Lecture 3 (January 16, 2009) - Sheaves and Maps of Sheaves

Maps of sheaves

Recall from last time that if we start with X a topological space and F a presheaf, we can define $|F|$, the etale space. Furthermore, sections of $|F| \xrightarrow{\pi} X$ form a sheaf $F^{\#}$. For all open sets $U \subset X$, $f \in F(U)$ induces a continuous map $U \to |F|$ and thence we have a map

$$\begin{array}{ccc} F(U) & \to & F^{\#}(U) \\ \uparrow & & \uparrow \\ F(V) & \to & F^{\#}(V) \end{array}$$

with $U \subset V$. **Exercise.** Check that this commutes (by using the definition of $F^{\#}$).

So we have a map of $F \to F^{\#}$ presheaves, where $F^{\#}$ is a sheaf.

Lemma. If F is a sheaf and $F^{\#}$, a sheaf of sections of $|F|$, is canonically isomorphic to F itself.

Proof. We need a definition: if $x \in X$ and F is a (pre)-sheaf over X, the *stalk* F_x of F at x is $\lim_{U \ni x} F(U)$, i.e., $\coprod_{x \in U} F(U)/\sim$, where if $x \in U \subset V$ and $f \in F(V)$, then $f \sim f|_U \in F(U)$, i.e., this is $\pi^{-1}(x)$ for $\pi : |F| \to X$, so $f \sim g$ if there is an $x \in W \subset U \cap V$

Theorem. *If G is any sheaf of sets and $\varphi : F \to G$ is a map of presheaves is a map of presheaves, it factors uniquely through $F^{\#}$.*

$$\begin{array}{c} F \xrightarrow{\varphi} G \\ \eta \backslash F^{\#} / \overline{\varphi} \end{array}$$

Proof. First, let's show that φ induces a continuous map $|F| \xrightarrow{\varphi} |G|$. For every U, we have $\varphi_U : F(U) \to G(U)$. Hence,

$$|\varphi_U| : \varphi_U \times \mathrm{id}_U : F(U) \times U \to G(U) \times U.$$

Here we have to check that these maps are compatible with the equivalence relations defining $|F|$, $|G|$. But this follows from $F \to G$ being a map of presheaves. ☐

Lecture 4 (January 21, 2009) - Sheaves and Schemes

Recall from last time that giving a sheaf F is equivalent to giving $|F|$, an etale space. We have $U \hookrightarrow X$ with $|F| \xrightarrow{\pi} X$ and $U \xrightarrow{f} |F|$ (in here, $|F| \Leftrightarrow$ sections of π over U).

Maps of presheaves

This simply states for a map $F \xrightarrow{\phi} G$, then $\forall U \subset V$,

$$\begin{array}{ccc} F(U) & \overset{\phi_U}{\to} & G(U) \\ \uparrow & & \uparrow \\ F(V) & \overset{\phi_V}{\to} & G(V). \end{array}$$

If F and G are presheaves of abelian groups, then ker$(\phi) : U \mapsto \ker \phi_U$, coker$(\phi)$: $U \mapsto$ coker ϕ_U, and Im$(\phi) : U \mapsto$ Im(ϕ_U).

Recall that ker(ϕ) is by definition the pullback of fiber products:

$$\begin{array}{ccc} \ker \phi & \to & 0 \\ \downarrow & & \downarrow \\ A & \overset{\phi}{\longrightarrow} & B \end{array}$$

Similarly (with $\downarrow$ replaced with $\uparrow$, 0 and ker ϕ switched, and ker ϕ replaced with coker ϕ), we can characterize the cokernel if we take $0 \to Q$, coker $\phi \to Q$ and $B \overset{\beta}{\to} Q$.

Furthermore, (1) im $\phi \to B$ is injective, i.e., ker $= 0$. (2) if ϕ factors through any other subobject J of B, then im $\phi \subseteq J$. If $\phi : F \to G$ is a map of sheaves, what are ker ϕ, coker ϕ, and im ϕ in the category of sheaves? Well, (1) presheaf kernel of ϕ is already a sheaf and hence is the sheaf kernel, so ϕ is injective as a map of presheaves if and only if it is injective as a map of sheaves. But the presheaf cokernel need not be a sheaf. Rather, the sheaf cokernel is the sheafification of presheaf cokernel.

Sheaf cokernel of $\phi = 0 \Leftrightarrow \phi$ is an epimorphism in the category of sheaves $\Leftrightarrow$ $|\phi| : |F| \to |G|$ is surjective $\Leftrightarrow$ $\forall x \in X$, $\phi_x : F_x \to G_x$ is surjective.

Example. Let $X = \mathbb{C}\backslash\{0\}$ and let $\mathcal{O} =$ sheaf of holomorphic functions on X, and $\mathcal{O}^* =$ sheaf of non-vanishing functions on X. Take $\mathcal{O} \overset{\exp}{\to} \mathcal{O}^*$, with $U \subset \mathbb{C}^*$, $f \in \mathcal{O}(U) \mapsto \exp(f)$. This is surjective, i.e., $\forall x \in \mathbb{C}^*$ and for all open neighborhoods $x \in U \subset \mathbb{C}^*$ and $g \in \mathcal{O}^*(U)$. Further, $\exists$nbhd V of x with $x \in V \subset U$ and $f \in \mathcal{O}(U)$ such that $\exp(f) = g$. If we take $U = \mathbb{C}^*$ and $g = z$, then $\neg\exists\, f$ on $\mathbb{C}^*$ s.t. $\exp(f) = z$. In other words, $\mathcal{O}(\mathbb{C}^*) \overset{\exp}{\to} \mathcal{O}^*(\mathbb{C}^*)$ so $\mathcal{O} \to \mathcal{O}^*$ has non-zero cokernel as a map of presheavs. This is the starting point for sheaf cohomology.

Example. The fundamental group $\pi_1(\mathbb{C}^*) \cong H_1(\mathbb{C}^*, \mathbb{Z}) \cong \mathbb{Z}$, since we just take paths around the punctured disk (on the Argand diagram).

Definition. We can have an exact sequence of sheaves on X:

$$0 \to F \overset{i}{\to} G \overset{e}{\to} H \to 0$$

i.e., i is injective and is the kernel of e, e is surjective & is the cokernel of i (all in the category of sheaves). But for a given $U \subset X$ open,

$$0 \to F(U) \to G(U) \to H(U) \to \text{"}H^1(U, F)\text{"} \to ...$$

where the last $\to$ need not be surjective.

Schemes

A scheme is going to be a topological space X equipped with a sheaf of rings $\mathcal{O}_X$ such that stalks $\mathcal{O}_{X,x}$ at each $x \in X$ are local rings ("locally ringed space").

Example. If M is a C^∞-manifold and $U \subset M$, let $C^\infty(U)$ be C^∞ functions on U. Then if $x \in M$, $C_x^\infty =$ germs of C^∞ functions on U. If we look at the maximal ideal $m_x \subset C_x^\infty$, then m_x/m_x^2 is related to the tangent space.

Affine schemes

If R is a commutative ring (always with 1), then $\text{Spec}(R)^{\text{sp}} =$ set of prime ideals in R, with the following topology: closed sets are $V(\mathfrak{a})$ with $\mathfrak{a} \subset R$ an ideal and $V(\mathfrak{a}) = \{\mathfrak{b} \mid \mathfrak{b} \text{ prime and } \mathfrak{a} \subset \mathfrak{b}\}$. If $f \in R$, then $V(f) = \{\mathfrak{b} \mid f \in \mathfrak{b}\}$ "f vanishes at $\mathfrak{b}$."

Lemma. (1) $\mathfrak{a}$ and $\mathfrak{b}$ are ideals with $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$,

(2) $V(\sum a_i) = \bigcap V(\mathfrak{a}_i)$.

(3) $V(\mathfrak{a}) \subset V(\mathfrak{b}) \Leftrightarrow \sqrt{\mathfrak{a}} \supset \sqrt{\mathfrak{b}}$ with the radical defined

$$\sqrt{\mathfrak{a}} = \{f \in R \mid \exists n \in \mathbb{N}\, f^n \in \mathfrak{a}\}.$$

(4) $\emptyset$, X are closed with $\emptyset = V(R)$ and $X = V(\{0\})$.

Exercise. Show that $\sqrt{a} = \bigcap_{\substack{\mathfrak{b} \text{ prime} \\ \mathfrak{a} \subset \mathfrak{b}}} \mathfrak{b}$.

Lecture 5 (January 23, 2009) - Spec(R)

Let R be a commutative ring. We write $X = \text{Spec}(R)$ to be the set of prime ideals $\mathfrak{p} \subset R$. Then we have the Zariski topology. A set $U \subset X$ is open means if $U = X \backslash V(\mathfrak{a})$ with $\mathfrak{a} \subset R$ an ideal, then in particular for $f \in R$, $X_f = X \backslash V((f)) = \{\mathfrak{p} \mid f \notin \mathfrak{p}\}$ (where $f \notin \mathfrak{p}$ is equivalent to saying $f \not\mapsto 0$ in $R/\mathfrak{p}$).

