Lecture 6 (January 26, 2009) -

Rational functions

If \( f(z) = P(z)/Q(z) = \frac{a_0 + a_1 z + \ldots + a_n z^n}{b_0 + b_1 z + \ldots + b_m z^m} \) with \( n = \deg P \) and \( m = \deg Q \).

Fundamental Theorem of Algebra

If \( P(z) \) is a complex polynomial of degree \( > 0 \) then \( P \) has a root in \( \mathbb{C} \) (\( \exists \alpha \in \mathbb{C} \) s.t. \( P(\alpha) = 0 \)). Hence, we can factor

\[
P(z) = (z - \alpha)g(z),
\]

where \( g \) is a polynomial of degree \( \deg P - 1 \). Inductively, we obtain a factorization

\[
P(z) = c \prod_{i=1}^{n} (z - \alpha_i) \quad (c \in \mathbb{C}).
\]

The multiplicity of a root \( \alpha \) of \( P \) is the number of \( \alpha_i \)'s which equal \( \alpha \). In fact,

\[
\#\{ p^{-1}(w) \} = \deg P
\]

for any \( w \in \mathbb{C} \), where preimages are counted with multiplicities.

Basic fact. A rational function extends continuously to a function \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) where \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \). This extension for \( f = P/Q \) is given by \( f(z_0) = \infty \) if \( Q(z_0) = 0 \) and \( f(\infty) = \lim_{z \to \infty} f(z) \). We define \( R(z) = f(\frac{1}{z}) \) for \( z \neq 0 \), and this will be a rational function in \( \frac{1}{z} \):

\[
R(z) = \frac{\sum a_i(\frac{1}{z})^j}{\sum b_i(\frac{1}{z})^j} = \frac{z^m \cdot (\text{poly in } z)}{z^n \cdot (\text{poly in } z)}.
\]

Set

\[
f(\infty) = R(0) = \begin{cases} 
0 & \text{if } m > n \\
\infty & \text{if } m < n \\
a_n/b_m & \text{if } m = n.
\end{cases}
\]

We can also compute the multiplicity of \( f \) as a zero, or a pole, or a preimage of \( a_n/b_m \). In conclusion, in \( \hat{\mathbb{C}} \), \( f \) has exactly \( d \) zeroes and \( d \) poles, counted with multiplicity (with \( d = \deg f = \max\{n, m\} \)).

Examples. If \( \deg f = 1 \), then it is of the form

\[
f(z) = \frac{az + b}{cz + d} \quad \text{with } a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0.
\]

It defines a homeomorphism of \( \hat{\mathbb{C}} \) to itself. (for example, \( f(z) = 1/z \) rotates the Riemann sphere around the \( x_1 \) axis). Exercise. What does \( f(z) = z^2 \) do to the Riemann sphere?

Ahlfors Ch 2. §1.4

What is the general form of a rational function \( f \) with \( |f(z)| = 1 \ \forall z \) with \( |z| = 1 \)? What is the relation between \( f \)'s set of zeroes and its set of poles?
Consider such a function with deg \(1\): \(f(z) = e^{i\theta}z\) (rotation) or \(f(z) = 1/z\). We also saw in class we can have \(f(z) = \frac{a-z}{1-\overline{a}z}\) with \(|a| \neq 1\). **Exercise.** Show this satisfies the above condition.

For \(\operatorname{deg} > 1\), take for example \(z^n\) for \(n > 1\).

Now, taking the product:

\[
f(z) = e^{i\theta}z^n \prod_{i=1}^{k} \frac{a-z}{1-\overline{a}z},
\]

we claim this is *all* such functions (classifying functions that preserve the unit circle, \(f(\{z : |z| = 1\}) = \{z : |z| = 1\}\)).

**Partial fraction expansion**

Start with \(f(z) = P(z)/Q(z)\). Our goal is to express \(f\) as a sum of rational functions, where each sum has \(\leq 1\) pole:

\[
f(z) = P_0(z) + \sum R_i(z)
\]

where \(P_0(z)\) is a polynomial, and \(R_i\) are single-poled rat'l functions.

