a generic point of Z . For a variety X , let $k(X)$ be the field of rational functions.

Let X be a variety with $Z \subset X$ a subvariety of codimension one. Then $\mathcal{O}_{Z, \mathfrak{p}}$ has dimension n , and $k(\mathcal{O}_{Z, \mathfrak{p}}) \subset k(X)$. For any $\alpha \in k(X) \setminus k(\mathcal{O}_{Z, \mathfrak{p}}) \subset k(X)$ with $\alpha \notin \mathcal{O}_{Z, \mathfrak{p}}$, for $\mathfrak{p} \in Z$. For any $\alpha \in k(X) \setminus k(\mathcal{O}_{Z, \mathfrak{p}})$, $\text{ord}_Z(\alpha) = -\text{ord}_Z(\alpha^{-1})$.

Definition. Let X be a variety and $Z \subset X$ a subvariety with codimension one. Then there is a well-defined homomorphism $\text{ord}_Z : k(X)^\times \rightarrow \mathbb{Z}$ such that (for $\alpha \in k(X) \setminus k(\mathcal{O}_{Z, \mathfrak{p}}) \subset k(X)$, $\alpha \notin \mathcal{O}_{Z, \mathfrak{p}}$, with $\mathfrak{p} \in Z$):

- (i) $\text{ord}_Z(\alpha \beta) = \text{ord}_Z(\alpha) + \text{ord}_Z(\beta)$,
- (ii) $\text{ord}_Z(\alpha) = \text{ord}_Z(\alpha \cdot \mathcal{O}_{Z, \mathfrak{p}})$, and
- (iii) $\text{ord}_Z(\alpha) = \text{ord}_Z(\alpha \cdot \mathcal{O}_{Z, \mathfrak{p}})$.

Example. If X is a variety which is regular in codimension one, this means for any $Z \subset X$ a subvariety with codimension one, $\mathcal{O}_{Z, \mathfrak{p}}$ is a discrete valuation ring.

Cycles and rational equivalence

Definition. Let X be a scheme. A 5-cycle on X is a finite formal sum

$$\sum \alpha_i [Z_i] \quad \alpha_i \in \mathbb{Z}$$

where $Z_i \subset X$ are 5-dimensional subvarieties of X . Let

$$\mathcal{Z}_5(X) \subset \left\{ \sum \alpha_i [Z_i] \mid Z_i \text{ are 5-dimensional subvarieties} \right\},$$

$$\mathcal{Z}_5(X) \subset \bigoplus_{\mathbb{Z}} \mathcal{Z}_5(X).$$

Any element $\alpha \in \mathcal{Z}_5(X) \subset \bigoplus_{\mathbb{Z}} \mathcal{Z}_5(X)$ is called a cycle.

For X a scheme, let $\{X_i\}$ be irreducible components of X . For each X_i , we have geometric multiplicity $m_i = j(\mathcal{O}_{X_i})$. Notice X_i has a generic point ζ_i (\mathcal{O}_{ζ_i} is an Artinian ring).

Notation. We define $[X] \in \sum_{i \geq 0} \mathbb{Z}[X_i] = \hat{\mathbb{Z}}[X]$ (this is a cycle).

We want to give an equivalence relation on cycles. Let X be a scheme and $Z \subset X$ a $(5 - \dim Z)$ -dimensional subvariety. For $\langle Z \rangle \in \mathbb{Z}[X]$, we define

$$[\text{div}(\langle Z \rangle)] \in \sum_{\substack{Z \subset X \\ \dim Z = 5 - \text{codim}_X Z}} \text{ord}_Z(\langle Z \rangle) [Z].$$

Notice a 5-cycle $\langle Z \rangle$ is rationally equivalent to 0, written $\mu \langle Z \rangle$, if there are a finite number of $(5 - \dim Z)$ -dimensional subvarieties $Z_i \subset X$ and $\langle Z_i \rangle \in \mathbb{Z}[X]$ such that $\langle Z \rangle \in \sum [\text{div}(\langle Z_i \rangle)]$.

All the cycles equivalent to 0 are written $\text{Rat}_5 X$. We can now define the cycle class,

$$E_5 X \in \hat{\mathbb{Z}}[\text{Rat}_5 X], \text{ with } E_{\dagger} X \in \bigoplus_5 E_5 X \in \bigoplus_5 \hat{\mathbb{Z}}[\text{Rat}_5 X].$$

Examples. (1) $E_5(X) \cong E_5(X_{\text{red}})$, since X and X_{reduced} have the same subvarieties.

(2) If $X \cong \coprod_{i=1}^n X_i$, then

$$\hat{\mathbb{Z}}[X] \cong \bigoplus_{i=1}^n \hat{\mathbb{Z}}[X_i] \text{ and } E_{\dagger} X \cong \bigoplus_{i=1}^n E_{\dagger} X_i.$$

(3) If $\dim X = 8$, then $E_8 X \cong \hat{\mathbb{Z}}[X]$, then there are no "5"-dimensional subvarieties, so there is nothing to mod out by.

(4) If X' and X'' are subschemes of X , then

$$E_5(X' \cup X'') \rightarrow E_5(X') \oplus E_5(X'') \rightarrow E_5(X' \cap X'') \rightarrow 0$$

is an exact sequence.

(5) If Z is an irreducible component of X , then for any cycle class $\alpha \in H^*(X, \mathbb{Z})$, we define the coefficient of Z in α to be the coefficient of $[Z]$ in any cycle which represents α (since $\alpha \in \sum \mathbb{Z}[Z_3]$).

(6) If $X \cong \text{spec } \mathbb{O}$, then $H^*(X, \mathbb{Z}) \cong \mathbb{Z}$ (just one point).

(7) If $X \cong \mathbb{P}^5$, then $H^*(X, \mathbb{Z}) \cong \mathbb{Z}^6$ and $H^*(X, \mathbb{Z}) \cong \text{Pic } \mathbb{P}^5 \cong \mathbb{Z}$.

(8) Let Z be a 5-dimensional variety. Assume we have $\pi: X \rightarrow Y$ a dominant function $(\pi(Z) = Y)$, or other way to think about it is that π maps a generic point of Z to a generic point of Y , or even $\pi = \mathbf{V}(Z)^\dagger$. Let's look at the fibers $\pi^{-1}(y)$ and $\pi^{-1}(y')$ (i.e., if $\pi = (B_1 \rightarrow B_2)$, we have $\pi^{-1}(y) = (A \rightarrow y)$ and $\pi^{-1}(y') = (A \rightarrow y')$). Then $\pi^{-1}(y)$ and $\pi^{-1}(y')$ are purely 5-dimensional subschemes, so $[\pi^{-1}(y) - \pi^{-1}(y')] \in \text{div}(\pi)$.