Recall localization: if $S \subset R$ is a multiplicative set, then

$$S^{-1}R = \left\{\tfrac{r}{s} \mid r \in R, s \in S\right\} / \tfrac{r}{s} \sim \tfrac{r'}{s'} \text{ if } \exists t \in S \text{ s.t. } ts'r = tsr'.$$

If $f \in R$, then $\frac{r}{f^n} \in R_f = \{f^n \mid n \in \mathbb{N}\}^{-1}R$. Recall that

$$\{\text{primes of } S^{-1}R\} \overset{1-1}{\longleftrightarrow} \{\text{primes of } R \text{ disjoint from } S\}.$$

If $S \subset R \to S^{-1}R$ and $A^* \subset A$ with $R \overset{f}{\to} A$ a ring homomorphism ($f(s) \subset A^*$), then it factors uniquely through localization, $S^{-1}R \overset{\overline{f}}{\to} A$. So, X_f is homeomorphic to $\text{Spec}(R_f)$. If $\mathfrak{a} \subset S^{-1}R$ is an ideal with $V(\mathfrak{a}) = \{\mathfrak{p} \mid \mathfrak{p} \supset \mathfrak{a}\} \overset{1-1}{\longleftrightarrow} \{\text{primes of } S^{-1}R\,/\,\mathfrak{a}\}$ with $R \to \{\text{primes of } S^{-1}R/\mathfrak{a}\}$ and $\{\text{primes of } S^{-1}R/\mathfrak{a}\} \to S^{-1}R/\mathfrak{p}$.

i.e., Primes in R containing $\ker(R \to S^{-1}R/\mathfrak{a})$ correspond with primes in $S^{-1}R$ containing $\mathfrak{a}$.

For $U \subset X$ a general open set, $U = X \setminus V(\mathfrak{a})$, $V(\mathfrak{a}) = \bigcap_{f \in \mathfrak{a}} V((f))$ with $\mathfrak{a} = \sum_{f \in \mathfrak{a}} (f)$ so then $U = \bigcup_{f \in \mathfrak{a}} X_f$. Here, we think of $X_f = \{\mathfrak{p} \mid f \notin \mathfrak{p}\}$ - points where f "does not vanish". Furthermore, $X_f \cap X_g = X_{fg}$. Hence, $fg \notin \mathfrak{p} \Leftrightarrow f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$. In other words, the X_f form a basis of the topology on X. To each X_f, we have the ring R_f, and if $X_f \supset X_g$. We have a ring homomorphism $\varphi : R_f \to R_g$. Why? Certainly have $R \to R_g$, so need that under this map f maps to a unit in R_g. If this is not the case, there is a maximal ideal $m \subset R_g$. Since $f/1 \in m$ that then implies $f \in \varphi^{-1}(m)$ would contradict $X_g \subset X_f$.

Now we have a presheaf defined on the X_f.

(1) $X_f \mapsto R_f$

(2) $X_f \supset X_g$, have the restriction $\frac{r}{f^n} \mapsto \frac{r}{f^n} \in G$.

We want to define "functions" on any open set $U \subset X$ such that r_i & r_j have the same image in R_{f_i}, R_{f_j}.

Definition. $\mathcal{O}_X$ is the sheaf of sections of the étale space associated to $X_f \to R_f$, i.e., sections of $\coprod_{f \in R} X_f \times R_f / \sim$, where $\sim$ is the equivalence relation generated by $\left(x, \frac{r}{f}\right) \sim \left(y, \frac{r'}{g}\right)$. If (1) $x \in X_f$, $y \in X_g$, (2) $x = y$, $X_f \subset X_g$, (3) $\frac{r'}{g} \mapsto \frac{r}{f}$ under $R_g \to R_f$. If $\mathfrak{p} \in X$, i.e. a prime ideal, $\mathcal{O}_{X,\mathfrak{p}}$, the stalk of $\mathcal{O}_X$ at $\mathfrak{p}$, is defined as

$$\mathcal{O}_{X,\mathfrak{p}} = \lim_{p \in X_f} R_f \text{ (a direct limit)}$$

with respect to inclusions $X_f \subset X_g$, i.e., $\frac{r_1}{f_1} \in R_{f_2}$ have some imagine in limit $\Leftrightarrow \exists \mathfrak{p} \in X_g \subset X_{f_1} \cap X_{f_2} = X_{f_1 f_2}$ s.t. $\frac{r_1}{f_1}$ & $\frac{r_2}{f_2}$ have same image in Rng. So then the direct limit above just becomes

Lemma. The map $R \to \prod_{\mathfrak{p}} R_{\mathfrak{p}}$ is injective.

Proof. Ker$(R \to S^{-1}R) = \{r \in R \mid \exists s \in S, rs = 0\} = \{r \in R \mid S \cap \text{Ann}(r) \neq \emptyset\} =$ ideal $\{b \in R \mid ba = 0\}$. If $a \in R, a \neq 0$, then $\exists m$ a maximal ideal such that Ann$(a) \subset m$ (where Ann$(a) \cap (R \setminus m) = \emptyset$) $\equiv a \not\to 0$ in R_m.

And, since if $f \in R$, $\mathfrak{p} \not\ni f$, $R_{\mathfrak{p}} = (R_f)_{\tilde{\mathfrak{p}}}$ (where $\tilde{p}$ is corresponding prime in R_f). We get $R_f \to \prod_{f \notin \mathfrak{p}} R_{\mathfrak{p}}$ is injective. □

Lemma. If $X_f \subset X$ is one of our basic opens, the map $R_f \to \mathcal{O}_X(X_f)$ is an isomorphism.

Proof (in a simple case) We want to show $R \xrightarrow{\sim} \mathcal{O}_X(X) \ni \alpha$ (where $\alpha : X \to |F|$). This implies $\exists$open cover $\{X_{f_i}\}$ of X and $\alpha_i \in R_{f_i}$ induce α and this cover can be taken to be finite and we have $\exists g_i$ such that $\sum_{i=1}^{n} g_i f_i = 1$. Without loss of generality, (1) $\alpha_i = a_i/f_i$, (2) α_1 and α_2 have the same image in $R_{f_1 f_2}$.

(Justification: Consider $X =$ Spec R, which is quasi-compact. Notice Spec $R = \bigcup U_i = X \setminus V(\mathfrak{a}_i) \Leftrightarrow \bigcap V(\mathfrak{a}_i) = \emptyset \Leftrightarrow \sum \mathfrak{a}_i = R \;\Leftrightarrow \exists \alpha_i \in \mathfrak{a}_i \; \sum \alpha_i = 1$ and this is a finite sum $\Rightarrow \exists$ finite subset of $\{\mathfrak{a}_i\}$ s.t. $X = \bigcup_i U_i$.)

Now, set $a = \sum g_i a_i \in R$. Then $f_j a = \sum g_i a_i f_j = \sum g_i f_i a_j = (1) a_j$. This implies $\frac{a}{1} = \frac{a_j}{f_j} \in R_{f_j}$. i.e., the α_i determine a unique $a \in R$ and $R \xrightarrow{\sim} \mathcal{O}_X(X)$.

Lecture 6 (January 26, 2009) - Locally Ring Topological Spaces, and Schemes

Answering homework questions

Spec$(R) = \left(\text{Spec}(R)^{\text{space}}, \mathcal{O}_{\text{spec}(R)\text{ [sheaf of rings]}}\right)$. Then $X_f = \text{Spec}(f^{-1}R) \rightsquigarrow f^{-1}R$. Then

$$\text{Basic Opens } \subset \text{All opens}$$

by taking $X_f \mapsto f^{-1}R$.

Spec R

Let $R \to$ Spec(R) be a locally ringed topological space, i.e., $\mathcal{O}_{\text{spec}(R)}$ is a sheaf of rings such that all stalks are local rings (a stalk at $\mathfrak{P}$ is $R_{\mathfrak{P}}$).

If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally ringed topological spaces, then a morphism $\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of locally ringed topological spaces consists of

(1) a continuous map $\varphi : X \to Y$,

(2) a homomorphism of sheaves of rings $\varphi^{\#} : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$.

If $\mathcal{F}$ is a (pre)sheaf on X, with $\varphi_* \mathcal{F} : U \mapsto \mathcal{F}(g^{-1}(U))$ $(U \subset Y)$ (**Exercise.** If $\mathcal{F}$ is a sheaf then so is $\varphi_* \mathcal{F}$.) So $\varphi^{\#}$ is equivalent to: $\forall U \subset Y$, a ring homomorphism

$$\varphi^{\#}_U : \mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U)).$$

compatible with restriction maps. For all $x \in X$, this induces maps

$$\varphi^{\#} : \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$$

s.t. $\left(\varphi^{\#}\right)^{-1}(\mathfrak{m}_x) = \mathfrak{m}_{\varphi(x)}$.

Then $x \in \varphi^{-1}(U)$ if and only if $\varphi(x) \in U$.

$$\mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U)) \to \mathcal{O}_{X,x}.$$

$[\mathcal{O}_{Y,f(x)} = \lim_{U \ni f(x)} \mathcal{O}_Y(U)]$

[For local rings $(R, \mathfrak{m}), (S, \mathfrak{n})$ a local homomorphism ψ between them is one such that $\psi^{-1}(n) = \mathfrak{m}$.]

Theorem. *To give a ring homomorphism $f : A \to B$ is equivalent to giving a morphism of locally ringed spaces,*

$$\text{Spec}(B) \to \text{Spec}(A).$$

Proof. Given $f : A \to B$, we get $\text{Spec}(f) : X = \text{Spec}(B) \to Y = \text{Spec}(A)$ such that if $\mathfrak{p} \subset B$ is prime, then $\text{Spec}(f)(\mathfrak{p}) = f^{-1}(\mathfrak{p}) = \ker(A \to B/\mathfrak{p})$.

