Lecture 3 (January 16, 2009) -

Example. Let K be a field, V a 1-dimensional K-vector-space, and W a 2-dimensional K-vector space. Take

$$(\mathrm{id}_K,\ \text{embedding of v.s.}), (K,V) \to (K,W).$$

As we will see, $(K,V) \vDash \theta$ and $(K,W) \vDash \theta$. Consider the formula $\varphi(v)$, which will say

$$\varphi(v) = \forall w\, \exists \lambda\ \lambda \bigstar v \approx w.$$

Then $\varphi(v)^{(K,V)} = \{a \in V \mid a \neq 0_{\mathrm{V}}\}$, since all non-zero elements of a 1-dimensional vector space span that vetor space. On the other hand, $\varphi(v)^{(K,W)} = \emptyset$, since there is no single basis element for a 2-dimensional vector space.

Definition. *We say that $\mathcal{N} \subseteq \mathcal{M}$ is an elementary substructure* (*written $\mathcal{N} \preceq \mathcal{M}$*) *if for all L-formulae $\varphi(\overline{x})$, $\varphi^{\mathcal{M}} \cap \mathcal{N} = \varphi^{\mathcal{N}}$.* [Familiarize yourself with this last part.]

Definition. *An embedding $\mathcal{N} \hookrightarrow \mathcal{M}$ is* elementary *if $f(\mathcal{N}) \preceq \mathcal{M}$.*

We now claim that if $f : \mathcal{N} \to \mathcal{M}$ is a bijection, then f is an embedding if and only if f is an elementary embedding.

Definition. *We say $\mathcal{M}, \mathcal{N}$ are* fundamentarily equivalent *if $Th(\mathcal{M}) = Th(\mathcal{N})$ and we write $\mathcal{M} \equiv \mathcal{N}$.*

Remarks. The above definition is a weaker version of isomorphism.

Exercise. *If $\mathcal{N} \preceq \mathcal{M}$, then $\mathcal{N} \equiv \mathcal{M}$.*

Example. There are $\mathcal{M} \subseteq \mathcal{N}$ substructures such that $\mathcal{M} \cong \mathcal{N}$ but $\mathcal{M} \npreceq \mathcal{N}$. Take a 1-sorted language, with one binary relation symbol R, and $\mathcal{M}$'s universe to be the natural numbers $\mathbb{N}$, with $R^{\mathcal{M}} := <$ (less than).

Compactness: If T is a set of sentences and if $\forall T_0 \subseteq_{\text{fin}} T$ there is $\mathcal{M}_{T_0} \vDash T_0$, then $\exists \mathcal{M} \vDash T$. ($\mathcal{M}_{T_0} \vDash T_0$ means $\forall \theta \in T_0$, $\mathcal{M} \vDash \theta$).

Upward Löwenheim-Skolem: *If T has an infinite model $\mathcal{M}$, then $\forall \kappa > |\mathcal{M}|$, $\exists \mathcal{N} \succeq \mathcal{M}$ such that $|\mathcal{N}| = \kappa$.*

Proof. "$\mathcal{N}$ is huge": define $L^+ := L \cup \{C_i \mid i \in \kappa\}$ (L with κ many constant symbols), and $T_1 = \{C_i \not\approx C_j \mid i \neq j\}$. We also need "$\mathcal{M} \preceq \mathcal{N}$". So, we grow our language again:

$$L^{++} := L^+ \cup \{C_m \mid m \in \mathcal{M}\}.$$

Let $T_2 := \left\{\varphi(C_{m_1}, \ldots, C_{m_n}) \mid \varphi(x_1, ..., x_n)^{\mathcal{M}} \ni (m_1, ..., m_n)\right\}$. Then let $T := T_1 \cup T_2$. This completes the proof. ☐ [**Homework.** Understand this...]

Downward Löwenheim-Skolem. *If $A \subseteq \mathcal{M}$, then there is $\mathcal{N} \preceq \mathcal{M}, A \subseteq \mathcal{N}$, $|A| = |\mathcal{N}|$.*

Proof. **Exercise.** (to look it up) It involves Skolem functions. ☐

We now claim if $\mathcal{M} \equiv \mathcal{N}$, then there is $\mathcal{U} \succeq \mathcal{M}$ and $\mathcal{U} \succeq \mathcal{N}$.