Push-forward of cycles

Let $\pi: X \rightarrow Y$ be a proper morphism of two schemes. Then for a unique $\pi_*: H^*(X, \mathbb{Z}) \rightarrow H^*(Y, \mathbb{Z})$, we want to understand $\pi_*: H^*(X, \mathbb{Z}) \rightarrow H^*(Y, \mathbb{Z})$. Assume α is a 5-cycle with $\alpha \in \sum \mathbb{Z}[Z_3]$ with Z_3 a 5-dimensional subvariety. We want $\pi_*[\alpha] \in H^*(Y, \mathbb{Z})$ to have some (covariant) functorial property relating π and π' . If $\pi: X \rightarrow Y$ and $\pi': X' \rightarrow Y'$ ($\pi = \mathbf{V}(Z)$ ($\mathbf{V}(Z)$ is a subvariety of Y)). We want

(1) $\dim \pi_*[\alpha] = \dim Z$ means $\pi_*[\alpha] \in H^{\dim Z}(Y, \mathbb{Z})$.

(2) $\dim \pi_*[\alpha] = \dim Z$ means $\mathbf{V}(\pi_*[\alpha]) \subset \mathbf{V}(\pi)$ has degree $[\mathbf{V}(\pi) : \mathbf{V}(\pi_*[\alpha])]$.

We then call $\pi_*[\alpha] \in H^{\dim Z}(Y, \mathbb{Z})$ (so π_* is well-defined).

Theorem. [1.4] *If $\pi: X \rightarrow Y$ is a proper morphism and α is a 5-cycle on X which is rationally equivalent to 0, then $\pi_*[\alpha]$ is rationally equivalent to zero on Y .*

Proof. We write $\alpha \in \text{div}(\pi)$ for $\pi = \mathbf{V}(Z)$ for some 5-dimensional subvariety of X . That is,

$$\alpha \in \text{div}(\pi) \iff \alpha \in \sum_{Z_3 \subset Z} \mathbb{Z}[Z_3].$$

$$\text{codim}_Z Z_3 \geq 1$$

For this cycle, we can assume $\mathbb{A}^1 \times \mathbb{A}^1 \xrightarrow{\alpha} \mathbb{A}^1$ and $\mathbb{A}^1 \xrightarrow{\beta} \mathbb{A}^1$. We can assume \mathbb{A}^1 is a variety and \mathbb{A}^1 is a variety and $\mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ is a proper surjection. Now we use the below proposition.

Proposition. [1.4] *Let $\mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ be a proper, surjective morphism and let $\alpha \in \mathbb{A}^1$. Then*

$$(a) \mathbb{A}^1[\text{div}(\alpha)] \cong \mathbb{A}^1 \text{ if } \dim \mathbb{A}^1 = \dim \mathbb{A}^1.$$

$$(b) \mathbb{A}^1[\text{div}(\alpha)] \cong \mathbb{A}^1[\text{div}(\mathbf{R}(\alpha))] \text{ if } \dim \mathbb{A}^1 = \dim \mathbb{A}^1,$$

where $\mathbf{R}(\alpha)$ is determinant of $\mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ as an \mathbb{A}^1 -linear morphism ($\mathbf{R}(\mathbb{A}^1 \times \mathbb{A}^1) = \mathbb{A}^1$).

Proof. *Case 1.* Let $\mathbb{A}^1 \xrightarrow{\alpha} \mathbb{A}^1$ for \mathbb{A}^1 a field. Let $\mathbb{A}^1 \xrightarrow{\beta} \mathbb{A}^1$ with $\mathbb{A}^1 \xrightarrow{\beta} \mathbb{A}^1$ \mathbb{A}^1 \mathbb{A}^1 . Since $\mathbb{A}^1 \xrightarrow{\beta} \mathbb{A}^1$ \mathbb{A}^1 \mathbb{A}^1 , and $\alpha \in \mathbb{A}^1$ ($\alpha \in \mathbb{A}^1$, with $\beta, \gamma \in \mathbb{A}^1$), we can assume $\alpha \in \mathbb{A}^1$ is an irreducible polynomial (since $[\text{div}(\alpha)] \cong [\text{div}(\beta)] + [\text{div}(\gamma)]$, so if we prove it for $[\text{div}(\beta)]$ and $[\text{div}(\gamma)]$ (and we know β and γ are polynomials), then we have shown it for $[\text{div}(\alpha)]$). Furthermore, let $\mathfrak{p} \in \mathbb{A}^1$ be a maximal prime ideal ($\mathfrak{p} \in \mathbb{A}^1$ is a closed point). In this case, $\text{ord}_{\mathfrak{p}}(\alpha) \in \mathbb{A}^1$ (since $\mathbb{A}^1 \xrightarrow{\beta} \mathbb{A}^1$), and $\text{ord}_{\mathfrak{p}}(\alpha) \in \mathbb{A}^1$ for $\mathbb{A}^1 \xrightarrow{\beta} \mathbb{A}^1$ with $\mathbb{A}^1 \xrightarrow{\beta} \mathbb{A}^1$.

Finally, consider the point $\mathfrak{p} \in \mathbb{A}^1$. Then $\mathfrak{p} \in \mathbb{A}^1$ \mathbb{A}^1 \mathbb{A}^1 (it corresponds to the prime ideal generated by \mathfrak{p} in the ring \mathbb{A}^1). The way to visualize this is that \mathbb{A}^1 is for everything but \mathfrak{p} , and \mathbb{A}^1 is for everything but \mathfrak{p} . Call degree $\alpha \in \mathbb{A}^1$. Then $\alpha \in \mathbb{A}^1$ \mathbb{A}^1 , some poly. For example, $\alpha \in \mathbb{A}^1$ \mathbb{A}^1 \mathbb{A}^1 . Then on \mathbb{A}^1 , we have

$$[\text{div}(\alpha)] \cong [\text{div}(\beta)] + [\text{div}(\gamma)] \text{ and } \mathbb{A}^1[\text{div}(\alpha)] \cong \mathbb{A}^1[\text{div}(\beta)] + \mathbb{A}^1[\text{div}(\gamma)] \cong \mathbb{A}^1,$$

with $\mathbb{A}^1 \xrightarrow{\beta} \mathbb{A}^1$.

Case 2. If $\mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ is finite, set with $\mathbb{A}^1 \xrightarrow{\beta} \mathbb{A}^1$ and $\mathbb{A}^1 \xrightarrow{\gamma} \mathbb{A}^1$ and $\mathbb{A}^1 \xrightarrow{\alpha} \mathbb{A}^1$. Also, assume all varieties in $[\text{div}(\alpha)]$ map to a variety $[\mathbb{A}^1]$ of codimension \mathbb{A}^1 . Let $\mathbb{A}^1 \xrightarrow{\beta} \mathbb{A}^1$ (the generic point of this image) and $\mathbb{A}^1 \xrightarrow{\gamma} \mathbb{A}^1$. Consider the commutative diagram

$$\begin{array}{ccc} \text{Spec } P \xrightarrow{\mathbb{A}^1} \text{Spec } E \xrightarrow{\mathbb{A}^1} \text{Spec } O \\ \downarrow \mathbb{A}^1 \quad \downarrow \mathbb{A}^1 \quad \downarrow \mathbb{A}^1 \\ \text{Spec } O \xrightarrow{\mathbb{A}^1} \text{Spec } E \xrightarrow{\mathbb{A}^1} \text{Spec } O \end{array}$$

with $E \cong F \cong P$ and $\dim F \cong \dim E \cong n$. Finally, F has finitely many maximal ideals \mathfrak{m}_3 such that $\mathfrak{m}_3 \cap E \cong \mathfrak{m}$. There is a one-to-one correspondence between $\{Z_3 \subset \mathbb{A}^1 \mid \mathfrak{m}_3 \cap E \cong \mathfrak{m}\}$ and $\{\mathfrak{m}_3 \in F \mid \mathfrak{m}_3 \cap E \cong \mathfrak{m}\}$. Also, from the diagram, $F \cong E \cong O$ and $F_{\mathfrak{m}_3} \cong E_{\mathfrak{m}_3} \cong O_{\mathfrak{m}_3}$.