[Exercise. The inverse image of an open set is open [so that img of closed set is closed]. Hence, you can do this for just the basis of the topology, i.e., one of these X_f's. Then if $g \in A$, we have $\text{Spec}(f)^{-1}(Y_g)$.]

Recall the relationship between Y_g and primes!

$$X_{f(g)} : g^{-1}A \to f(g)^{-1}B.$$

For $\mathfrak{p} \in B$ prime, the induced local homomorphism $A_{f^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$.

On the other hand, given $\varphi : X = \text{Spec}(B) \to Y = \text{Spec}(A)$, we get a ring hom

$$\varphi^{\#} : A = \mathcal{O}_Y(Y) \to \mathcal{O}_X(g^{-1}(Y)) = \mathcal{O}_X(X) = B.$$

These two functions are mutually inverse (we claim): start with $f : A \to B$. Then since $\varphi_{\text{Spec}(A)}(\text{Spec}(A)) = A$, the induced map $\text{Spec}(f)^{\#} : A \to B$ is equal to f. Given $\varphi : \text{Spec}(B) \to \text{Spec}(A)$, we get a ring homomorphism $\varphi^{\#} : A \to B$ and claim $\text{Spec}\left(\varphi^{\#}\right) = g$. Suppose $\mathfrak{p} \in \text{Spec}(B)$. Then $\varphi(B) \in \text{Spec}(A)$.

$$\begin{array}{ccc} \mathcal{O}_Y(Y) = A & \xrightarrow{\varphi^{\#}} & B = \mathcal{O}_X(X) \\ \downarrow & & \downarrow \\ \mathcal{O}_{Y,\varphi_B} = A_{\varphi(B)} & \xrightarrow{\varphi^{\#}_{\mathfrak{p}}} & B_{\mathfrak{p}} = \mathcal{O}_{X,\mathfrak{p}}, \end{array}$$

where the subscripts of the rings (bottom row) mean localization. Since $\varphi^{\#}_{\mathfrak{p}}$ is a local homomorphism, $\varphi_{\mathfrak{p}}^{-1}(\mathfrak{p}\, B_{\mathfrak{p}}) = (\mathfrak{a} A_{\mathfrak{a}}) \implies \mathfrak{p} = \mathfrak{a}$. ☐

This gives us an equivalence

$$\{\text{Rings}\} \longleftrightarrow \{\text{Affine schemes}\}^{\text{op}}$$

$$R \longleftrightarrow \text{Spec}(R).$$

Definition. *A scheme is a locally ringed topological space $(X, \mathcal{O}_X)$ such that X has an open cover $X = \bigcup_\alpha U_\alpha$ such that each $(U_\alpha,\ \mathcal{O}_X|_{U_\alpha})$ is an affine scheme.*

Examples. [of affine schemes] (1) Consider $\text{Spec}(\mathbb{Z})$. This is just prime ideals (i.e., (p) for p a prime number). For $n \in \mathbb{Z}$, $V((n)) = \{\text{prime ideals } \mathfrak{p}, n \in \mathfrak{p}\}$. Then if $n \neq 0$, then this is a finite set of prime divisors of n. If $n = 0$, then it's just everything. The point (0) is not closed, rather, $\overline{\{(0)\}} = \text{Spec}(\mathbb{Z})$ is a "generic point" (i.e., a single point in this topological space whose Zariski closure is the whole space).

Lecture 6 (January 26, 2009) - More Schemes

Schemes

Recall these are $(X, \mathcal{O}_X)$ locally ringed spaces with an open cover $X = \bigcup_i U_i = \text{Spec}(R_i) \implies \supset \text{Spec}(g^{-1}R_i)$, with each $(U, \mathcal{O}_X|U_i)$ by affine schemes.

Definition. *Morphisms between schemes. A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ with $(X, \mathcal{O}_X) \supset f^{-1}(U)$ and $U = \text{Spec}(R) \subset (Y, \mathcal{O}_Y)$, and*

$$\begin{array}{ccc} f^{-1}(U) & \to & U = \text{Spec}(R) \\ \uparrow & \nearrow & \\ V_i & & \end{array}$$

with $f^{-1}(U) = \bigcup_j V_j = \text{Spec}(S_j)$. We thus get a map of locally ringed spaces,

$$V_j = \text{Spec}(S_j) \to \text{Spec}(R) = U.$$

Hence, we have ring homomorphism $R \to S_j$. Then $X = \bigcup \text{Spec}(S_j) = V_j$, and $Y = \bigcup U_i = \bigcup \text{Spec}(R_i)$ s.t. $\forall j, f(V_j) \subset U_i$ for some i, $S_j \leftarrow R_i$. We want maps that coincide on $V_{j_1} \cap V_{j_2} = \bigcup W_k = \text{Spec}(A_k)$, i.e.

$$R_i \to S_{j_r} \to W_k$$

commutes ($r = 1, 2$).

Examples. (1) If $f : X \to \text{Spec}(A)$, where A is some ring (e.g., a field). Then for all affine opens, $U \subset X, U = \text{Spec}(R)$, we have a ring homomorphism $f^* : A \to R$, and if $U_1 \subset U_2$, then $\text{Spec}(R_1) \subset \text{Spec}(R_2) \to \text{Spec}(R)$, so $R_1 \leftarrow R_2 \leftarrow A$. This is equivalent to saying that $\mathcal{O}_X$ is a sheaf of A-algebras (i.e., $\forall U \subset X$ open, we have a ring homomorphism $A \to \mathcal{O}_X(U)$ compatible with restriction.

(2) Have $f : T = \text{Spec}(A) \to X = \bigcup_i \text{Spec}(S_i) = \bigcup U_i$. Then $f^{-1}(U) \subset \text{Spec}(A)$ is open with $f^{-1}(U) = \bigcup_{\text{some collection of } f \in A} T_f$, so we have $f^{-1}A \leftarrow S_i$.

If k is a field, for simplicity let k algebraically closed, then $\mathbb{A}^1_{k,t} = \text{Spec}(k[t])$.

(a) Consider $\{(t-a) \mid a \in k\}$. Each $(t-a)$ is closed. Further, $\overline{\{(0)\}}$ is everything. Finally, proper closed subsets are $V(\mathfrak{a})$, with $\mathfrak{a} \neq 0$. Hence this is just

$$\begin{aligned} V((\mathfrak{a})) = V((f)) &= \{(p_i) \mid p_i \text{ is irreducible factor of } f\} \\ &= \text{finite set of points: } t - a_i \text{ if } a_i\text{'s are zeroes of } f. \end{aligned}$$

If $X \to \text{Spec}(k)$ is a scheme over k, then $X_1 \xrightarrow{g} X_2$ with $X_1 \xrightarrow{f_1} S$ and $X_2 \xrightarrow{f_2} S$ (on a category $\mathcal{C}$ over S), then $(x_1, f_1) \to (x_2, f_2)$ with $g : X_1 \to X_2$ such that $f_2 g = f_1$. Then for every k-algebra R, look at maps of schemes over k, $\text{Spec}(R) \to X$. We write $X(R)$ for this set, and we call this the "R-valued points of X (over k)."

R-valued points v.s. points of the scheme

Let $\mathbb{A}^1_k(R)$ for R a k-algebra be such that homomorphisms of k-algebras: $k[t] \to R$. Note that $\mathbb{A}^1_k(R) \equiv$ Hom of k-algebras $k[t] \to R$ ($f \leftrightarrow f(t)) \equiv R$. So if $R = k$ itself, $\mathbb{A}^1_k(k) \cong k$.

If k is not algebraically closed, e.g. $k = \mathbb{R}$ or $k = \Gamma_p$, the points $\mathbb{A}^1_k \cong$ generic point $\cup$ set of monic irreducible polynomials in $k[t]$.

While (1) $\mathbb{A}^1_k(k) = k$ (corresponds to linear polys $\{t - a \mid a \in k\}$). (2) If K/k is a finite extension, then $\mathbb{A}^1_k(k) \cong k$ with $(\varphi : k[t] \to k) \to \varphi(t)$. If K/k is finite, then ker$(\varphi) = (p(t))$, with p an irreducible monic polynomial. Further, $k[t]/p(t)$ is a field (and in fact $k \subset k[t]/p(t)$). If φ : Spec$(k) \to X$, k a field, then to get a continuous map we need $(*, k) \mapsto x \in X$, and if $x \in$ Spec$(A) \subset X$, then we need $A \overset{\varphi^*}{\to} k$ with ker(φ^*) a prime ideal in A equal to x. (So we have maps $X(k) \to$ points of X). In the general situation, $X(k) \to X^{\text{sp}}$ (as a top space) with $(\varphi$: Spec $k \to x) \mapsto x \in X$ $(K \leftarrow k(x))$. Then $X(k) \leftrightarrow (x \in X,\, k(x) \hookrightarrow k)$. For the affine line,

$$K = \mathbb{A}^1(k) \to \text{set of monic irreducibles in } k[t]$$

and $\mathbb{A}^1(k) \longleftrightarrow (K \hookleftarrow k[t]/p(t)) = k((p))$.