Proof. Eldiag($\mathcal{M}$) $\cap$ Eldiag($\mathcal{N}$). Assume Eldiag($\mathcal{M}$) $\supseteq \mathcal{O}_{\mathcal{M}} := \bigwedge F_{\mathcal{M}}$ and similarly for $\mathcal{O}_N$.

Lecture 5 (January 23, 2009) -

Last time, we showed that given a theory T, then the following are equivalent:

- T is model-complete.
- All modules of R are existentially closed (e.c.) models of T.
- Every existential is equivalent to a universal.

Every formula is equivalent to an existential and to an universal.

What fields are e.c. fields?

Axioms of Fields

The language we will use will be $\mathcal{L}_{\text{fields}} = \{ +, \cdot, 0, 1\}$. We will use

$\forall x\, \forall y\, \forall z\; x + (y + z) \approx (x + y) + z \wedge x \cdot (y \cdot z) = (x \cdot y) \cdot z \wedge x \cdot (y + 1) = x \cdot y + x \cdot z$

$\forall x\, \forall y\; (x + y = y + x \wedge x \cdot y = y \cdot x)$

$\forall x\; (x + 0 = x \wedge x \cdot 1 = x)$

$\forall x\, \exists y\; (x + y = 0)$

$\forall x\, \exists y\, (x = 0 \vee x \cdot y = 1)$.

The theory of fields is an $\forall\exists$-theory.

Lemma. If T is an $\forall\exists$-theory, then $A_i \vDash T\; \forall i \in I$ ordered, $\forall i < j$, $A_i \leq A_j$, then

$$\bigcup_{i \in I} A_i \vDash T.$$

Proof. It suffices to prove this for $T = \{\theta\}$.

Note. If $(I, <),\; A_i \preceq A_j\, \forall i < j$, then $\forall i\; A_i \preceq \bigcup_{i \in I} A_i$.

Definition. Given A_i for $i \in (I, c)$, define

$$\bigcup_{i \in I} A_i \text{ : universe : union of universes (sortwise)}$$

For example, if f is an n-ary function symbol and $a_1, ..., a_n \in \bigcup_{i \in I} A_i$ of the right sort,

$$f^{\bigcup_{i\in I} A_i}(a_1, ..., a_n) := f^{A_j}(a_1, ..., a_n).$$

Let $j : \forall i\; a_i \in A_j$. If $j' : \forall i\; a_i \in A_{j'}$, then

$$f^{A_j}(a_1, ..., a_n) = f^{A_{j'}}(a_1, ..., a_n),$$

where $j < j',\; A_j \leq A_{j'}$ implies the above.

We claim that an e.c. field is algebraically closed:

Proof. If F is not algebraically closed, there is a polynomial $p(x) \in F(x)$ such that $F \nvDash \exists x \; p(x) = 0$ but $\overline{F} \geq F, \; \overline{F} \vDash \exists x \; p(x) = 0$. □

Exercise. Find a T which has no e.c. models of any size.

Exercise. Find a theory T, $\mathcal{M}$ an e.c. model of T, $\mathcal{M} \preceq \mathcal{N}$ which is not an e.c. model of T.

Proposition. Suppose that T is an $\forall\exists$-theory, $A \vDash T$, $\lambda \geq |A|, \lambda \geq |L|$, then $\exists\mathcal{M}$ is an e.c. model of T such that $A \leq \mathcal{M}$ and $|\mathcal{M}| = \lambda$.

Proof. We have a chain $A = \mathcal{M}_0 \leq \mathcal{M}_1 \leq \mathcal{M}_2 \leq \ldots$ Our induction hypothesis will be $|M_i| \leq \lambda$. The base case $\mathcal{M}_0 = A$ is trivial. We want to list

$$\left\{ \left(\varphi^j(x_1, \ldots, x_n); \left(m^j, \ldots, m^j_{n_j} \right) \right) \middle| \, j \in \lambda \right\},$$

the pairs of L-formula, tuples of elements for $\mathcal{M}$. For the limit cardinal case we have

$$\alpha : M_\alpha = \bigcup_{i \in \alpha} \mathcal{M}_i.$$

The successor case: Given $\mathcal{M}_i$, we look at φ^i, $(\overline{m}^i) \in \mathcal{M}_i$. If $\mathcal{M}_i \vDash \varphi^i(\overline{m}^i)$, then $\mathcal{M}_{i+1} = \mathcal{M}_i$. Otherwise, if there is $\mathcal{N} \vDash T, \mathcal{M}_i \leq \mathcal{N}, \mathcal{N} \vDash \varphi^i(\overline{m}^i) = \exists y \, \varphi(\overline{m}^i, y)$. So we need to find $\alpha \in \mathcal{N}$ s.t. $\mathcal{N} \vDash \varphi(\overline{m}, y)$.

