Lecture 6 (January 26, 2009) -

Rational functions

If $f(z) = P(z)/Q(z) = \frac{a_0+a_1z+\ldots+a_nz^n}{b_0+b_1z+\ldots+b_mz^m}$ with $n = \deg P$ and $m = \deg Q$.

Fundamental Theorem of Algebra

If $P(z)$ is a complex polynomial of degree >0 then P has a root in $\mathbb{C}$ ($\exists\alpha\in\mathbb{C}$ s.t. $P(\alpha)=0$). Hence, we can factor

$$P(z) = (z-\alpha)g(z),$$

where g is a polynomial of degree $\deg P - 1$. Inductively, we obtain a factorization

$$P(z) = c\prod_{i=1}^{n}(z-\alpha_i) \ \ (c\in\mathbb{C}) .$$

The *multiplicity* of a root α of P is the number of α_i's which equal α_i. In fact,

$$\#\{p^{-1}(w)\} = \deg P$$

for any $w\in\mathbb{C}$,where preimages are counted with multiplicities.

Basic fact. A rational function extends continuously to a function $f:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ where $\widehat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$. This extension for $f = P/Q$ is given by $f(z_0)=\infty$ if $Q(z_0)=0$ and $f(\infty) = \lim\limits_{z\to\infty} f(z)$. We define $R(z) = f\left(\frac{1}{z}\right)$ for $z\neq 0$, and this will be a rational function in $\frac{1}{z}$:

$$R(z) = \frac{\sum a_i\left(\frac{1}{z}\right)^i}{\sum b_j\left(\frac{1}{z}\right)^j} = \frac{z^m\cdot(\text{poly in } z)}{z^n\cdot(\text{poly in } z)}.$$

Set

$$f(\infty) = R(0) = \begin{cases} 0 & \text{if } m>n \\ \infty & \text{if } m<n \\ a_n/b_m & \text{if } m=n. \end{cases}$$

We can also compute the multiplicity of f as a zero, or a pole, or a preimage of a_n/b_m. In conclusion, in $\mathbb{C}$, f has exactly d zeroes and d poles, counted with multiplicity (with $d = \deg f = \max\{n,m\}$).

Examples. If $\deg f = 1$, then it is of the form

$$f(z) = \frac{az+b}{cz+d} \text{ with } a,b,c,d\in\mathbb{C} \text{ and } ad-bc\neq 0.$$

It defines a homeomorphism of $\widehat{\mathbb{C}}$ to itself. (for example, $f(z) = 1/z$ rotates the Riemann sphere around the x_1 axis). **Exercise.** What does $f(z) = z^2$ do to the Riemann sphere?

Ahlfors Ch 2. §1.4

What is the general form of a rational function f with $|f(z)| = 1\ \forall z$ with $|z|=1$? What is the relation between f's set of zeroes and its set of poles?

Consider such a function with deg 1: $f(z) = e^{i\theta} z$ (rotation) or $f(z) = 1/z$. We also saw in class we can have $f(z) = \frac{a-z}{1-\bar{a}z}$ with $|a| \neq 1$. **Exercise.** Show this satisfies the above condition.

For deg > 1, take for example z^n for $n > 1$.

Now, taking the product:

$$f(z) = e^{i\theta} z^n \prod_{i=1}^{k} \frac{a-z}{1-\bar{a}z},$$

we claim this is *all* such functions (classifying functions that preserve the unit circle, $f(\{z : |z| = 1\})) = \{z : |z| = 1\}$).

Partial fraction expansion

Start with $f(z) = P(z)/Q(z)$. Our goal is to express f as a sum of rational functions, where each sum has ≤ 1 pole:

$$f(z) = P_0(z) + \sum R_i(z)$$

where $P_0(z)$ is a polynomial, and R_i are single-poled rat'l functions.

Examples. (1) If $f(z) = \frac{2z+1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$ with $A, B \in \mathbb{C}$. We then compute

$$A(z-2) + B(z-1) = 2z + 1.$$

This implies $A = -3$ and $B = 5$ (by comparing coefficients of z^1 and $z^0 = 1$).

(2) Let $f(z) = \frac{2z+1}{(z-1)(z-2)^3}$ (deg $P <$ deg Q). Then

$$\frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2} + \frac{D}{(z-2)^3} \text{ with } A, B, C, D \in \mathbb{C}.$$

If $f(z) = P(z)/Q(z)$ when deg $P \geq$ deg Q, then the polynomial term $P_0(z)$ has degree deg $P -$ deg Q. Then

$$f(z) = \frac{z^3+2}{(z-1)^2} = Az + B + \frac{C}{z-1} + \frac{D}{(z-1)^2}$$

and this gives us four equations, and we can solve to show this

$$= z + 2 + \frac{4}{z-1} + \frac{4}{(z-1)^2}.$$

Lecture 12 (February 11, 2009) - Linear Fractional Transformations

Linear Fractional Transformations

Theorem. If a LFT f takes a circle C_1 to a circle C_2 (circle here means circle or a line -- i.e. a circle on the Riemann sphere), then f takes pairs of symmetric points (z, z^*) about C_1 to symmetric points $(f(z), f(z^*))$ about C_2.