What if $[K : k] > 1$? e.g.in $k = \mathbb{R}$ and $K = \mathbb{C}$,

$$\mathbb{A}^1_{\mathbb{R}} = \text{Spec}\, \mathbb{R}[t] = \{0\} \cup \{(t - a) \mid a \in \mathbb{R}\} \cup \{\text{monic irred quadratic}\}.$$

Then $\mathbb{A}^1_{\mathbb{R}}(\mathbb{R}) \leftrightarrow \{(t - a) \mid a \in \mathbb{R}\}$ and $\mathbb{C} \leftrightarrow \mathbb{A}^1_{\mathbb{R}}(\mathbb{C}) = \mathbb{C}$-valued points of $A^1_{\mathbb{R}}$. Further,

$$t \mapsto \alpha \ \ (q = (t - \alpha)(t - \overline{\alpha})),\ \ \mathbb{C} \underset{\leftarrow}{\leftarrow} \mathbb{K}((q)).$$

Grothendieck's EGA: Want to study solutions of polynomial equations over a field or ring k:

$$\begin{gathered} f_1(t_1, ..., t_n) = 0 \\ \vdots \\ f_m(t_1, ..., t_n) = 0. \end{gathered}$$

Consider the functor

$$k\text{-algebras} \to \text{Sets}.$$

$$R \mapsto \text{set of } n\text{-tuples } (r_1, ..., r_n) \in R^n$$

satisfying the equations

$$\equiv k[t_1, ..., t_n]/(f_1, ..., f_m) \to R$$

which is equivalent to giving a homomorphism of k-algebras. But that is equivalent to giving Spec( ) $\leftarrow$ Spec$(\mathbb{R})$.

Lecture 8 (January 30, 2009) -

Given a scheme X and R a ring, we have $X(R) = R$-valued points of $X =$ morphisms Spec$(R) \to X$. e.g., If $X = \mathbb{A}^n_{\mathbb{Z}} =$ Spec$(\mathbb{Z}[t_1, ..., t_n])$ and $X(R) =$

Then $X =$ Spec(A) and $X(R) =$ ring hom $A \to R$.

$X(R) =$ Ring Homs $\mathbb{Z}[t_1, ..., t_n] \to R \equiv R^n(\varphi \mapsto (\varphi(t_1), ..., \varphi(t_n)))$. Recognize that $R \to X(R)$ is a functor from rings to sets.

If $f : R \to S$ is a ring homomorphism, this induces a map

$$X(R) \to X(S)$$
$$(\varphi : \text{Spec}\, R \to X) \mapsto (\varphi \circ \text{Spec}(f)).$$

Question. Given a functor from rings to sets (or if k is a field, k-algebras $\to$ sets), we can ask if there is a scheme X s.t. $F(R) = X(R)$. We say "F is represented by X."

For example, we could take $F : R \to R^\times$ (the group of units). In fact, this is represented by the affine scheme:

A homomorphism $\mathbb{Z}[x,y]/(xy = 1) \to R$ is the same as giving elements $r = \varphi(x), s = \varphi(y)$ s.t $rs = \varphi(xy) = 1$ (i.e. giving a unit $r \in R^*$).

That is, $\mathbb{Z}\left[x, \frac{1}{x}\right]$, or $\{x^n \mid n \in \mathbb{N}\}^{-1}\mathbb{Z}[x]$.

We have shown that $R \to R^\times$ is represented by Spec$(\mathbb{Z}[t, t^{-1}])$, denoted $\mathbb{G}_m$. We have a group operation, $\mu : R^\times \times R^\times \to R^\times, 1_R \in R^\times$, and some axioms, like, $\forall r, s, t \in R^\times,\ \mu(r, \mu(s,t)) = \mu(\mu(r,s), t)$.

How can we get such an operation μ? If X, Y are two affine schemes, "Spec(A)" and "Spec(B)", then $X(R) \times Y(R) =$ pairs of homomorphisms $(A \to R, B \to R)$, which is the same as giving a homomorphism $(A \otimes B \to R)$: Spec $A \times$ Spec $B =$ Spec $A \otimes B$. For example,

$$\mathbb{A}^n_{\mathbb{Z}} \times \mathbb{A}^m_{\mathbb{Z}} \cong \mathbb{A}^{n+m}_{\mathbb{Z}} \text{ because}$$
$$\text{Spec}(\mathbb{Z}[t_1, ..., t_n]) \times \text{Spec}([\mathbb{Z}[u_1, ..., u_m]]) = \text{Spec}(\mathbb{Z}[t_i] \otimes \mathbb{Z}[u_i]) = \text{Spec}(\mathbb{Z}[t_i, u_i]).$$

Exercise. Verify the previous statement.

Example. The functor $R \to R^\times \times R^\times$ is represented by Spec$(\mathbb{Z}[t, t^{-1}] \otimes \mathbb{Z}[u, u^{-1}])$. The map $\mu : R^\times \times R^\times \to R^\times$ is a natural transformation. (**Verify this!**)

$$\begin{array}{ccc} \text{Spec}(\mathbb{Z}[t, u, t^{-1}, u^{-1}]) & \overset{x \mapsto tu}{\longleftarrow} & \text{Spec}(\mathbb{Z}[x, x^{-1}]) \\ \downarrow \text{ (by } t \mapsto r, u \mapsto s \text{ w/ } r, s \in R^\times) & & \\ R & & . \end{array}$$

With Spec$(\mathbb{Z}[x, x^{-1}]) \to R$ ($x \mapsto rs$). Hence, we have

$$\mathbb{Z}[t, u, t^{-1}, u^{-1}] \leftarrow \mathbb{Z}[x, x^{-1}]$$
$$\mu : \mathbb{G}^t_m \times \mathbb{G}^n_m \to \mathbb{G}^x_m,$$

with $1 : \text{Spec}\, \mathbb{Z} \to \mathbb{G}_m$ with $t \mapsto 1$. We can also give an associative law:

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m & \overset{1 \times \mu}{\longrightarrow} & \mathbb{G}_m \times \mathbb{G}_m \\ \downarrow \mu \times 1 & & \downarrow \mu \\ \mathbb{G}_m \times \mathbb{G}_m & \underset{\mu}{\longrightarrow} & \mathbb{G}_m. \end{array}$$

In general, a **group scheme** is a scheme G, together with maps

$$\mu : G \times G \to G,\ 1 : \text{Spec}\, \mathbb{Z} \to G$$

such that the associative law, identity, inverses hold (with identity given by

$$\text{Spec}\, \mathbb{Z} \times \mathbb{G}^{(1,\text{id})} \to \mathbb{G} \times \mathbb{G} \overset{\mu}{\to} \mathbb{G}).$$

Exercise. What is the correct diagram to express the existence of inverses?

It is denoted $\mathbb{G}_m$ for multiplicativity. You can also have $\mathbb{G}_a$ as an additive group ($R \to R$) (indeed, we can give $\mathbb{G}_a = \text{Spec}(\mathbb{Z}[x])$ by $\sigma : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$ through $x \mapsto x \otimes 1 + 1 \otimes x$).

An elliptic curve is an example of a group scheme!

Example. If $E/\mathbb{C}$ is an elliptic curve, then the addition law on E corresponds to E being a group scheme over $\mathbb{C}$: $E \times E \to \mathbb{C}$.

Example. If R is a ring, a finitely generated projective module P over R is one a direct summand of a free module ($\exists Q$ s.t. $P \oplus Q$ is free). Fix an integer $n \geq 2$. If R is a local ring, then any finitely generated projective module is free (P is projective iff $\forall p$ prime ideals in $R,$ the localization P_p is free). Then that implies there exists a function $\text{Spec}(R) \to \mathbb{N}$ which takes $x \mapsto \text{Rank}(P_x)$. (**Exercise.** Figure out why this is related to Cartier divisors (Henri claims it is)). Now, (with $n \geq 2$) we can look at the set of projective rank 1 quotients of R^n: $R^n \longrightarrow P = Q$ (P proj of rank 1), with $R^n \longrightarrow Q$ as well (where the bold arrows $\longrightarrow$ represent surjection).

Fact. If $f : R \to S$ is a ring homomorphism, then $(R^n \longrightarrow P) \mapsto (S^n \to S \otimes_R P)$ gives a functor from rings to sets. This is represented by: $\mathbb{P}_{\mathbb{Z}}^{n-1}$. (next time, we will see two different ways of constructing projective space, and we will see why the previous is true)

Lecture 9 (February 2, 2009) -

How do we express the inverse property of groups with a diagram? We have the fiber product

$$\begin{array}{ccc} P & \to & \text{Spec}(k) \\ \downarrow & & \downarrow e \\ G \times G & \underset{\mu}{\to} & G \end{array} .$$

Then take $P \overset{\varphi}{\to} G$ with $G \times G \overset{\pi_1}{\to} P$, and the condition is that φ is an isomorphism.

Projective modules & projective spaces

If R is a commutative ring with unity, an R-module P is projective if and only if there is an R-module Q such that $P \oplus Q$ is free.

Lemma. *Suppose R is a local ring with maximal ideal $\mathfrak{m}$, residue field k, and P is a finitely generated projective R-module. Then R is free.*

Proof. Since P is finitely generated, this implies that there is a surjective map $R^n \overset{\varphi}{\to} P$, and $R^n \cong P \oplus Q$ with $Q = \ker(\varphi)$.

$$\begin{array}{ccc} R^n & \xrightarrow{\varphi} & P \\ & \iota \searrow & \downarrow \\ & & P. \end{array}$$

If we let $\pi = \iota \circ \varphi : R^n \to R^n$ so $\text{im}(\pi) = \iota(P)$, then (notice $\pi^2 = \pi$)

$$k^n = k \otimes_R R^n \cong (k \otimes_R P) \oplus (k \otimes_R Q).$$

Call $k \otimes_R P = \overline{p}$ and $k \otimes_R Q = \overline{q}$. Then $\overline{p}$ and $\overline{q}$ are f.d. k vector spaces of dimension m and n (respectively).