Now assume $\langle \cdot \rangle = F$. Then computationally,

$$\begin{aligned} \text{div}(\langle \cdot \rangle) &\cong \sum_{Z_3} \text{ord}_{Z_3}(\langle \cdot \rangle) \cdot [Z_3], \text{ and} \\ \mathbb{0}_{\dagger}[\text{div}(\langle \cdot \rangle)] &\cong \sum_{Z_3} \text{ord}_{Z_3}(\langle \cdot \rangle) \cdot [\mathbb{V}(Z_3) \wedge \mathbb{V}(\langle \cdot \rangle)] \cdot [\langle \cdot \rangle] \\ &\cong \sum_{Z_3} j(F_{\mathfrak{m}_3} \hat{\mathbb{I}} \langle \cdot \rangle) \cdot [\mathbb{V}(Z_3) \wedge \mathbb{V}(\langle \cdot \rangle)] \cdot [\langle \cdot \rangle]. \end{aligned}$$

Now we use two lemmas from the back of the book (A.2.3 and A.2.2), that state if $(\mathbb{7} \mathbb{B} E) \xrightarrow{\mathbb{A}^1} (F_{\mathfrak{m}_3} \mathbb{B} \mathfrak{m}_3)$ is a push-forward,

$$\begin{aligned} j_E(F_{\mathfrak{m}_3} \hat{\mathbb{I}} \langle \cdot \rangle) &\cong [F_{\mathfrak{m}_3} \hat{\mathbb{I}} \mathfrak{m}_3 \wedge E \hat{\mathbb{I}} \mathfrak{m}_3] \cdot j(F_{\mathfrak{m}_3} \hat{\mathbb{I}} \langle \cdot \rangle) \\ &\cong \sum_{Z_3} \text{ord}_E(\mathbb{R}(\langle \cdot \rangle)) \cdot [\langle \cdot \rangle] \cong \dagger \text{ord}_E(\mathbb{R}(\langle \cdot \rangle)) \cdot [\langle \cdot \rangle]. \end{aligned}$$

Alternate Definition of Rational Equivalence

Let $Z \cong \mathbb{A}^n \setminus \{0\}$ with Z a 5 -dimensional variety and $\mathbb{A}^1 \setminus \{0\} \cong \mathbb{A}^1 \setminus \{0\}$. Then the fibers $\mathbb{0} \cong \mathbb{A}^1 \setminus \{0\}$ and $\mathbb{0} \cong \mathbb{A}^1 \setminus \{0\}$ are purely 5 -dimensional subvarieties of Z . Then if $\mathbb{0} \cong \mathbb{A}^1 \setminus \{0\}$, we can define

$$\begin{aligned} \text{div}(\mathbb{0}) &\cong [\mathbb{0} \cong \mathbb{A}^1 \setminus \{0\}] \cdot [\mathbb{0} \cong \mathbb{A}^1 \setminus \{0\}] \text{ and} \\ \dagger \text{div}(\mathbb{0}) &\cong \dagger [\mathbb{0} \cong \mathbb{A}^1 \setminus \{0\}] \cdot \dagger [\mathbb{0} \cong \mathbb{A}^1 \setminus \{0\}] \end{aligned}$$

where $\dagger [\mathbb{0} \cong \mathbb{A}^1 \setminus \{0\}] \cong Z \cong \mathbb{A}^1 \setminus \{0\}$ and $\dagger [\mathbb{0} \cong \mathbb{A}^1 \setminus \{0\}] \cong Z \cong \mathbb{A}^1 \setminus \{0\}$.

Proposition 1.6. *A cycle in $\mathbb{A}^5 \setminus \{0\}$ is rationally equivalent to zero if and only if there are $(5$ -dimensional)-dimensional subvarieties $Z_3 \subset \mathbb{A}^1 \setminus \{0\}$ such that projections from Z_3 to $\mathbb{A}^1 \setminus \{0\}$ are dominant with*

$$\cong \sum_{Z_3 \subset \mathbb{A}^1 \setminus \{0\}} [Z_3 \cong \mathbb{A}^1 \setminus \{0\}] \cdot [Z_3 \cong \mathbb{A}^1 \setminus \{0\}].$$

Proof. We have $\mathcal{O}_{\mathbb{A}^1}(\text{div}(\langle \cdot \rangle)) \cong \mathcal{O}_{\mathbb{A}^1}(-V(\langle \cdot \rangle))$, \mathbb{A}^1 a 1-dimensional subvariety of \mathbb{A}^1 . Then consider \mathbb{A}^1 and we have $\mathbb{Z} \cong \mathbb{Z}$, \mathbb{A}^1 and $\text{div}[\langle \cdot \rangle] \cong \text{div}(\mathbf{0}) \cong [\mathbb{Z}(\cdot)] \cong [\mathbb{Z}(\cdot)]$ (where $\mathbf{0} \in \mathbb{Z} \cong \mathbb{A}^1$).

Lecture 2 (January 27, 2009)

Recall from last time that for \mathbb{A}^1 a scheme, $\sum \mathbf{8}_3[\mathbb{Z}_3]$ is a 5-cycle, and \mathbb{A}^1 is

$$\mathbb{A}^1 \cong \left\{ \sum \mathbf{8}_3[\mathbb{Z}_3] \mid \mathbb{Z}_3 \text{ are 5-dimensional subvarieties} \right\}.$$

Furthermore, $\mathbb{A}^1 \cong \bigoplus_5 \mathbb{A}^1$. For $\mathbb{A}^1 \cong \mathbb{A}^1 \times \mathbb{A}^1$ -dimensional, $\langle \cdot \rangle = V(\langle \cdot \rangle)$,

$$[\text{div}(\langle \cdot \rangle)] \cong \sum_{\text{codim } \mathbb{Z} \cong \mathbb{A}^1} \text{ord}_{\mathbb{Z}(\langle \cdot \rangle)}[\mathbb{Z}].$$

Furthermore, recall that $\text{Rat}_V \cong \{ \text{Eldiv}(\langle \cdot \rangle) \} \cong \mu^*$. We defined

$$\mathbb{E}_5 \cong \mathbb{A}^1 \hat{\otimes} \text{Rat}_5$$

with

$$\mathbb{E}_5 \cong \bigoplus_5 \mathbb{E}_5.$$

Theorem 1.4. For $\mathbf{0} \in \mathbb{A}^1 \cong \mathbb{A}^1$ a proper morphism, we define

$$\mathbf{0} \in \mathbb{A}^1 \cong \mathbb{A}^1$$

by $\mathbf{0}_T[\mathbb{Z}] \cong \text{deg}(\mathbb{Z} \hat{\otimes} [\cdot])[\cdot]$ ($[\cdot] \cong \mathbf{0}(\mathbb{Z})$). Then for μ^* on \mathbb{A}^1 , $\mathbf{0}_T \mu^*$ on \mathbb{A}^1 .