Note: Symmetric here for a line means symmetric about the line (image under reflection across the line).

The idea is for z_1, z_2, z_3 distinct points on the circle C_1, then the cross ratio

$$X(z_1, z_2, z_3, z^*) = \overline{X(z_1, z_2, z_3, z)},$$

because if the points are sent to $0, 1, \infty$ then the "circle" will be the line $\mathbb{R}$ and so $z^* = \overline{z}$.

Example. p83 #6 in Ahlfors. Suppose a linear fractional transformation takes a pair of concentric circles to concentric circles. Show that the ratios of the radii are the same.

Solution. The idea is to show the center is taken to the center and ∞ to ∞ (so all we can do is scale).

By pre-composing f with an affine transformation ($az + b$), we can assume the center is at 0, the inner radius is 1, and the outer radius is $R > 1$. Similarly, we can compose the image with some $a'z + b'$ and have center 0, inner radius 1, and outer radius $S > 1$. By further post-composing with S/z if necessary (if they switch), we may also assume that the inner circle is sent to the inner circle (circle of radius 1 goes to circle of radius 1). We then claim that $f(0) = 0$ and $f(\infty) = \infty$. Well, the image of the circle of radius $1/R$ is the circle of radius $1/S$. Continue reflecting circles, getting closer to 0, and they are sent to circles with radii tending to 0. By continuity, $f(0) = 0$. Similarly, reflecting on the Riemann sphere yields $f(\infty) = \infty$. Now, remember $f(z) = \frac{az+b}{cz+d}$ (a LFT), and so $f(0) = 0$ and $f(\infty) = \infty$ will imply $f(z) = \frac{a}{d}z$ and $f(S) = S$ so that f is a rotation and so $R = S$.

Look up **Schwartz Reflection Principle**.

Example. p83 #7 in Ahlfors. Find a LFT which carries $|z| = 1$ and $\left|z - \frac{1}{4}\right| = \frac{1}{4}$ to a pair of concentric circles. What is the ratio of the radius?

Solution. We can assume the inner circle has radius R. We have to find an LFT sending our circles to $\{|z| = 1\} \cup \{|z| = R\}$ and compute R.

Note $X\left(-1, 0, \frac{1}{2}, 1\right) = X(-R, -1, 1, R)$ because the cross product is an invariant under LFT's. So, find LFT such that

$$-1 \mapsto 0, 0 \mapsto 1, 1/2 \mapsto \infty, 1 \mapsto \text{crossproduct},$$

and one such that

$$-R \mapsto 0, -1 \mapsto 1, 1 \mapsto \infty, R \mapsto \text{crossproduct}.$$

The answer is $R = 2 + \sqrt{3}$.

Example. $f(z) = z + \frac{1}{z} = \frac{z^2+1}{z}$. Where is it conformal? What does it do? Well, notice

$$f'(z) = 1 - \tfrac{1}{z^2} = 0 \Leftrightarrow z = \pm 1$$

We defined conformal as derivative does not vanish. So, this one is not. We can easily see that if $z = e^{i\theta}$, then $z + \frac{1}{z} = z + \overline{z}$ (since $\overline{z} = \frac{1}{z}$) $= 2 \operatorname{Re} z = 2 \cos\theta$. In general,

$$|z| = r \Longrightarrow z\overline{z} = r^2 \Longrightarrow \tfrac{1}{z} = \tfrac{\overline{z}}{r^2} \Longrightarrow f(z) = z + \tfrac{\overline{z}}{r^2} = \left(x + \tfrac{y}{r^2}\right) + i\left(y - \tfrac{y}{r^2}\right).$$

This is an ellipse!

Hence, f takes $\mathbb{D}\backslash\{0\}$ or $\mathbb{C}\backslash\overline{\mathbb{D}}$ by conformal homeomorphism to $\mathbb{C}\backslash[-2,2]$.