Since $R \to k$ is surjective and P is projective, P is flat (see commutative algebra). That is, $P - \gg k \otimes P$. Choose bases $\alpha_1, ..., \alpha_m$ of $P \otimes k$ and $\alpha_{m+1}, ..., \alpha_n$ of $Q \otimes k$, and elements $\overline{\alpha}_1, ..., \overline{\alpha}_m \in P, \overline{\alpha}_{m+1}, ..., \overline{\alpha}_n \in Q$ mapping to these basis elements. Hence, we have a map

$$R^n \xrightarrow{A} R^n$$

$$e_i \mapsto \overline{\alpha}_i$$

$$A(R^n) \subset P,\ A(R^{n-m}) \subset Q.$$

The matrix X of A in $M_n(R)$ has image in $M_n(k)$, the matrix of the map

$$k^n \to k^n = (k \otimes P) \oplus (k \otimes Q)$$

$$e_i \mapsto \alpha_i,$$

which is an isomorphism. Hence, the image $\overline{X}$ of X in $M_n(k)$ is invertible if and only if $\det\left(\overline{X}\right) \in k^* \Leftrightarrow \det(X) \notin \mathfrak{m} \Leftrightarrow \det(X) \in R^* \Leftrightarrow X \in GL_n(R)$. Hence, this matrix being invertible implies A is an isomorphism. Hence, $R^m \to P$ is an isomorphism.

If we look at the proof, suppose that R is no longer local. Then P is still finitely generated projective as an R-module, and $\mathfrak{p} \in \text{Spec}(R)$ is a prime ideal. By the lemma, $P_{\mathfrak{p}} = R_{\mathfrak{p}} \otimes_R P = (R \setminus p)^{-1}\mathfrak{p}$ is free. Thus, there exist elements, $\alpha_1, ..., \alpha_m \in P_{\mathfrak{p}}$ which form a basis. That is, $\alpha_i = a_i / f_i$ with $f_i \in R \setminus \mathfrak{p}$. We get a map

$$\varphi : (f_1 ... f_m)^{-1} R^n \to (f_1 ... f_m)^{-1} P^n$$

$$e_i \mapsto \alpha_i.$$

We know that if we localize at $\mathfrak{p}$, this is an isomorphism.

(**Exercise.** If A is a Noetherian commutative ring, and $\varphi : M \to N$ is a homomorphism of R-modules, and there exists $\mathfrak{p} \in \text{Spec}(A)$ such that $\varphi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is an isomorphism, then $\exists g \notin \mathfrak{p}$ such that $M\left[\frac{1}{g}\right] \to N\left[\frac{1}{g}\right]$ is an isomorphism of $g^{-1}A$-modules.)

Hence, φ induces an isomorphism from $(g^{-1})R^n \to (g^{-1})P$ for some $g \notin \mathfrak{p}$. Now, a finitely generated projective R-module P is "locally free" if $X = \bigcup X_{f_i} = \text{Spec}(f_i^{-1})R$ such that $f_i^{-1}P$ is free (where X is $\text{Spec}(R)$). □

Question. Fix $n \geq 1$. Consider the functor

$$P_n : \text{Rings} \to \text{Sets}$$
$$R \mapsto \text{Rank 1 projective quotientso f } R^{n+1}.$$

i.e., equivalence classes of surjective maps $R^{n+1} \to P$ with P projective, $P_{\mathfrak{p}}$ free of rank 1 for all $p \in \text{Spec}(R)$ (so that $R^{n+1} \to P$ and $R^{n+1} \to P'$ are equivalent if and only if they have the same kernel).

Example. Let k be a field, and have $P^n(k) : k^{n+1} \to L$ with L a one-dimensional vector space. For each basis element $\ell \in L$, $L \cong k$ through $1 \mapsto k$ gives us a matrix for $\varphi : \overline{a} = [a_0, ..., a_n]$. Since φ is surjective, not all the φ are zero. A different choice of ℓ : $\ell^1 = \lambda\ell$ ($\ell \in k^*$) w/ matrices $[a_0, ..., a_n']$ with respect to ℓ is $\lambda^{\pm}$ times a'. If R is a local ring, $P^n(R) = n + 1$ tuples, $\overline{a} = [a_0, ..., a_n]$ such that not all $a_i \in \mathfrak{m}$ which is "at least one is a unit" modulo by R^*.

If A is a general commutative ring, L need not be free! However, $X = \text{Spec}(A) = \bigcup X_{g_i} = \text{Spec}(g_i^{-1}A) = g^{-1}L$, and g_i^{-1} is free. So the map $g_i^{-1}R^{n+1} \to L$ is represented by a vector $\underline{a} = [a_0, ..., a_n]$ unique to a unit in $g_i^{-1}R$. We can also assume that at least one of the a_i is a unit.

Next time, we will show there is a scheme $\mathbb{P}^n_{\mathbb{Z}}$ (not affine) which "represents" P^n, i.e., there is a map $\text{Spec}(A) \to \mathbb{P}^n_{\mathbb{Z}}$ with $\text{Spec}(A) \leftrightarrow$ giving a rank 1 projective quotient of A^{n+1}.

Lecture 11 (February 6, 2009) -

Examples. (2) In particular, if k is a field,

$$\textstyle\prod_k^r(k) = (k^{n+1} \setminus \{0\}),\ k^* = \bigcup_{i=0}^n \mathbb{A}^n(k) = \{(a_0, ..., a_{i-1}, 1, a_{i+1}, ..., a_n) \mid a_i \in k\}.$$

(3) $\mathbb{P}^n_k = \left(\mathbb{A}^{n+1}_k \setminus \{0\}\right) / \mathbb{G}_{m,k}$

(4) $\mathbb{P}^n_R = \bigcup_{i=0}^{n+1} \mathbb{A}^n_R / \sim$ (where we will specify the equivalence class $\sim$).

Recall $\mathbb{G}_{m,k} = \text{Spec}(k[t,\ t^{-1}])$. This is an affine group scheme. $\mathbb{G}_{m,k}(R) = R^*$.

Review. Group actions. Classically, for G a group and X a set, a group action is $\mu : G \times X \to X$, $(g, x) \mapsto gx$ s.t. $g(hx) = (gh)x\ \forall g, h$ and $e \cdot x = x\ \forall x$.

For G a group object in a category $\mathcal{C}$, an action of G on an object X is a map

$$\rho : G \times X \to X \text{ such that (1) } \begin{array}{ccc} G \times G \times X & \xrightarrow{1_G \times \rho} & G \times X \\ \mu \times 1 \downarrow & & \downarrow \rho \\ G \times X & \xrightarrow{\rho} & X \end{array}$$

$$\text{(2) } X \xrightarrow{e \times 1_X} G \times X \xrightarrow{\rho} X.$$

Exercise. This is equivalent to giving, for any object $T \in \mathcal{C}$, an action of the group $G(T)$ (morphism $T \to G$) on $X(T)$ compatible with maps $T' \to T$.

Fact. A scheme X is determined by the functor $R \to X(R)$ (with R a ring).

Remark. In any category $\mathcal{C}$, an object X is completely determined by the contravariant functor $h_X : T \to \text{Hom}(T, X)$. Given X, Y, we have the functors h_X and h_Y. Those are functors $\mathcal{C} \to$ Sets. Then by the Yoneda lemma (natural transformations $h_X \to h_Y$ are in one-to-one correspondence with maps $X \to Y$), we look at X or the functor h_X represented by X are the same thing.

Actions of $\mathbb{G}_m$ on "other things"

If $X = \text{Spec}(A)$ is an affine scheme (over $\mathbb{Z}$), then an action of $\mathbb{G}_m$ on X is a map $\rho : \mathbb{G}_m \times X \to X$. Notice giving $\mathbb{G}_m \times X$ is equivalent to giving $\text{Spec}(\mathbb{Z}[t, t^{-1}] \otimes A)$ and to give X is equivalent to giving $\text{Spec}(A)$. Hence,

$$A \overset{\rho^*}{\to} A[t, t^{-1}] = \mathbb{Z}[t, t^{-1}]$$

$$a \mapsto \sum_{i=-\infty}^{\infty} \rho_i(a) t^i,$$

where $\rho^*(ab) = \sum \rho_i(ab)\, t^i$ and (if ρ is a ring homomorphism)

$$\rho^*(a)\rho^*(b) = \sum \rho_j(a) t^i \sum \rho_k(b)\, t^b = \sum_{i=-\infty}^{\infty} \sum_{j+k=i} [\rho_j(a)\rho_k(b)] t^i.$$

If ρ is an action, we have "reversal of the diagram," thati s

$$\begin{array}{ccc} A[x, x^{-1}, y, y^{-1}] & \leftarrow & A[t, t^{-1}] \\ \uparrow & & \uparrow \\ A[t, t^{-1}] & \longleftarrow & A \end{array}$$

Where the maps are $\sum a_i t^i \mapsto \sum_{i,j} \rho_j(a) x^j y^i$, $a \mapsto \sum \rho_j(0) t^i, a \mapsto \sum \rho_k(a) t^k,$ and $\sum p_k(a) t^k \mapsto \sum_k p_k(a)(xy)^k$, given in clockwise order starting from the top. Thus

$$\rho_j(a_i) = \begin{cases} 0 & j \neq i \\ a_i & j = i. \end{cases}$$

Exercise. The fact that $e \in \mathbb{G}_m(\mathbb{Z})$ acts as identity so $\sum_{j=-\infty}^{\infty} e_j(a) = a$, i.e., the map $a \mapsto \sum e_j(a)$ is a grading.