We will motivate this theorem by seeing its application to Bezout's Theorem on plane curves.

Bezout's Theorem on plane curves

For \mathbb{A}^2 plane curves in \mathbb{A}^2 with $\text{deg}(\mathbb{J}) \cong 7$ $\text{deg}(\mathbb{K}) \cong 8$, if \mathbb{J} \mathbb{K} intersect with no common roots, then

$$\sum \mathbf{3}(\mathbb{T} \mathbb{J} \dagger \mathbb{K}) \cong 7 \dagger 8,$$

the intersection multiplicity of \mathbb{J} and \mathbb{K} at \mathbb{T} .

Assume \mathbb{J} is irreducible. Then for all $\mathbb{K}^{\#}$ plane curves, $\frac{\mathbb{K}}{\mathbb{K}^{\#}} = V(\mathbb{J})$, so

$$\sum \mathbf{3}(\mathbb{T} \mathbb{J} \dagger \mathbb{K}) = \sum \mathbf{3}(\mathbb{T} \mathbb{J} \dagger \mathbb{K}^{\#}) \cong \sum \text{ord}_{\mathbb{T}} \left(\frac{\mathbb{K}}{\mathbb{K}^{\#}} \right).$$

Then for all $J^\#, \frac{J}{J^\#} = V(K^\#)$,

$$\sum 3(TBJ \dagger K^\#) = \sum 3(TBJ^\# \dagger K^\#) \cong \sum \text{ord}_T \left(\frac{J}{J^\#} \right).$$

Hence,

$$\sum 3(TBJ \dagger K) \cong \sum 3(TBJ^\# \dagger K^\#)$$

Given J a plane curve, $< = V(J)$,

$$[\text{div}(<)] \cong \sum \text{ord}_T(<)[T].$$

We want to show $\sum \text{ord}_T(<) \cong !$.

Take $1 \rightarrow J \rightarrow \mathbb{A}^2$. Then

$$1_{\dagger}[\text{div}(<)] \cong \sum \text{ord}_T(<) \cdot [1T] \text{ in } \mathbb{A}^2,$$

(with \cdot the degree of J) and notice $\text{ord}_T(<) \cong [\text{div}(\cdot)](\cdot; -V(\mathbb{A}^2))$.

Alternate Definition of Rational Equivalence

Proposition 1.6. If $\mu \in \mathbb{A}^5 \setminus$ if and only if

$$\cong \sum ([Z_3(!)] - [Z_3(\quad)])$$

$Z_3 \circledast \setminus$, \mathbb{A}^2 with $0_3 \rightarrow Z_3 \rightarrow \mathbb{A}^2$ dominant with $T \rightarrow \setminus$, $\mathbb{A}^2 \setminus$, satisfying $Z_3(!) \cong T \cdot J \cdot (!)$ and $Z_3(\quad) \cong T \cdot J \cdot (\quad)$.

Theorem 1.7. If $0 \rightarrow \setminus \mathbb{A}^2 \rightarrow \mathbb{A}^5$ is a flat morphism (of relative dimension 8), define

$$0^\dagger \rightarrow \mathbb{A}^5 \rightarrow \mathbb{A}^8 \setminus$$

by $0^\dagger[Z] \cong [0 \rightarrow Z] \rightarrow \mathbb{A}^8$ (Z a variety). Then for $\mu \in \mathbb{A}^5$, $0^\dagger \mu \in \mathbb{A}^8 \setminus$.

Note. Ramin says: "Keep in mind, for flat morphisms, it essentially means that the dimensions of fibers are constant. This is what it means to be flat, it makes these morphisms nice in the aforementioned sense." [not really a quote, just paraphrase] Specifically, for $0 \rightarrow \mathbb{C} \rightarrow \mathbb{A}^2$ a fiber of a flat morphism $0 \rightarrow \setminus \mathbb{A}^2 \rightarrow \mathbb{A}^5$, the dimension is given by $\dim \setminus - \dim \mathbb{A}^2$.

Lemma 1.7.1. For all subschemes $J^\# \circledast J \rightarrow 0^\dagger[J^\#] \cong [0 \rightarrow (J^\#)]$.

Proposition 1.7. Consider the fiber square with 1 flat and 0 proper:

$$\begin{array}{ccc} \setminus & \mathbb{A}^2 & \setminus \\ 0^\# \rightarrow & \mathbb{A}^5 & \rightarrow 0 \end{array}$$

$$\mathbb{A}^1 \times \mathbb{A}^1$$

$\mathbb{A}^1 \times \mathbb{A}^1$ is flat, \mathbb{A}^1 is proper, and for all $\mathcal{O}_Z \rightarrow \mathcal{O}_Z \otimes \mathcal{O}_Z \rightarrow \mathcal{O}_Z$ in $\mathbb{A}^1 \times \mathbb{A}^1$.

Note. Ramin says "What is a fiber square? Think about a product structure. What is a product? The product of two sets \mathbb{A}^1 and \mathbb{A}^1 is the Cartesian pairs (B, C) such that you have two projections onto B and onto C . The above is the situation when B and C don't have any maps onto any other thing. Now, if you did have a third map, then you'd want the collection of pairs (B, C) which actually map into the same thing in D . Set theoretically, for a diagram

$$\begin{array}{ccc} \mathbb{A}^1 & \times & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & & \mathbb{A}^1 \end{array}$$

the fiber product is $\mathbb{A}^1 \times_{\mathbb{A}^1} \mathbb{A}^1$ given by $\mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{A}^1$. Suppose you have $\text{Spec } V \rightarrow \text{Spec } W$ and $\text{Spec } X \rightarrow \text{Spec } W$ and you want to construct something $\mathbb{A}^1 \times \text{Spec } V$ and $\mathbb{A}^1 \times \text{Spec } X$ (diagram). First thing you do is reverse the arrows so you get $W \rightarrow V$ and $W \rightarrow X$, and then the ring you construct is $V \otimes_W X$ and $X \otimes_W V$. Then you consider the spectrum of these rings and that is the fiber product. Problem is, scheme-theoretically you have to do this locally, so you have to make sure the data glues together." \square

Proof. For \mathbb{A}^1 and \mathbb{A}^1 varieties with $\mathcal{O}_Z \rightarrow \mathcal{O}_Z$, assume surjective. Then

$$\mathcal{O}(\mathbb{A}^1) \otimes \mathcal{O}(\mathbb{A}^1) \cong \mathcal{O}(\mathbb{A}^1).$$

We want $\mathcal{O}_Z[\mathbb{A}^1] \cong \mathcal{O}_Z[\mathbb{A}^1]$. Take $\mathbb{A}^1 \cong \text{Spec}(\mathbb{C})$ and $\mathbb{A}^1 \cong \text{Spec}(\mathbb{C})$ with $\mathbb{C} \cong \mathbb{C}$ fields, and let $\mathbb{A}^1 \cong \text{Spec}(\mathbb{C})$ with \mathbb{C} local Artinian, and $\mathbb{A}^1 \cong \text{Spec}(\mathbb{C})$ with $\mathbb{C} \cong \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$. Then we are done by Lemma A.1.3. \square

We are now ready to prove the main theorem.