Lecture 16 (February 20, 2009) -

Let γ be a closed curve in $\mathbb{C}$. The winding number is

$$\eta(\gamma,z)=\frac{1}{2\pi i}\int_\gamma \frac{d\rho}{\rho-z}\in\mathbb{Z} \text{ for } z\notin\gamma.$$

We now look at Ahlfors's proof that $\eta(\gamma,z)$ is constant on connected components of $\mathbb{C}\backslash\gamma$ (as a function of z). It suffices to show that $\eta(\gamma,z)$ is constant along straight paths in $\mathbb{C}\backslash\gamma$. Consider the linear fractional transformation

$$\frac{z-z_0}{z-z_1}.$$

Then $L(z)=\log\left(\frac{z-z_0}{z-z_1}\right)$ is well-defined and analytic for all $z\notin[z_0,z_1]$. Further,

$$L'(z)=\frac{1}{z-z_0}-\frac{1}{z-z_1}$$

so that

$$\int_\gamma\left(\frac{1}{z-z_0}-\frac{1}{z-z_1}\right)=0.$$

If γ is a closed loop on a domain of L, then

$$2\pi i\,\eta(\gamma,z_0)=\int_\gamma\frac{1}{z-z_0}\,dz=\int_\gamma\frac{1}{z-z_1}\,dz=2\pi i\,\eta(\gamma,z_1).$$

Lemma. *If z lies in the unbounded connected component of $\mathbb{C}\backslash\gamma$, then $\eta(\gamma,z)=0$.*

Proof. Choose a disk D containing γ. Then if you take a $z_0\notin D$, then the function $\frac{1}{z-z_0}$ is analytic on D, and hence by Cauchy's Theorem,

$$\int_\gamma\frac{dz}{z-z_0}=0.$$

This is the same as saying $\eta(\gamma,z_0)=0$. By the previous lemma, $\eta(\gamma,z)=0$ for all z in the unbounded component. ☐

Then the goal is to show Cauchy's Integral Formula: if f is analytic on a disk D and γ is a closed curve on D with $z_0\notin\gamma$, then

$$\eta(\gamma,z_0)f(z_0)=\frac{1}{2\pi i}\int_\gamma\frac{f(z)}{z-z_0}\,dz.$$

Theorem 5. [Ahlfors] *Let f be analytic on a disk, except possibly at finitely many points $\zeta_1,\zeta_2,\ldots,\zeta_n\in D$. Assume*

$$\lim_{z\to\zeta_j}(z-\zeta_j)f(z)=0.$$

Then $\int_\gamma f=0$ for any closed loop γ in $D\backslash\{\zeta_1,\ldots,\zeta_n\}$.

Proof. (sketch) It suffices to consider integrals over rectangles. Because $\int_{\partial R}f=0$ if f has no singularities on R (a rectangle), we can consider the following special case. Let R be a square centered at one of the ζ_j's (singularities). We have

$$F(z)=\int_{z_0}^z f$$

with $z \in D \backslash \{\zeta_1, ..., \zeta_n\}$. $\square$

For $z \in \partial R$, $|f(z)| \leq \frac{\varepsilon}{|z-z_0|}$, note $\left|\int_{\partial R} \frac{dz}{z-\zeta_j}\right| \leq \int_{\partial R} \frac{|dz|}{|z-\zeta_j|} \leq \frac{1}{\ell} \cdot 4 \cdot 2\ell \leq 8$. Hence, we can make the integral arbitrarily small,

$$\left|\int_{\partial R} f(z)\, dz\right| \leq \int_{\partial R} |f(z)|\, |dz| \leq \varepsilon \cdot 8.$$

Thus, $\int_{\partial R} f = 0$ for any rectange with boundary avoiding ζ_j.

Cauchy Integral Formula

Fix γ in the disk D and $z_0 \in \gamma$. Consider $F(z) = \frac{f(z)-f(z_0)}{z-z_0}$ on D. Then F is analytic in $D \backslash \{z_0\}$, and

$$\lim_{z \to z_0} (z - z_0) F(z) = 0.$$

By Theorem 5,

$$\int_\gamma F\, dz = \int \frac{f(z)-f(z_0)}{z-z_0}\, dz = \int \frac{f(z)}{z-z_0}\, dz - f(z_0) \int_\gamma \frac{dz}{z-z_0}.$$

Then $f(z_0)\, \eta(\gamma_1 z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-z_0}\, dz$. $\square$

Example. Exercise 2.2 #2 is to compute

$$\int_{|z|=2} \frac{dz}{z^2+1}.$$

We write $\frac{1}{z^2+1} = \frac{A}{z+i} + \frac{B}{z-i} = \frac{i/2}{z+i} + \frac{-i/2}{z-i}$. Hence,

$$\int_{|z|=2} \frac{dz}{z^2+1} = \frac{i}{2} \int_{|z|=2} \frac{dz}{z+i} - \frac{i}{2} \int_{|z|=2} \frac{dz}{z-i} = \frac{i}{2}(2\pi i) - \frac{i}{2}(2\pi i) = 0.$$

Theorem. *If F is analytic on a disk D centered at z_0, and suppose γ is a circle around z_0. Then all derivatives $f^{(n)}$ are analytic on D, and satisfy*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

for any z inside γ (notice $\eta(\gamma, z) = 1$).

Lemma. *Let φ be a continuous function defined on some curve γ. Then*

$$F(z) = \int \frac{\varphi(\zeta)}{\zeta - z}\, dz$$

is analytic.