We get $\mathcal{N}_2$ from $\mathcal{N}_1$ like $\mathcal{N}_1$ from $\mathcal{N}_0$. We claim that $\mathcal{M}$ is an e.c. model of T (T is $\forall\exists$). Suppose not. If there is $\mathcal{N} \vDash T, \mathcal{M} \vDash \mathcal{N}$, then this is existential $\varphi(x), \overline{m} \in \mathcal{M}$ such that $\mathcal{N} \vDash \varphi(\overline{m}), \mathcal{M} \nvDash \varphi(\overline{m})$, $\overline{m} \in \mathcal{N}_i$, $(\varphi, \overline{m}) = (\varphi^j, \overline{m}^j)$ for some $j \leq \lambda$ in the construction of $\mathcal{N}_{i+1}$, and this was fixed.

E.c. fields

- Are algebraically closed

- For any characteristic p, for any $\lambda \geq \aleph_0$, there is an e.c. field of char p, of size λ.

- For $\lambda > \aleph_0$, for each char p, there is a unique algebraically closed field of size λ, char p (transcendence basis)

Therefore, for uncountable field k, k is an e.c. field if and only if k is algebraically closed.

Lecture 6 (January 26, 2009) - Algebraically Closed Fields

Definition. $\text{ACF} = \text{Th(fields)} \cup \{\forall \overline{y} \text{ if } y = 0\}$

$\text{ACF}_p = \text{ACF} \cup \{\theta_p : 1 + 1 + \ldots + 1 = 0\}$

$\text{ACF}_0 = \text{ACF} \cup \{\, \exists\theta_p \;:\; p \in \mathbb{N}\}$.

Facts about algebraically closed fields. (1) Infinite.

(2) [Can't read board at that angle from here, copy Ramin's notes again... :( ]

Proposition. *Given an $\forall\exists$ theory T with no finite models, which is λ-categorical for some $\lambda \geq |L|$, then T is model-complete.*

Proof. Suppose $A \vDash T$ is not an existentially closed model of T, i.e., $\exists B \vDash T, a \in A$ existential $\varphi(x)$ such that $A \leq B$, $A \nvDash \varphi(a), B \vDash \varphi(a)$. Then $L^+ = L \cup \{P, C\}$. Consider an L^+-structure $\mathcal{M}$ with $\mathcal{M}|_L := B$, $P^{\mathcal{M}} = A$, $C^{\mathcal{M}} := a$. $\text{Th}_{L^+}(\mathcal{M}) \supseteq T$ with "$P \vDash T$" (relativization). Now we find $\mathcal{N} \vDash T^+$ s.t. $|P(\mathcal{N})| = \lambda$. By compactness, $\geq \lambda$. Then by the down-ward Löwenheim-Skolem this $= \lambda$. Now $P(\mathcal{N})|_L \vDash T$, size λ, not .c. but since T is $\forall\exists$ and $\lambda \geq |L|$, there is some e.c. closed model of T of size L. But T is λ-categorical. ☐

Corollary. *If* $K \vDash$ ACF and Σ is a system of polynomials over K and Σ has a solution in some field $F \geq K$, then Σ has a solution in $\overline{F} \geq K \vDash$ ACF then Σ has a solution in K (Hilbert's Nullstelensatz).

Definition. *T eliminates quantifiers* (*has quantifier elimination*) *if $\forall\varphi(x)$, $\exists$ a q.f. $\psi(x)$ s.t. $\varphi(x) \Leftrightarrow \psi(x)$.*

Proposition. *Suppose T is model-complete, and $\{$substructures of models of $T\}$ has the amalgamation property (A.P.) then T has q.e.*