Theorem 1.7. *If $\mathbb{A}^1 \times \mathbb{A}^1$ is a flat morphism (of relative dimension 8), define*

$$\mathcal{O}_Z \otimes \mathcal{O}_Z \rightarrow \mathcal{O}_Z \otimes \mathcal{O}_Z$$

by $\mathcal{O}_Z[\mathbb{A}^1] \cong \mathcal{O}_Z[\mathbb{A}^1]$ (Z a variety). Then for $\mu^! \in \mathbb{A}^1$, $\mathcal{O}_Z \otimes \mu^!$ in $\mathbb{A}^1 \times \mathbb{A}^1$

Proof. Let $\mu^! \in \mathbb{A}^1$. Then $\mathcal{O}_Z \otimes \mu^! \cong \mathcal{O}_Z(\mu^!)$ with

$$\mathcal{O}_Z(\mu^!) \cong \mathcal{O}_Z(\mu^!), \text{ for}$$

Let $Z \subset \mathbb{A}^n$ and $Z' \subset \mathbb{A}^n$ with $(0, \dots, 0) \in Z \setminus Z'$. Take $\mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ a subscheme. Then $\mathbb{A}^1 \times Z \rightarrow \mathbb{A}^1$ and $\mathbb{A}^1 \times Z' \rightarrow \mathbb{A}^1$ so that we can construct a fiber square

$$\begin{array}{ccc} \mathbb{A}^1 \times Z & \xrightarrow{\quad} & \mathbb{A}^1 \times Z' \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \xrightarrow{\quad} & \mathbb{A}^1 \end{array}$$

So

$$H^0(\mathbb{A}^1 \times Z, \mathcal{O}(\mathbb{A}^1 \times Z)) \cong H^0(\mathbb{A}^1 \times Z', \mathcal{O}(\mathbb{A}^1 \times Z'))$$

and then by the previous proposition, this equals

$$H^0(\mathbb{A}^1, \mathcal{O}(\mathbb{A}^1)) \otimes H^0(Z, \mathcal{O}(Z))$$

$$\cong H^0(\mathbb{A}^1, \mathcal{O}(\mathbb{A}^1)) \otimes H^0(Z', \mathcal{O}(Z'))$$

and so by Theorem 1.4 it suffices to show $H^1(\mathbb{A}^1 \times Z, \mathcal{O}(\mathbb{A}^1 \times Z)) = 0$ for $Z \subset \mathbb{A}^n$. Need $H^1(\mathbb{A}^1 \times Z, \mathcal{O}(\mathbb{A}^1 \times Z)) = 0$ with $H^1(\mathbb{A}^1 \times Z, \mathcal{O}(\mathbb{A}^1 \times Z)) = 0$ for $T \subset \mathbb{A}^1$.

1.8 An Exact Sequence

See Proposition 1.8 and Example 1.9.3.

Lecture 3 (February 3, 2009)

Recall that if V is a variety and $Z \subset V$ is of codimension r , then we defined $\mathfrak{a}_Z = \ker(\mathcal{O}_Z \rightarrow \mathcal{O}_Z)$ (regular functions in the local ring), then $\mathfrak{a}_Z \cong \mathfrak{a}_Z$.

Furthermore, recall that a \mathbb{Z} -cycle is a finite formal sum $\sum_{Z \subset V} n_Z [Z]$ with $n_Z \in \mathbb{Z}$.

Divisors

Definition. Let V be a variety (n -dimensional). Then a **Weil divisor** is an $(n - r)$ -cycle on V . Furthermore, $\text{Div}(V) \cong \text{Div}(V)$ is an abelian group of Weil divisors.

Definition. A **Cartier divisor** on V is defined by $\{(Y, \mathcal{O}_Y)\} \in \mathcal{H}$, where $Y \subset V$ is open, and $\mathcal{O}_Y \cong \mathcal{O}_Y$, and \mathcal{O}_Y are non-zero functions in $\mathcal{O}_Y \cong \mathcal{O}_Y$ that satisfy (1) $\mathcal{O}_Y \cong \mathcal{O}_Y$ on Y , (2) $\mathcal{O}_Y \cong \mathcal{O}_Y$ is a unit on Y (a unit is nowhere-vanishing, regular).

Definition. The \mathcal{O}_Y 's are called the local equation of \mathcal{H} .

Definition. If H is a Cartier divisor of X , and $Z \subset X$ is a subvariety of codimension ≥ 2 then we can define an order function on the divisor,

$$\text{ord}_Z(H) = \text{ord}_Z(f)$$

This is well-defined (independent of our choice of local equations) because they differ by a unit: $\text{ord}(f) = \text{ord}(uf)$, where u is a unit in $\mathcal{O}_{X, Z}$.

Definition. The associated Weil divisor to H is $[H] = \sum_{Z \subset X, \dim Z = n-1} \text{ord}_Z(H) [Z]$.

Note. There are only finitely many $Z \subset X$ of order $\text{ord}_Z(H) \neq 0$.

Definition. Let $\text{Div}(X)$ be group of Cartier divisors. Let $H \in \text{Div}(X)$. Then we define $|H| = \{D \in \text{Div}(X) \mid D \sim H\}$.

We also induce a homomorphism $\text{Div}(X) \rightarrow \text{Pic}(X)$ with $H \mapsto [H]$.

Definition. For any $\mathcal{O}_X(D)$, we define a Principal Cartier Divisors $\text{div}(f)$ by all local equations $f = 0$.

Definition. Two Cartier divisors H and H' are linearly equivalent if there exists an $\mathcal{O}_X(D)$ such that $H' = H + \text{div}(f)$.

Definition. $\text{Pic}(X) = \text{Div}(X) / \sim$, where \sim is the above linear equivalence.

Hence, $H' = H + \text{div}(f) \iff [H'] = [H] + [D]$ as (D) -cycle. This induces $\text{Pic}(X) \cong \text{Div}(X) / \sim$.

Definition. The support of a Cartier divisor $H \in \text{Div}(X)$ denoted $\text{supp}(H)$, or $|H|$ is the union of all subvarieties $Z \subset X$ such that if $H \in \text{Div}(X)$ then $\mathcal{O}_X(D)$ is not a unit of $\mathcal{O}_{X, Z}$. Or $\mathcal{O}_X(D)$ somewhere on some $Z \subset X$ (that is, $\mathcal{O}_X(D)$ has a zero or a pole on Z).

Example. If $X = \mathbb{P}^2$ is defined by $D^2 = BC$, then $B \in D \in \mathbb{P}^2$ is a Weil divisor, but not a Cartier divisor. This is because there is no way to have local equations about the cycles define $B \in D \in \mathbb{P}^2$ exclusively. If $D \in \mathbb{P}^2 \iff B \in D \in \mathbb{P}^2$ or $C \in D \in \mathbb{P}^2$. No matter what local equation you choose, you will always "get another line."

§2.1 - Line bundles as pseudo-divisors

Definition. A pseudo-divisor is a triple (P, S) where P is line bundle, $S \subset X$ is closed subset of X ("support"), and s is nowhere vanishing section on $X \setminus S$.