**Examples.**

1. If \(f(z) = \frac{2z+1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}\) with \(A, B \in \mathbb{C}\). We then compute

\[
A(z-2) + B(z-1) = 2z + 1.
\]

This implies \(A = -3\) and \(B = 5\) (by comparing coefficients of \(z^1\) and \(z^0 = 1\)).

2. Let \(f(z) = \frac{2z+1}{(z-1)(z-2)^2}\) (deg \(P < \deg Q\)). Then

\[
\frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2} + \frac{D}{(z-2)^3}
\]

with \(A, B, C, D \in \mathbb{C}\).

If \(f(z) = P(z)/Q(z)\) when \(\deg P \geq \deg Q\), then the polynomial term \(P_0(z)\) has degree \(\deg P - \deg Q\). Then

\[
f(z) = \frac{z^3+2}{(z-1)^3} = Az + B + \frac{C}{z-1} + \frac{D}{(z-1)^2}
\]

and this gives us four equations, and we can solve to show this

\[
= z + 2 + \frac{4}{z-1} + \frac{4}{(z-1)^2}.
\]

**Lecture 12 (February 11, 2009) - Linear Fractional Transformations**

**Linear Fractional Transformations**

**Theorem.** If a LFT \(f\) takes a circle \(C_1\) to a circle \(C_2\) (circle here means circle or a line -- i.e. a circle on the Riemann sphere), then \(f\) takes pairs of symmetric points \((z, z^*)\) about \(C_1\) to symmetric points \((f(z), f(z^*))\) about \(C_2\).

**Note:** Symmetric here for a line means symmetric about the line (image under reflection across the line).
The idea is for \( z_1, z_2, z_3 \) distinct points on the circle \( C_1 \), then the cross ratio

\[
X(z_1, z_2, z_3, z^*) = \frac{X(z_1, z_2, z_3)}{X(z_1, z_2, z_3, z)},
\]

because if the points are sent to 0, 1, \( \infty \) then the "circle" will be the line \( \mathbb{R} \) and so \( z^* = \overline{z} \).

**Example.** p83 #6 in Ahlfors. Suppose a linear fractional transformation takes a pair of concentric circles to concentric circles. Show that the ratios of the radii are the same.

**Solution.** The idea is to show the center is taken to the center and \( \infty \) to \( \infty \) (so all we can do is scale).

By pre-composing \( f \) with an affine transformation \( (az + b) \), we can assume the center is at 0, the inner radius is 1, and the outer radius is \( R > 1 \). Similarly, we can compose the image with some \( a'z + b' \) and have center 0, inner radius 1, and outer radius \( S > 1 \). By further post-composing with \( S/z \) if necessary (if they switch), we may also assume that the inner circle is sent to the inner circle (circle of radius 1 goes to circle of radius 1). We then claim that \( f(0) = 0 \) and \( f(\infty) = \infty \). Well, the image of the circle of radius \( 1/R \) is the circle of radius \( 1/S \). Continue reflecting circles, getting closer to 0, and they are sent to circles with radii tending to 0. By continuity, \( f(0) = 0 \). Similarly, reflecting on the Riemann sphere yields \( f(\infty) = \infty \). Now, remember \( f(z) = \frac{az+b}{cz+d} \) (a LFT), and so \( f(0) = 0 \) and \( f(\infty) = \infty \) will imply \( f(z) = \frac{a}{d}z \) and \( f(S) = S \) so that \( f \) is a rotation and so \( R = S \).

Look up **Schwartz Reflection Principle**.

**Example.** p83 #7 in Ahlfors. Find a LFT which carries \( |z| = 1 \) and \( |z - \frac{1}{2}| = \frac{1}{4} \) to a pair of concentric circles. What is the ratio of the radius?