Proof. (1) It suffices to show q.e. for existential $\varphi(x)$ (induction). Fix existential φ. Take

$$S_\varphi := \{(A, a) \mid A \vDash T, A \vDash \varphi(a)\}$$

$$F_{(A,a)} = \{\psi(x) \mid \psi \text{ is q.f. and } A \vDash \psi(a)\}.$$

We claim if $(B, b) \vDash T \cup F_{(A,a)}$ then $B \vDash \varphi(b)$. For a q.f. type of b in B and q.f. type of a in A, then $\langle b \rangle_B \cong \langle a \rangle_A = D$ so there is some $C_0 \subseteq C \vDash T$ s.t. $D \cong \langle a \rangle_A \hookrightarrow A$ and $D \cong \langle b \rangle_B \hookrightarrow B$ with $A \hookrightarrow C_0$ and $B \hookrightarrow C_0$ with $C_0 \hookrightarrow C \vDash T$. Now, $A \vDash \varphi(a)$ so (since φ is existential), $C \vDash \varphi(a), C \vDash \varphi(b)$. Since T is model-complete, this implies $B \vDash \varphi(b)$.

Lecture 7 (January 28, 2009) -

Organizational:

- No class on Friday Feb 6th, 27th. (class 2-4 on Mon Feb 2nd, 23rd)
- Grad student logic conference in Ubrana on 18/19 April.
- ASL conference Notre Dame May 20-23 Apply for funding NOW.

Quantifier Elimination for A.C.F

Corollary. *If $\mathcal{M} \vDash ACF$, and $\varphi(x)$ is a formula with 1 free variable, then $\varphi(\mathcal{M})$ is finite or cofinite.*

Proof. If φ is quantifier-free, without loss of generality (d.n.f)

$$\varphi = \bigvee_{j=1}^{m_i} \left(\bigwedge_{j=1}^{m_i} \varphi_{ij}(x) \right)$$

where φ_{ij} is a poly equation or inequality. It suffices to show that $\bigwedge_{i=1}^{m_j}\varphi_{ij}(k)$ is finite/confinite. Hence, it suffices to show $\varphi_{ij}(x)$ is finite/cofinite: finite if equal, cofinite is ineq. $\square$

Definition. *A theory T is strongly minimal if the previous corollary holds for T in place of* ACF.

Examples. (1) $L = \emptyset$, theory of infinite sets.

(2) Distinguishable, torsion-free abelian groups ($\mathbb{Q}$-vector space).

(3) For any prime p, $\mathbb{F}_p$-vector spaces.

Zilber's Trichotomy Conjecture.

Counterexample: Read Hrushovski's article "A new strongly minimal set."

From this point, we let T be a strongly minimal theory.

Lemma. (about uniform finiteness/cofiniteness) *Take $\varphi(x;y)$ with x, y single variables. Then $\exists n$ such that for all but finitely many $b \in \mathcal{M}$, $|\varphi(\mathcal{M};b)| = n$ (finiteness), or for all but finitely many $b \in \mathcal{M}$, $|\mathcal{M} \setminus \varphi(\mathcal{M};b)| = n$ (cofiniteness).*

Proof. Note that for each n, "$|\varphi(\mathcal{M},y)| = n$" is 1-st order, call it $f_n(y)$ (finite), and "$|\varphi(\mathcal{M},y)| = n$" is 1-st order, call it $c_n(y)$ (cofinite). Then

$$\mathcal{M} = (\textstyle\bigsqcup_n f_n(\mathcal{M})) \sqcup (\bigsqcup_n c_n(\mathcal{M})).$$

We claim that not all f_n, c_n are finite. Further, we claim that $\forall i \neq j$, $f_i \cap f_j$, $f_i \cap c_i$, $f_i \cap c_j$, $c_i \cap c_j = \emptyset$. We claim there are infinitely many non-empty f_i or infinitely many non-empty c_i. Take $\Sigma = T \cup \{\varphi(x,C)$ is infinite, co-infinite$\}$. By strong minimality of T, Σ is inconsistent. So a finite part of Σ is inconsistent.

$$\Sigma = \{\exists_{\geq n}\, x\, \varphi(x,C) \mid n \in \mathbb{N}\} \cup \{\exists_{\geq n}\, x\, \neg\varphi(x,C) \mid n \in \mathbb{N}\}$$

(we are adding one constant C to the language).