In conclusion, we can say

Proposition. *An action of $\mathbb{G}_m$ on an affine scheme $X = \text{Spec}(A)$ is the same as a $\mathbb{Z}$-grading of the ring A.*

Remark. An algebraic action of $\mathbb{C}^*$ on a $\mathbb{C}$-vector space $V$is simply a grading of V, that is, $\bigoplus_{i \in \mathbb{Z}} V_i$ where V_i is an eigenspace (λ acts by λ^i).

Lecture 12 (February 9, 2009) -

Recall the action of $\mathbb{G}_m = \text{Spec}\, k[t, t^{-1}]$ on $\text{Spec}(A) \equiv$ grading of A, e.g., $A = k[t_0, \ldots, t_n]$ (obvious grading $(\lambda, P) \mapsto (\lambda a_0 - \lambda a_n) = \lambda P, x_i \mapsto t x_i)$

Recall $\mathbb{P}^n(k) = (k^{n+1} \backslash \{0\})/k^*$.

In general, let S be a graded ring, i.e., $\mathbb{G}_m$ acts on Spec(S).

We can ask, what does it mean *scheme-theoretically* to delete $\{0\}$? The origin corresponds to the ideal $(x_0, ..., x_n)$. In general, the complement in Spec(A) of $V((f_1, ..., f_n)) = \{\mathfrak{p} \,|\, (f_1, ..., f_n) \subset \mathfrak{p}\}$ (but this last set just says $\forall i,\ f_i \in \mathfrak{p}$). Hence,

$$X \setminus V((f_1, ..., f_n)) = \{\mathfrak{p} \,|\, \exists i\, f_i \notin \mathfrak{p}\} = \bigcup_i \{\mathfrak{p} \,|\, f_i \notin \mathfrak{p}\} = \bigcup_i \text{Spec}(A)_{f_i},$$

where the localization $\text{Spec}(A)_{f_i} = \text{Spec}(f^{-1}A)$.

So $\mathbb{A}^n \backslash \{0\} = \bigcup \text{Spec}(k[x_0, ..., x_i, 1/x_i, ..., x_n])$. This is the set of points $\mathfrak{p}$ such that at least one X_i is a unit in $k[x_0, ..., x_n]_{\mathfrak{p}}$.

<u>Notice</u> the grading on $k[x_0, ..., x_n]$ extends to each localization -- true whenever we localize with respect to $\{f^n\}/f$ homogeneous. $\equiv$ Action of $\mathbb{G}_m$ on $\mathbb{A}^{n+1}$ induces an action.

Now look at $\mathbb{G}_m$ acting on U_i. We take

Categorical quotient

Let $G \times X \xrightarrow{\lambda} X$ with $(g, x) \mapsto gx$. We also have the second projection $G \times X \to X$. If then $X \to Y$, do these two maps have the same image in Y? This is precisely what it means to be the quotient! So, it is the universal object with the property such that $X \to X/G$ from second projection, $X \to X/G$ from λ, and $X/G \to Y$. Indeed, X/G is universal for maps $f : X \to Y$ s.t. $f \cdot \lambda = f \cdot \text{proj}$, if it exists. For example, $\mathbb{G}_m$ acting on $\mathbb{A}^1_{\mathbb{C}}$ (with $(\lambda, a) \mapsto \lambda a$) are orbits of closed $\mathbb{C}$-rational points--namely the origin, and everything else. This quotient in general is **not** a scheme.

Moduli problems deal with parametrizing isomorphic classes of elliptic curves over $\mathbb{C}$. We can embed any elliptic curves $E \hookrightarrow \mathbb{P}^2(\mathbb{C})$. Then $\text{Aut}(\mathbb{P}^2(\mathbb{C})) \cong PGL_3(\mathbb{C})$ acts on the space of all cubic homogenous polynomials with non-zero discriminant. The space of cubic curves $\cong \mathbb{P}^9$ (there are ten coefficients in a degree 3 homogeneous polynomial). Furthermore, $\Delta \neq 0 \Longrightarrow$ Zariski open subset of $\mathbb{P}^9$ is a quotient by action of PGL_3.

Returning from our digression, $\mathbb{P}^n$ is going to be quotient $(\mathbb{A}^{n+1} \backslash \{0\})/\mathbb{G}_m$. We form this as follows. Have $\mathbb{G}_m$ act on each U_i:

(1) Form the quotient $U_i/\mathbb{G}_m = \text{Spec}(\text{subring of } \mathcal{O}(U_i), \text{ invariant under } \mathbb{G}_m)$. We shall see that in this case the quotient is a "good" object.

(2) Define $\mathbb{P}^n = \bigcup V_i$. This involves constructing a scheme by gluing open subschemes together.

Let S be any graded ring, and let f be a homogeneous element of positive degree. Then $(f^{-1}S)_0 =$ subting of degree zero elements in $f^{-1}S$.

Notice this is still a graded ring. We want to construct the quotient of Spec$(f^{-1}S)$ by $\mathbb{G}_m$. In general, A is a graded ring: $\mathbb{G}_m \times \text{Spec}(A) \rightrightarrows \text{Spec}(A)/\mathbb{G}_m = \text{Spec}(A_0)$, given by $\sum a_i t^i \leftarrow \sum a_i$, where $\sum a_i t^i \in A \otimes \mathbb{Z}[t, t^{-1}] = A[t, t^{-1}]$.

So, take Spec$(f^{-1}S)$.

Proposition. There is a one-to-one correspondence between prime ideals in $(f^{-1}S)_0$ and prime ideals in $f^{-1}S$ which are "invariant under the action of $\mathbb{G}_m$".

What does "invariant under the action" mean? Well, $V \subset \mathbb{A}^{n+1}$ invariant under $k^* \Leftrightarrow$ it's a cone $\Leftrightarrow$ ideal is homogeneous. In our case, $G \times X \to X$ where $G \times V \to V$ (this factors through V). Hence, "invariant under the action" is saying these are homogeneous prime ideals.

Proof. (of proposition) For simplicity, assume f has degree 1, so that $f^{-1}S$ has a unit of degree 1. Let $\mathfrak{p}$ be a homogeneous prime ideal, with

$$p = \sum p_i \in \mathfrak{p} \Leftrightarrow p_i \in p \,\forall i \Leftrightarrow \mathfrak{p}_i/f_i \in \mathfrak{p}\,\forall i.$$

Hence, $\mathfrak{p}_i/f_i \in (f^{-1}S)_0 \cap \mathfrak{p}$, so hence

$$p \in \mathfrak{p} \Leftrightarrow \text{ it is a sum of elements of the form } f^i q_i \text{ with } q_i \in (f^{-1}S)_0 \cap \mathfrak{p}.$$

Lecture 13 (February 11, 2009) - Graded rings?

Recall $S = \bigoplus_{d \in \mathbb{Z}} S_d$ is a graded ring.

Proposition. If $\exists u \in S_d, d \geq 1$, u a unit, then there is a $1-1$ correspondence between homogeneous prime ideals in S and prime ideals in S_0.

So Spec(S_0) is Spec$(S)/\mathbb{G}_m$.

Now, given S a graded ring, consider the localizations $f^{-1}S$ for f homogeneous of positive degree. So Spec$((f^{-1}S)_0)$ is the quotient Spec$(f^{-1}S)/\mathbb{G}_m$.

If f, g are two such elements, then $(f^{-1}S)_0 \subset \left((fg)^{-1}S\right)_0 \supset (g^{-1}S)_0$, and so

$$\text{Spec}(f^{-1}S) \supset \text{Spec}\left((fg)^{-1}S\right) \subset \text{Spec}(g^{-1}S).$$

Then Proj(S) := union Spec$((f^{-1}S)_0)/\sim$ given by the inclusions $*$. Observe that the points of Proj(S) are simply the set of homogeneous prime ideals in S such that $\exists f$ homogeneous of positive degree s.t. $f \notin \mathfrak{p}$. Thus

$$S_+ \nsubseteq \mathfrak{p} \text{ where } S_+ = \bigoplus_{d>0} S_d.$$

Note: For $\mathfrak{p} \lhd (f^{-1}S)_0$ (a prime ideal) we have a 1-1 correspondence $\tilde{\mathfrak{p}} \lhd f^{-1}S$ as well as $\tilde{p} \lhd S$ s.t. $f \notin \tilde{\mathfrak{p}}$.