We say $(\mathcal{P} \otimes \mathcal{L} \cong \mathcal{P} \otimes \mathcal{L}^{\otimes n}) \iff \mathcal{L} \cong \mathcal{L}^{\otimes n}$ if (i) $\mathcal{L} \cong \mathcal{L}^{\otimes n}$ (ii) $\exists \mathcal{P} \in \mathcal{P}$ s.t. $\mathcal{L}|_{\mathcal{P}} \cong \mathcal{L}^{\otimes n}|_{\mathcal{P}}$. In other words, they are equal if on the closed subsets we have isomorphic line bundles. (Recall isomorphism of line bundles!)

Definition. For X an algebraic scheme, \mathcal{L} a Cartier divisor, a divisor H determines a line bundle $\mathcal{O}_X(H)$ by the sheaf of sections on the \mathcal{O}_X -subsheaf generated by $\mathcal{O}_X(H)$ on Y_3 (with $Y_3 \in \bigcup_3 Y_3$).

Definition. A Cartier divisor H is effective if the $\mathcal{O}_X(H) \cong \mathcal{O}_X(D)$ for $D \in \text{Div}_X$, where $\mathcal{O}_X(D)$ is the canonical section (i.e., if H has a canonical section that is regular, it is effective).

Definition. A Cartier divisor H determines a pseudo-divisor $(\mathcal{O}_X(H) \otimes \mathcal{L} \cong \mathcal{O}_X(H) \otimes \mathcal{L}^{\otimes n})$. If $(\mathcal{P} \otimes \mathcal{L} \cong \mathcal{P} \otimes \mathcal{L}^{\otimes n})$ if $|\mathcal{L}| \cong \mathcal{L}^{\otimes n}$ and $\mathcal{O}_X(H) \in \mathcal{P}$.

Lemma. If X is a variety, any pseudo-divisor $(\mathcal{P} \otimes \mathcal{L} \cong \mathcal{P} \otimes \mathcal{L}^{\otimes n})$ is represented by a Cartier divisor H such that

- (1) If $\mathcal{L} \cong \mathcal{L}^{\otimes n}$, the operation is unique.
- (2) If $\mathcal{L} \cong \mathcal{L}^{\otimes n}$, the operation is unique up to linear equivalence.

Definition. If H is a pseudo-divisor on an n -dimensional variety with support $|H|$, then the Weil divisor class is $[H] \in \text{Cl}_n(X) \cong \text{Cl}_n(X)$.

Definition. If $E \in (\mathcal{P} \otimes \mathcal{L} \cong \mathcal{P} \otimes \mathcal{L}^{\otimes n})$ and $F \in (\mathcal{P} \otimes \mathcal{L}^{\otimes m} \cong \mathcal{P} \otimes \mathcal{L}^{\otimes mn})$ are pseudo-divisors, then

$$E \otimes F \in (\mathcal{P} \otimes \mathcal{P} \otimes \mathcal{L}^{\otimes n+m} \cong \mathcal{P} \otimes \mathcal{L}^{\otimes n+m}), \text{ and}$$

$$E \in (\mathcal{P} \otimes \mathcal{L}^{\otimes n} \cong \mathcal{P} \otimes \mathcal{L}^{\otimes n}).$$

where the tensor product is taken over the trivial bundle, $(\mathcal{P} \otimes \mathcal{P} \otimes \mathcal{L}^{\otimes n+m} \cong \mathcal{P} \otimes \mathcal{L}^{\otimes n+m})$.

Definition. Let $\mathcal{O} \in \mathcal{P} \otimes \mathcal{L}^{\otimes n} \cong \mathcal{O} \otimes \mathcal{L}^{\otimes n}$. Then $\mathcal{O}^{\otimes n} \in (\mathcal{O} \otimes \mathcal{L}^{\otimes n} \cong \mathcal{O} \otimes \mathcal{L}^{\otimes n})$.

Notice

$$\mathcal{O}^{\otimes n} \otimes (\mathcal{H} \otimes \mathcal{L}^{\otimes m}) \in (\mathcal{O} \otimes \mathcal{P} \otimes \mathcal{L}^{\otimes n+m} \cong \mathcal{O} \otimes (\mathcal{P} \otimes \mathcal{L}^{\otimes n+m})) \cong (\mathcal{O} \otimes \mathcal{P} \otimes \mathcal{L}^{\otimes n+m}) \cong (\mathcal{O} \otimes \mathcal{P} \otimes \mathcal{L}^{\otimes n+m}) \cong (\mathcal{O} \otimes \mathcal{P} \otimes \mathcal{L}^{\otimes n+m})$$

$$\cong (\mathcal{O} \otimes (\mathcal{P} \otimes \mathcal{L}^{\otimes n+m})) \cong (\mathcal{O} \otimes (\mathcal{P} \otimes \mathcal{L}^{\otimes n+m})) \cong (\mathcal{O} \otimes (\mathcal{P} \otimes \mathcal{L}^{\otimes n+m})) \cong (\mathcal{O} \otimes (\mathcal{P} \otimes \mathcal{L}^{\otimes n+m})).$$

Intersecting with divisors

Definition. If H is a pseudo-divisor on a scheme X and Z is a 5-dimensional sub-variety of X , define $H[Z]$ or $H \cdot Z$ in $E_5(X, Z)$ to be

$$4^{\dagger}H, \text{ a pseudo-divisor on } Z \text{ with support } |H| \cap Z.$$

Then $H \cdot Z$ is the Weil divisor class of $4^{\dagger}H$ in $[4^{\dagger}H]$.

Note: If H is a Cartier divisor, and $Z \not\subset |H|$, then H restricts (pullback) to a Cartier divisor by $4^{\dagger}H$.

Definition. Let $\alpha \in \sum \mathbf{8}_Z[X]$ be a 5-cycle. Then the support of α , $|\alpha|$ is the union of subvarieties with non-zero coefficients $\mathbf{8}_Z$.

Definition. Each $H \cdot [Z]$ is a class in $E_5(X, |H| \cap Z)$. We define the intersection class

$$H \cdot \alpha \in \sum \mathbf{8}_Z H \cdot [Z].$$

Proposition. If H is a pseudo-divisor on X and α is a 5-cycle, then

- (1) $H \cdot (\alpha \cdot \beta) \in H \cdot \alpha \cdot H \cdot \beta$,
- (2) $(H \cdot H^w) \cdot \alpha \in H \cdot \alpha \cdot H^w$
- (3) If $0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0$ is a morphism, then there is

$$1_{\dagger} \cdot 0 \in (|H| \cap |\alpha|) \cdot \mathcal{A} \cdot |H| \cdot 0 \in (|\alpha|)$$

with $1_{\dagger}(0 \cdot H \cdot \alpha) \in H \cdot 0_{\dagger}(\alpha)$.

- (4) Same as above but other direction.
- (5) If H is a pseudo-divisor with $b_X(H)$ is trivial, then $H \cdot \alpha \in 0$!