Lemma. *Suppose $b, a_1, ..., a_n \in \mathcal{M}$. Suppose $b \in$ acl$(a_1, ..., a_n) \setminus$ acl$(a_1, ..., a_{n-1})$. Then $a_n \in$ acl$(a_1, ..., a_{n-1}, b)$. (Steinitz exchange)*

Proof. Say $\psi(x; y_1, ..., y)$ witnesses $b \in \text{acl}(a_1, ..., a_n)$, i.e., $\psi(x;\overline{a})$ is finite ($b \in \overline{a}$). Define $\psi(x, a_1, ..., a_{n-1}, y_n)$ as $\tilde{\psi}(x, y_n)$. Then for all but finitely many $c \in \mathcal{M}$, $\tilde{\psi}(x;c)$ has the same finite size, or the same cofinite size (see previous lemma). Similarly for $c \notin \text{acl}(a_1, ..., a_{n-1})$ $\tilde{\psi}$ has the same finite size as $\varphi(x;a_n)$. Also, all vertical slices $\tilde{\psi}(d;y_n)$ have the same size as $\tilde{\psi}(b, y_n)$. If almost all vertical slices are cofinite, then any N of them intersect.

Lecture 9 (February 2, 2009) -

(1) Given $A \subseteq \mathcal{M}$, and given $n \in \mathbb{Z}$, there is a unique n-type of dimension n.

Proof. Induct on n. If $n = 1$, the generic type of $\mathcal{M}$ over A will be

$$\forall\varphi(x) \text{ w/ parameters from } A,\ \varphi(\mathcal{M} \text{ finite}) \text{ or } \varphi(\mathcal{M}) \text{ cofinite.}$$

Now for the induction case (n to $n+1$), we need to show if $(\overline{b}, c), (\overline{d}, e)$ are $(n+1)$-tuples then $\dim_A = n+1$ (they have the same order over A). We need $bc \equiv_A de$, that is,

$$\text{type}(bc/A) = \text{type}(de/A).$$

First, note that b, d both satisfy the unique (by induction) n-type over A of dimension n.

$$p(x) = \{\varphi(\overline{x}, y) \mid \vDash \varphi(\overline{b}, c) \text{ w/ param from } A\},$$

$\exists y,\ \varphi(\overline{x}, y) \in \text{type}(\overline{b}/A) = \text{type}(\overline{d}/A)$. Then $p(y) = p(d, y)$ is a consistent type. Alice waves her hands and says "I refer you to Monster model." (**What is Monster model?**) Now we just need to show e and e' have the same type over $A\overline{d}$.

If $\overline{b}$ models the generic n-type over A, then $\overline{b}$ is a generic point of A^n (not the same as in algebraic geometry).

(2) $RM(p) = \dim(p)$. Take $\overline{a} \vDash p$, reorder s.t. $\overline{a} = \overline{b}\overline{c}$ with $\overline{b}$ a transcendence basis for $\overline{a}$, and for each c_i, take $\psi_i(x, y)$ that witnesses that c_i is algebraic over $\overline{b}$ ($\subset \text{acl}(\overline{b})$). We claim there exists $\varphi_i(x, y) \in$ generic length($\overline{x}$)-type, least possible n_i. Take

$$\Theta(\overline{x}, \overline{y}) = \textstyle\bigwedge_i [\psi_i(\overline{x}, y) \wedge \exists_{=n,y}\, \psi_i(\overline{x}, y)].$$

Then any $q \ni \Theta$ has $RM(q) \leq r$, and $\text{Rm}(p) = r$ implies $q = p$.

(3) The unique generic type p has $RM(p) = r$. We will induct on r. For $r = 1$, it is an easy exercises to show $RM(p) \neq 0$. Now, inductively, take $\varphi(\overline{x}, y)$ to be of generic $(r+1)$-type. We need $MR(\varphi(\overline{x}, y)) \geq r + 1$. We need

$$\{\psi_i(\overline{x}, y)\}_{i \in \aleph_0} \text{ s.t. } MR(\psi_i) \geq r,\ \ \psi_i \cap \psi_j = \emptyset \text{ so } \psi_i \implies \varphi.$$

Now, find $\{b_i\}_{i \in \omega}$ independent parameter with $\varphi_i(\overline{x}, y) = \varphi(\overline{x}, y) \wedge y = b_i$. Then it is an easy propostion of RM that $RM(\text{tp}(\overline{b}\overline{c}/A)) \geq RM(tp(\overline{b}/A))$ and this is equality if and only if $\overline{c} \in \text{acl}(A\overline{b})$.

Given $\varphi(\overline{y}, x), pk \leq \text{length}(\overline{x}),\ \{\overline{a} \mid MR(\varphi(\overline{a}; \overline{x})) \geq k\}$ is definable. [Couldn't read the board at this point.]