For example, for $S = k[x_0, ..., x_d]$ for k a commutative ring, then Proj(S) is the set of homogeneous prime ideals in S not containing all the x_i. Now, recall (here the overbars mean reduction mod $\mathfrak{p}$).

$$\mathfrak{p} \lhd S \text{ s.t. } \exists x_i \not\subset \mathfrak{p} \Leftrightarrow \text{the ideal } (\overline{x}_0, ..., \overline{x}_n) \text{ in } S_{\mathfrak{p}} \text{ is the unit ideal} =$$
$$\bigcup_{i=0}^n \text{Spec}\left(k\left[x_0, ..., x_i, \tfrac{1}{x_1}, ..., x_d\right]_{\text{deg } 0}\right) \cong \mathbb{A}^d \cong \text{Spec}\, k[x_1/x_0].$$

since $k\left[x_0, ..., x_i, \frac{1}{x_1}, ..., x_d\right]_{\text{deg } 0} = k\left[\frac{x_0}{x_i}, ..., 1, ..., \frac{x_0}{x_i}\right]$. In other words, this follows since a line which is not vertical is determined by its slope, and a line which is not horizontal is determined by the reciprocal of its slope.

Remark: If X is a scheme, a closed subscheme $Y \subset X$ is

(1) a closed subset $Y \subset X$ as topological spaces, and

(2) if $i : Y \to X$ is the inclusion, a surjective homomorphism of sheaves of rings, $\mathcal{O}_X \to i_* \mathcal{O}_Y \ \equiv$

sheaf of ideals $\mathcal{J}_Y \subset \mathcal{O}_X$ which is the kernel of this map.

Fact. For every affine open Spec$(A) \subset X$, an ideal $I \triangleleft A$ is compatible with localization (i.e. the ideal in $f^{-1}A$ is $f^{-1}I$).

Note: A subscheme of $\mathbb{P}_k^d = \bigcup U_i$ is equivalent to $Y_i \subset U_i$ s.t. $Y_i \cap (U_i \cap U_j) = Y_j \cap (U_i \cap U_j)$ which is also equiv to giving a homogeneous ideal $\mathfrak{a} \subset k[x_0, ..., x_d]$ s.t. $(x_0, ..., x_d) \not\subset \mathfrak{a}$. Thus, if k is Noetherian,

$\mathfrak{a} = (f_1, ..., f_k)$ with f_i homogeneous polynomials.

Sheaves of modules

If X is a topological space and $\mathcal{O}$ is a sheaf of rings on X, a sheaf of $\mathcal{O}$-modules is a sheaf element s.t. $\forall U \subset X$, $\mathcal{M}(U)$ is an $\mathcal{O}(U)$-module, and $\forall U \subset V$, $\mathcal{M}(V) \to \mathcal{M}(U)$ is a homomorphism of $\mathcal{O}(V)$-modules, where $\mathcal{M}(U)$ is an $\mathcal{O}(V)$-module via the map $\mathcal{O}(U) \to \mathcal{O}(V)$.

If $S =$ Spec(A) is an affine scheme and M is an A-module, we get a sheaf $\tilde{M}$:

$$\tilde{M}(S_f = \text{Spec}(f^{-1}A)) = f^{-1}M = f^{-1}A \otimes_A M.$$

Obviously if $f \mid g$ so $f^{-1}A \to g^{-1}A$ and get a map $f^{-1}M \to g^{-1}M$. Define $\tilde{M}$ to be the sheaf of sections of the etale space over Spec$(A) = \bigcup S_f \times f^{-1}M$ with obvious identifications.

Theorem. $\tilde{M}(S_f) = f^{-1}M$ for any $f \in A$.

Definition. If X is a scheme, a *quasi-coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{M}$* is a sheaf of $\mathcal{O}_X$-modules such that $\forall$ affine open sets $U =$ Spec$(A) \subset X$, $\mathcal{M}/U \cong \tilde{M}$ for M an A-module.

That is, for all Spec$(A) \subset X$ affine opens, we have an A-module M_A s.t. if Spec$(A) \subset$ Spec$(B) \subset X$, then $A \otimes_B M_B \xrightarrow{\sim} M_A$.

Examples of sheaves of modules

(1) $\mathcal{O}_X^n$ free of rank n.

(2) P locally free sheaf of rank n, i.e., $\forall U =$ Spec$(A) \subset X$ affine open, $P(U)$ is a projective A-module. $\equiv \forall x \in X$, P_x is a free A-module.

(3) Invertible sheaves $\equiv$ rank 1 locally free.

(4) Sheaf of ideals $\varphi \subset \mathcal{O}_X$.

(5) $X =$ Spec$(\mathbb{Z})$. Consider $\mathbb{Z}/2\mathbb{Z}$[tilde]. We want to draw the etale space corresponding to this. Then $U =$ Spec$\left(\mathbb{Z}\left[\frac{1}{n}\right]\right) =$ Spec$(\mathbb{Z}) \backslash \{(\mathfrak{p}_1), ..., (\mathfrak{p}_k)\}$. Then

$$(\widetilde{\mathbb{Z}/2\mathbb{Z}})(U) = n^{-1}\mathbb{Z}/2\mathbb{Z} = \begin{cases} 0 & 2 \nmid n \\ \mathbb{Z}/2\mathbb{Z} & 2 \mid n. \end{cases}$$

We are now in a position to prove the following.

Theorem. *There is a one-to-one correspondence between maps of schemes over k,*

$$X \to \mathbb{P}^d_k,$$

and rank 1 *locally free quotients* $\mathcal{O}_X^{d+1} - \gg \mathcal{L}$. *(* $\equiv$ *an invertible sheaf* $\mathcal{L}$ *on* X *together with* $d+1$ *elements* $s_0, ..., s_d \in \mathcal{L}(X)$ *s.t.* $\forall x \in X$, *the images of* $s_0, ..., s_d$ *in* $\mathcal{L}_x$ *generate this rank* 1 *free module* $\mathcal{O}_{X,x.}$

Lecture 15 (February 16, 2009) - Manifolds and Bundles (wha?)

Hatcher: Book on vector bundles on his website.

Definition. *A real / complex* C^k */* C^* *manifold* M *is a topological space with an equivalence class of "atlases" i.e. coverings by charts,* $M = \bigcup_\alpha U_\alpha$, *where a chart is a pair* (U, φ) *with* $U \subset M$ *open such that* $\varphi : U \to \mathbb{R}^n$ *or* $\mathbb{C}^n$ *is a homeomorphism onto an open subset, and if* $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$ *are two charts, we get a homeomorphism* $\mathbb{R}^n \supset \varphi_\beta(U_\alpha \cap U_\beta) \overset{\varphi_{\alpha\beta}}{\to} \varphi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$, *and we require that* $\varphi_{\alpha\beta}$ *be continuous, differentiable of order* k, *real analytic (or in* $\mathbb{C}$*-case, complex analytic).*

Notice $\varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} = \varphi_{\alpha\gamma} : \varphi_\gamma(U_\alpha \cap U_\beta \cap U_\gamma) \to \varphi_\alpha(U_\alpha \cap U_\beta \cap U_\gamma)$. Furthermore, an atlas (V_i, ψ_i) is a refinement of an atlas $(U_\alpha, \varphi_\alpha)$ if $\{(U_\alpha, \varphi_\alpha) \subset (V_i, \psi_i)\}$. We say two atlases are *equivalent* if they have a common refinement.

Definition. *If* $W \subset M$ *is open, we say that* $f : W \to \mathbb{R}$ *(resp* $\mathbb{C}$*) is continuous,* C^k, C^∞, *or complex analytic, if* $\forall \alpha$, $f \circ \varphi_\alpha^{-1} : \varphi_\alpha(W \cap U_\alpha) \to \mathbb{R}$ *(resp* $\mathbb{C}$*) has this property.*

For example, a map $f : M \to N$ of C^∞ manifolds is a continuous map of topological spaces such that for every pair of charts $N \supset U \overset{\varphi}{\to} \mathbb{R}^n$, $M \supset V \overset{\psi}{\to} \mathbb{R}^m$, we have

$$f \circ \psi^{-1} : \psi(V \cap f^{-1}(U)) \to \mathbb{R}^n.$$

We say $M \subset N$ is a submanifold if it is a closed subspace and the inclusion is C^∞.

A submanifold $M \subset \mathbb{R}^n$ can be described in various ways, especially by equations. For example, $f(x, y) = 0$ gives a curve in $\mathbb{R}^2$.

Examples of bundles

If we have $M \subset \mathbb{R}^n$, then we have $TM =$ tangent bundle to M. Recall if we have a bundle $\pi : E \to M$ (E over M), $E_x = \pi^{-1}(\{x\}) =$ vector space for $x \in M$. Then $(TM)_x$ is simply the tangent space to M in $\mathbb{R}^n$ at x. This is a subspace of $\mathbb{R}^n$. A morphism $f : E \to F$ of vector bundles over F is a continuous map such that (1) $f \circ \pi_F = \pi_E$, (2) f is a linear on the fibers, (3) f is C^k, C^∞, as necessary.

Give, a vector bundle $\pi : E \to M$ and $s : M \to E$ then s is a section if $\pi \circ s = 1_M$. Interstingly, sections of tangent bundles is the same as sections of vector spaces.

Lecture 17 (February 20, 2009) -

Last time, we looked at R as a ring, M as an R-module, and

$$S^*(M) = R \oplus M \oplus .... \oplus S^k M \oplus ...$$

the symmetric algebra. Now, any times we have a map

$$\begin{array}{c}\text{Spec}(S^*(\mathcal{M}))\\ \downarrow \pi \quad \uparrow s\\ \text{Spec}(R),\end{array}$$

then a section corresponds to a homomorphism of R-algebras:

$$S^*M \to R.$$

An R-algebra homomorphism from a symmetric algebra into any R-algebra, $S^*M \to A$, is induced by a unique R-linear map.