Lecture 4 (February 10, 2009)

Theorem 2.4 Let H, H^w be Cartier divisors on an 8-dimensional variety X . Then

$$H \cdot [H^w] \in H^w \cdot [H] \text{ in } E_8(X, |H| \cap |H^w|),$$

where $[H^w] \in \sum_{\text{codim } Z=3} \mathbf{8}_3[Z] \subset E_8(X)$.

Proof. Assume H and H^w are effective Cartier divisors that intersect properly (that is, there is no codimension 3 subvariety of X in their intersection, $|H| \cap |H^w|$). Recall the following fact:

Let E be a local domain of dimension $\#$. Take $\beta + \nu - E$. Define $\int_E(\beta E \hat{I} + \nu E)$ as follows:

$$! \ddot{A} \ker \ddot{A} E \hat{I} + \nu E \ddot{A} E \hat{I} + \nu E \ddot{A} \text{ coker } \ddot{A} !$$

Then let $\int_E(\beta E \hat{I} + \nu E) \simeq j_E(\text{coker}) - j_E(\ker)$ (length of cokernel minus length of kernel).

Next, we will need the lemmas:

Lemma A.27. $e_E(\beta E \hat{I} + \nu E) \simeq \sum_{\substack{\text{ht } \circledast \nu \\ : - \text{Spec } E}} j_E(E \circledast \hat{I} + \nu E) \dagger j_{E \hat{I} \circledast} (E \hat{I} \circledast + E)$.

Lemma A.28. $e_E(\beta E \hat{I} + \nu E) \simeq \int(\beta E \hat{I} + \nu E)$.

Note: For the above lemma, we want $H \dagger [H^\nu] \simeq \sum \mathbb{Z}_3[A_3]$ to be $H^\nu \dagger [H] \simeq \sum \mathbb{Z}_3[A_3]$.

Case 1 Assume H and H^ν are effective that intersect properly. Calculate the coefficient of $[\]$ in $H \dagger [H^\nu]$, with $\dim E \simeq \#$. Then all primes \circledast with $\text{ht } \nu$ correspond to a subvariety Z of codimension ν . Hence,

$$[H^\nu] \simeq \sum_{\substack{\text{ht } \circledast \nu \\ : - \text{Spec } E}} \mathbb{8}_3[Z_3] \quad \text{PPP}$$

Here, $\mathbb{8}_3 \simeq \text{ord}_{Z_3}(\nu)$. Since $\text{ord}_{Z_3} \nu \simeq E \circledast \beta$ we further have $\mathbb{8}_3 \simeq j_{E \circledast} (E \circledast \hat{I} + \nu E \circledast)$. To continue, we first must compute

$$H \dagger [H^\nu] \simeq \sum_{\circledast - E} \mathbb{8}_3 H \dagger [Z_3],$$

with

$$H \dagger [Z_3] \simeq \sum_{\substack{[\] \in \mathbb{S} Z_3 \\ \text{codim } [\] \simeq \nu}} \mathbb{Z}_3[[\]].$$

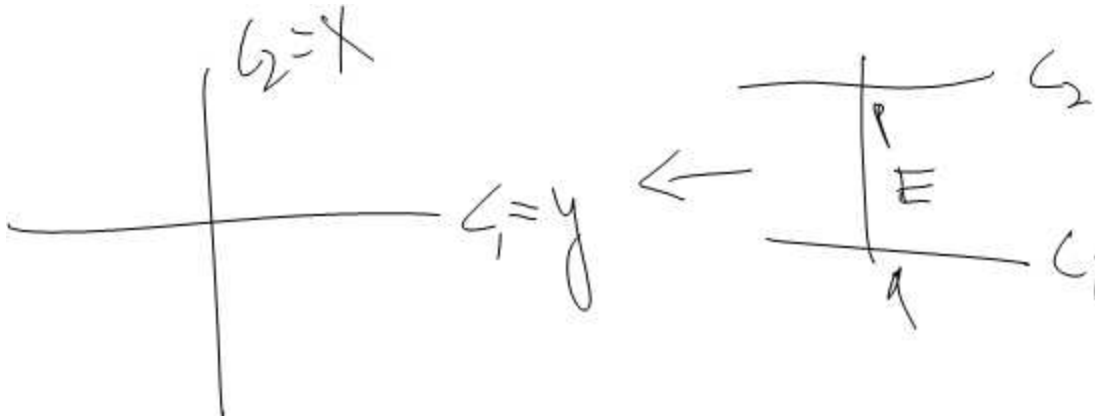
Well, the coefficient of $[\]$ in $H \dagger [Z_3]$ is

$$\sum_{\circledast} j_{E \hat{I} \circledast} (E \hat{I} + \nu E \circledast) \dagger j_{E \hat{I} \circledast} (E \hat{I} \circledast + E) \simeq \int_E(\beta E \hat{I} + \nu E)$$

by the lemma.

Case 2 Assume H and H^ν are effective. We will reduce to the case of proper intersection.

Example 2.4.1. Consider \mathbb{P}^2 . Consider the situation.



First, we have

$$H \in 1^+ G_{\infty} \in G_{\infty} \quad I \text{ (principal), and}$$

$$H^{\#} \in 1^+ G_{\#} \in G_{\#} \quad I \text{ (principal),}$$

where by principal we mean $b_{\zeta} \mathbf{z} b_{\zeta}(H)$ and $b_{\zeta} \mathbf{z} b_{\zeta}(H^{\#})$ respectively. We have

$$H \dagger [H^{\#}] \in H \dagger ([G_{\#} \quad I]) \in \underbrace{H \dagger [G_{\#}]}_{\cdot} \quad H \dagger [I],$$

where \cdot is just a point ($\cdot = - \wedge_1(|H| \quad |G_{\#}|)$) and $E_1(|H|)_{E_1(\cdot)}$. Notice $H \dagger [G_{\#}] = - \wedge_1(|H| \quad |G_{\#}|) \in - \wedge_1(I: I)$. We can also think of $H \dagger [G_{\#}] = E_1(|G_{\#}|)$ and in that case $\cdot \in !$, because $G_{\#} \in \cdot$. Thus, $H \dagger [G_{\#}] \in \cdot$. On the other hand, $H \dagger [I] \in !$. Why? Because $|I| \in \mathcal{S} |H|$ in $\wedge_1(I)$. Hence,

$$\underbrace{H \dagger [G_{\#}]}_{\cdot} \quad H \dagger [I] \in \cdot = - \wedge_1(I).$$

Notice that since $I \in \cdot$, the other one turns out to be \cdot . Hence, they are non-equal as cycles, but are as classes. \square

Definition. Let H_1, \dots, H_g be pseudodivisors on \mathbb{P}^2 . For any $\mathbf{z} \in \mathbb{P}^5 \setminus \mathbb{P}^2$, define

$$H_1 \dagger \dots \dagger H_g \dagger \in H_1 \dagger (H_2 \dagger \dots \dagger H_g \dagger) = E_{5-g}(|H_1| \quad \dots \quad |H_g| \quad | \mathbf{z} |).$$

More generally, for any homogeneous polynomial of degree d ,

$$T(X_1, \dots, X_g)$$

with integer coefficients. Then

$$T(H_1, \dots, H_g) \dagger = E_{5-d}(|H_1| \quad \dots \quad |H_g| \quad | \mathbf{z} |).$$