**Solution.** We can assume the inner circle has radius \( R \). We have to find an LFT sending our circles to \( \{ |z| = 1 \} \cup \{ |z| = R \} \) and compute \( R \).

Note \( X(-1, 0, \frac{1}{2}, 1) = X(-R, -1, 1, R) \) because the cross product is an invariant under LFT's. So, find LFT such that

\[
-1 \leftrightarrow 0, 0 \leftrightarrow 1, 1/2 \leftrightarrow \infty, 1 \leftrightarrow \text{crossproduct},
\]

and one such that

\[
-R \leftrightarrow 0, -1 \leftrightarrow 1, 1 \leftrightarrow \infty, R \leftrightarrow \text{crossproduct}.
\]

The answer is \( R = 2 + \sqrt{3} \).

**Example.** \( f(z) = z + \frac{1}{2} = \frac{z^2 + 1}{z} \). Where is it conformal? What does it do? Well, notice

\[
f'(z) = 1 - \frac{1}{z^2} = 0 \iff z = \pm 1
\]

We defined conformal as derivative does not vanish. So, this one is not. We can easily see that if \( z = e^{i\theta} \), then \( z + \frac{1}{2} = z + \overline{z} \) (since \( \overline{z} = \frac{1}{z} \)) = 2 \( \Re z = 2 \cos \theta \). In general,

\[
|z| = r \iff z\overline{z} = r^2 \iff \frac{1}{z} = \frac{z}{r^2} \iff f(z) = z + \frac{\overline{z}}{r^2} = (x + \frac{y}{r^2}) + i(y - \frac{x}{r^2}).
\]

This is an ellipse!
Hence, $f$ takes $\mathbb{D}\setminus\{0\}$ or $\mathbb{C}\setminus\overline{\mathbb{D}}$ by conformal homeomorphism to $\mathbb{C}\setminus[-2,2]$.

**Lecture 16 (February 20, 2009)**

Let $\gamma$ be a closed curve in $\mathbb{C}$. The winding number is

$$\eta(\gamma, z) = \frac{1}{2\pi i} \int_{\gamma} \frac{d\rho}{\rho - z} \in \mathbb{Z} \text{ for } z \notin \gamma.$$ 

We now look at Ahlfors's proof that $\eta(\gamma, z)$ is constant on connected components of $\mathbb{C}\setminus\gamma$ (as a function of $z$). It suffices to show that $\eta(\gamma, z)$ is constant along straight paths in $\mathbb{C}\setminus\gamma$. Consider the linear fractional transformation

$$\frac{z - z_0}{z - z_1}.$$ 

Then $L(z) = \log\left(\frac{z - z_0}{z - z_1}\right)$ is well-defined and analytic for all $z \notin [z_0, z_1]$. Further,

$$L'(z) = \frac{1}{z - z_0} - \frac{1}{z - z_1}$$

so that

$$\int_{\gamma} \left( \frac{1}{z - z_0} - \frac{1}{z - z_1} \right) = 0.$$

If $\gamma$ is a closed loop on a domain of $L$, then

$$2\pi i \eta(\gamma, z_0) = \int_{\gamma} \frac{1}{z - z_0} \, dz = \int_{\gamma} \frac{1}{z - z_1} \, dz = 2\pi i \eta(\gamma, z_1).$$

**Lemma.** If $z$ lies in the unbounded connected component of $\mathbb{C}\setminus\gamma$, then $\eta(\gamma, z) = 0$.

**Proof.** Choose a disk $D$ containing $\gamma$. Then if you take a $z_0 \notin D$, then the function $\frac{1}{z - z_0}$ is analytic on $D$, and hence by Cauchy's Theorem,

$$\int_{\gamma} \frac{dz}{z - z_0} = 0.$$ 

This is the same as saying $\eta(\gamma, z_0) = 0$. By the previous lemma, $\eta(\gamma, z) = 0$ for all $z$ in the unbounded component. $\square$