If $\mathcal{H} \in \mathcal{H}_5 \subset \mathbb{P}^5$ is complete, define the intersection # to be

$$(\mathcal{H} \cdot \mathcal{H}_5) \cdot \int_{\mathbb{P}^5} T(\mathcal{H} \cdot \mathcal{H}_5) \cdot \tilde{\omega}.$$

Note. The "integral" means if $\omega \in \sum \mathbb{Z}_3[T_3]$ with \int over \mathbb{P}^5 then

$$\int_{\mathbb{P}^5} \omega \in \sum \mathbb{Z}_3[5(\cdot) + 5].$$

If Z is a pure 5-dimensional subscheme of \mathbb{P}^5 , then

$$T(\mathcal{H} \cdot \mathcal{H}_5) \cdot Z \in T(\mathcal{H} \cdot \mathcal{H}_5)[Z].$$

Chern classes of line bundles

Let P be a line bundle on a scheme \mathbb{P}^5 . We can define a homomorphism

$$G_3(P) \in \sum \mathbb{Z}_3[Z_3] \xrightarrow{\cong} G_3(\mathbb{P}^5)$$

as follows. For any 5-dimensional subvariety $Z \subset \mathbb{P}^5$ $\in \mathcal{O}_Z(\mathcal{G})$ where \mathcal{G} is a Cartier divisor on \mathbb{P}^5 . Then

$$G_3(P) \cdot [Z] \in [G] - E_5(\mathbb{P}^5).$$

Note that if $P \in \mathcal{O}_{\mathbb{P}^5}(\mathcal{H})$ for a pseudo-divisor on \mathbb{P}^5 , then

$$G_3(P) \in \mathcal{H} \cdot \int_{\mathbb{P}^5}.$$

Proposition 2.5 (a) If μ is on \mathbb{P}^5 , then $G_3(P) \in \mathbb{Z}$. So we have

$$G_3(P) \in \sum \mathbb{Z}_3[E_5(\mathbb{P}^5) - E_5(\mathbb{P}^5)].$$

(b) If P, P' are line bundles on \mathbb{P}^5 , then

$$G_3(P + P') \in G_3(P) + G_3(P').$$

(c) If $\pi: \mathbb{P}^5 \rightarrow \mathbb{P}^5$ is a proper morphism, P is a line bundle on \mathbb{P}^5 , π^* is a 5-cycle on \mathbb{P}^5 , then

$$\pi_*(G_3(\pi^*P)) \in G_3(P) \cdot \pi_*$$

(d) If $\pi: \mathbb{P}^5 \rightarrow \mathbb{P}^5$ is flat of relative dimension 8, and P is a line bundle on \mathbb{P}^5 , with π_* a 5-cycle on \mathbb{P}^5 , then

$$G_3(\pi^*P) \cdot \pi^* \in \pi^*(G_3(P) \cdot \pi_*).$$

(e) If P, P' are line bundles with $\pi: \mathbb{P}^5 \rightarrow \mathbb{P}^5$ then

$$G_3(P \otimes P') \in G_3(P) + G_3(P')$$

Definition. If P_1, \dots, P_8 are line bundles on $\mathbb{P}^5 - E_5(\mathbb{P}^5)$, and we have a homogeneous polynomial $T(X_1, \dots, X_8)$ of degree d , then

$$T(G_*(P) \otimes \mathbb{P}^n \rightarrow G_*(P_8)) \cong E_5(\mathbb{P}^n).$$

In particular,

$$G_*(P) \cong E_5(\mathbb{P}^n).$$

Definition. Let H be an effective divisor on \mathbb{P}^n and let $\mathbb{P}^n \xrightarrow{H} \mathbb{P}^n$ be the inclusion. Define the Gysin homomorphism

$$\begin{aligned} \mathbb{P}^n \xrightarrow{H} \mathbb{P}^n &\rightarrow E_5(\mathbb{P}^n) \cong G_*(P) \\ \cong \mathbb{P}^n &\rightarrow G_*(P) \otimes H \end{aligned}$$

Proposition 2.6. (a) If μ_1 on \mathbb{P}^n , then $\mathbb{P}^n \otimes H \cong \mathbb{P}^n$. So we have

$$\mathbb{P}^n \xrightarrow{H} \mathbb{P}^n \rightarrow E_5(\mathbb{P}^n) \cong G_*(P).$$

(b) If $\mu_1 = Z_5$, then

$$\mathbb{P}^n \xrightarrow{H} \mathbb{P}^n \rightarrow G_*(b_B(H)) \cong \mathbb{P}^n.$$

(c) If $\mu_1 = \mathbb{P}^n \otimes H$, then

$$\mathbb{P}^n \xrightarrow{H} \mathbb{P}^n \rightarrow G_*(R) \cong \mathbb{P}^n,$$

where $R \cong \mathbb{P}^n \otimes H$.

(d) If \mathbb{P}^n is purely 8-dimensional, then

$$\mathbb{P}^n[\mathbb{P}^n] \cong [H].$$

(e) If P is a line bundle on \mathbb{P}^n , then

$$\mathbb{P}^n(G_*(P)) \cong G_*(P) \otimes \mathbb{P}^n$$

for any $\mu_1 = \mathbb{P}^n \otimes H$.

$$G_*(P) \otimes (G_*(P)) \cong G_*(P) \otimes (G_*(P)).$$

Lecture 5 (February 16, 2009)

(1) T is projective if and only if T is proper.

(2) $b_1(\mathbb{P}^n) \cong \text{proj } E[B_1 \otimes \mathbb{P}^n \otimes B_1]$.

$T(I) \xrightarrow{H} \mathbb{P}^n$, with $\mu_1 = E_1(\mathbb{P}^n)$.

So $\cong_8(I) \cong \mathbb{P}^n \otimes T^1$.

(1) $0 \rightarrow \mathbb{P}^n \otimes H \rightarrow 0_1(\cong_8(0^1 I)) \rightarrow \cong_8(I) \rightarrow 0_1$.

(a) (1) $\cong_8(I) \cong \mathbb{P}^n$ if $\mathbb{P}^n \cong \mathbb{P}^n$.

(2) $\cong_1(I) \cong \mathbb{P}^n$.

$$\backslash(b_{\setminus}(H)) \approx \deg\left(\begin{matrix} \text{H} \\ \text{O} \end{matrix}\right)_{\setminus}.$$

$$g_{\setminus} \approx H_{\setminus} \approx b_{\setminus}(\text{O}_{\setminus}) \approx b_{\setminus}(\text{O}_{\setminus}) \approx \dots$$

$$\begin{aligned} \backslash(I) \approx \deg\left(\begin{matrix} \text{H} \\ \text{H}^{\#} \\ \text{O} \\ \text{O}^{\#} \\ \text{G}^{\#} \end{matrix}\right) \approx \\ \deg\left(\begin{matrix} \text{HO} \\ \text{H}^{\#} \\ \text{O}^{\#} \\ \text{G}^{\#} \end{matrix}\right) \approx \deg\left(\begin{matrix} \text{H} \\ \text{H} \\ \text{O} \\ \text{O}^{\#} \\ \text{G}^{\#} \end{matrix}\right). \end{aligned}$$