Then the goal is to show Cauchy's Integral Formula: if $f$ is analytic on a disk $D$ and $\gamma$ is a closed curve on $D$ with $z_0 \notin \gamma$, then

$$\eta(\gamma, z_0)f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} \, dz.$$ 

**Theorem 5.** [Ahlfors] Let $f$ be analytic on a disk, except possibly at finitely many points $\zeta_1, \zeta_2, \ldots, \zeta_n \in D$. Assume

$$\lim_{z \to \zeta_j} (z - \zeta_j)f(z) = 0.$$ 

Then $\int_{\gamma} f = 0$ for any closed loop $\gamma$ in $D\setminus\{\zeta_1, \ldots, \zeta_n\}$.

**Proof.** (sketch) It suffices to consider integrals over rectangles. Because $\int_{\partial R} f = 0$ if $f$ has no singularities on $R$ (a rectangle), we can consider the following special case. Let $R$ be a square centered at one of the $\zeta_j$'s (singularities). We have

$$F(z) = \int_{z_0}^{z} f$$
with \( z \in D \setminus \{\zeta_1, \ldots, \zeta_n\} \).

For \( z \in \partial R \), \( |f(z)| \leq \frac{\varepsilon}{|z-z_0|} \), note \( \left| \int_{\partial R} \frac{dz}{z-z_0} \right| \leq \int_{\partial R} \frac{|dz|}{|z-z_0|} \leq \frac{1}{\ell} \cdot 4 \cdot 2 \ell \leq 8 \). Hence, we can make the integral arbitrarily small,

\[
|\int_{\partial R} f(z) \, dz| \leq \int_{\partial R} |f(z)| |dz| \leq \varepsilon \cdot 8.
\]

Thus, \( \int_{\partial R} f = 0 \) for any rectangle with boundary avoiding \( \zeta_j \).

**Cauchy Integral Formula**

Fix \( \gamma \) in the disk \( D \) and \( z_0 \in \gamma \). Consider \( F(z) = \frac{f(z)-f(z_0)}{z-z_0} \) on \( D \). Then \( F \) is analytic in \( D \setminus \{z_0\} \), and

\[
\lim_{z \to z_0} (z-z_0) F(z) = 0.
\]

By Theorem 5,

\[
\int_{\gamma} F \, dz = \int \frac{f(z)-f(z_0)}{z-z_0} \, dz = \int \frac{f(z)}{z-z_0} \, dz - f(z_0) \int_{\gamma} \frac{dz}{z-z_0}.
\]

Then \( f(z_0) \eta(\gamma, z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-z_0} \, dz \).

**Example.** Exercise 2.2 #2 is to compute

\[
\int_{|z|=2} \frac{dz}{z^2+1}.
\]

We write \( \frac{1}{z^2+1} = \frac{A}{z+i} + \frac{B}{z-i} = \frac{i/2}{z+i} + \frac{-i/2}{z-i} \). Hence,

\[
\int_{|z|=2} \frac{dz}{z^2+1} = \frac{i}{2} \int_{|z|=2} \frac{dz}{z+i} - \frac{i}{2} \int_{|z|=2} \frac{dz}{z-i} = \frac{i}{2} (2\pi i) - \frac{i}{2} (2\pi i) = 0.
\]

**Theorem.** If \( F \) is analytic on a disk \( D \) centered at \( z_0 \), and suppose \( \gamma \) is a circle around \( z_0 \). Then all derivatives \( F^{(n)} \) are analytic on \( D \), and satisfy

\[
F^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta-z_0)^{n+1}} \, d\zeta
\]

for any \( z \) inside \( \gamma \) (notice \( \eta(\gamma, z) = 1 \)).

**Lemma.** Let \( \varphi \) be a continuous function defined on some curve \( \gamma \). Then

\[
F(z) = \int \frac{\varphi(\zeta)}{\zeta-z} \, d\zeta
\]

is analytic.