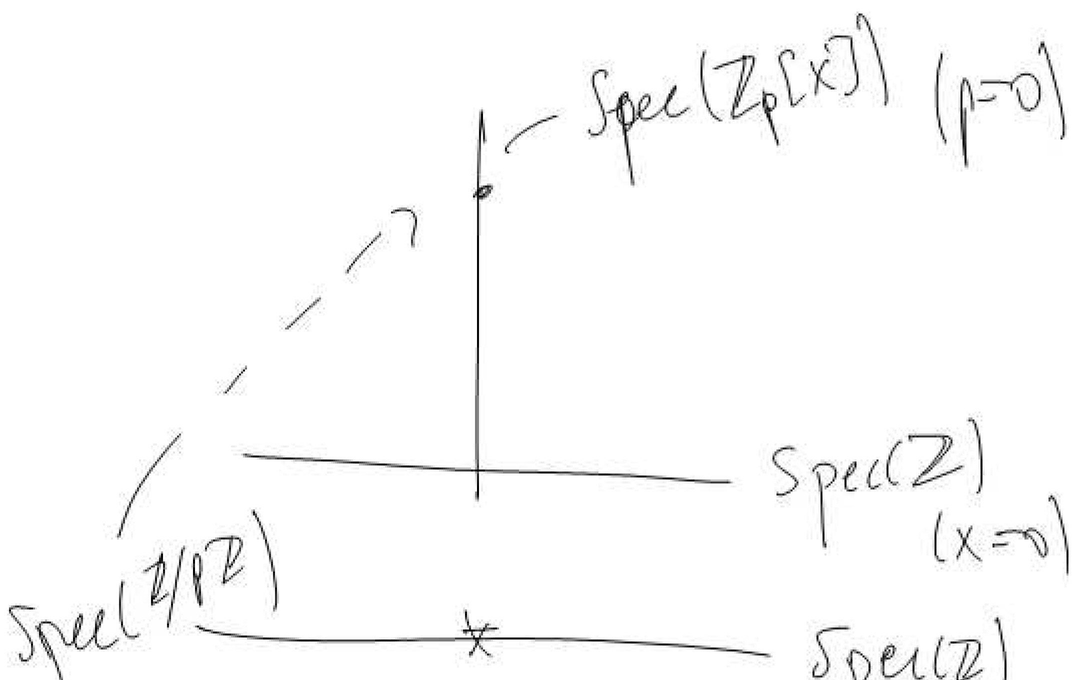
Now, in the previous direct sum for $S^*(M)$, each of the S^iM summands can be sent into A using f. Hence, sections correspond 1-1 with R-linear maps $M \to R$, i.e., elements of M^*. This gives a diagram

$$\begin{array}{ccc} & & \text{Spec}(S^*M)\\ & \sigma \swarrow & \downarrow \pi\\ \text{Spec}(A) & \underset{\varphi}{\to} & \text{Spec}(R),\end{array}$$

which tells us that maps σ such that $\pi \circ \sigma = \varphi$ are in one-to-one correspondence with R-linear maps MA.

Example. Let $M = \mathbb{Z}/p\mathbb{Z}$ with $p \neq 0$ a prime. Then

$$\text{Spec}(S^*M) \cong \text{Spec}(\mathbb{Z}[x]/(px)).$$



Example. Let $M = R^k/\{\text{submodule generated by } r_1, ..., r_l\}$, with x_i the generators of R^k. Then

$$S^*M \cong R[x_1, ..., x_k]/\{\text{ideal generated by the } r_i\}.$$

If $M(\cong R^k)$ is free, then

$$\begin{array}{c}\text{Spec}(S^*M) \cong \mathbb{A}^k_R \\ \downarrow \\ \text{Spec}(R)\end{array}$$

and sections are one-to-one with elements of the free rank k module M^*. If $M = \bigoplus Re_i$ sections are of the form $\sum a_i e^i$ (with e^i the dual basis of M^*).

Homework questions

Now, if G is a group and we have a functor taking

$$\text{Rings} \to \text{Sets}$$
$$R \mapsto \text{ Hom}(G, GL_n(R))$$
$$R \xrightarrow{\varphi} S,\ GL_n(R) \xrightarrow{GL_n(\varphi)} GL_n(S),\ \rho : G \to GL_n(R) \text{ with } \rho \mapsto GL_n(\rho) \cdot \rho.$$

We claim there is a ring $R(G)$ such that this functor is isomorphic to

$$R_i \mapsto \text{Ring homs}(R(G), R).$$

Example. Let $G \cong \mathbb{Z}$ be infinite cyclic. Then $\rho \equiv$ picking $\rho(g) \in GL_n(R)$. In this case the functor is just GL_n, represented by $\mathbb{Z}\left[\{x_{ij} \mid 1 \le i \le n, 1 \le j \le n\} \cdot \frac{1}{\det((x_{ij}))}\right]$, the polynomial ring of matrices with entries $x_{ij}/\det((x_{ij}))$. This is an affine open subset of $\mathbb{A}^{n^2}_{\mathbb{Z}}$. Then if we have a ring homomorphism from that ring to R given by φ, then

$$\varphi : x_{ij} \mapsto \varphi(x_{ij}) \in R$$

such that $\det(\varphi(x_{ij})) \in R^*$, i.e. an element of $GL_n(R)$.

If $G = F_n$, the free group on $\langle g_1, ..., g_n\rangle$ or more generally on a set, $\langle\, \Sigma\,\rangle$, then for any group H,

$$\text{Hom group}(G, H) = \text{Hom sets}(\Sigma, H) \approx H^\Sigma.$$

Then $\text{Hom}(G, GL_{\underline{n}}(R)) = GL_n(R)^\Sigma$, and so $R \to GL_n(R)^\Sigma$ represented by

$$GL_n\Sigma = \text{ Spec}\left(\mathbb{Z}\left[\left\{x^\sigma_{ij}\ \middle|\ \sigma \in \Sigma,\ 1 \le i, j \le n\right\}, \left\{\frac{1}{\det(x_{ij})}\ \middle|\ \sigma \in \Sigma\right\}\right]\right) \subset \mathbb{A}^{n^2\Sigma}_{\mathbb{Z}}.$$

If $G = \langle \Sigma \mid T\rangle$ (generators and relations), then if $t \in T,\ t = \sigma^{\pm 1}_{t(1)}, ..., \sigma^{\pm 1}_{t(n_t)}$. Then if

$$\begin{array}{ccc} F_\Sigma & \xrightarrow{\varepsilon} & G \\ \rho\cdot\varepsilon & \searrow & \downarrow \rho \\ & & GL_n(R). \end{array}$$

Then $\varphi : F_\Sigma \to GL_n(R)$ factors through G if and only if for all $t \in T$, $\varphi(t) = I_n$.

$$\rho\left(\sigma^{\pm 1}_{t(1)} .... \rho^{\pm 1}_{t(n_t)}\right) = I.$$

For each $t \in T$, we have n^2 equations corresponding to the entries of this matrix equation. Then $\text{Hom}(G, GL_n(R))$ is represented by

$$\frac{\text{Spec}\left(\mathbb{Z}\left[\left\{x^\sigma_{ij}\ \middle|\ \sigma\in\Sigma, 1\le i,j\le n\right\}, \left\{\frac{1}{\det\left(x_{ij}\right)}\ \middle|\ \sigma\in\Sigma\right\}\right]\right)}{\text{ideal generated by entries of matrices corresponding to the relations.}}.$$

Sub-Example. What are the representations of $\mathbb{Z}/2\mathbb{Z}$ in GL_2? Well, $GL_2 = \begin{pmatrix} x & z \\ y & w \end{pmatrix}$ such that the determinant is a unit. That is,

$$GL_2 = \mathbb{Z}\left[x, y, z, w, \tfrac{1}{xw - zy}\right] \subset \mathbb{A}^4_{x,y,z,w}.$$

Then the representations are equivalent to elements in GL_2 such that $A^2 = I$. Then the $\text{Hom}(\mathbb{Z}/2\mathbb{Z}, GL_2)$ is given by $x^2 + yz = 1, w^2 + zy = 1, xy + yw = 0, xz + wz = 0$.☐

If $\mathbb{A}^2_k \backslash \{(0,0)\}$, then $R^* \times R \cup R \times R^*$ is given by

$$\{(f,g) \in R^2 \,|\, (f,g) = R\},$$

i.e., $\exists a, b \in R$ such that $af + bg = 1$. For example, $\text{Spec}(\mathbb{Z}) \to \mathbb{A}^2_k \backslash \{(0,0)\}$, with $x \mapsto 2$ and $y \mapsto 3$.

Another homework question: What does it mean to show something has a natural scheme structure?

Given $G = \text{Spec}(R)$ with $G \times G \to G,\ R \otimes_k R \leftarrow R$, with $k[\varepsilon] \to k[\varepsilon \otimes 1, 1 \otimes \varepsilon]$. What are the elements in this ring? Well, if we let $\varepsilon \otimes 1 = \varepsilon'$ and $1 \otimes \varepsilon = \varepsilon''$, then they are of the form

$$a + b\varepsilon' + c\varepsilon'' + d\varepsilon'\varepsilon''.$$

Check associativity, identity, etc.

$$\begin{array}{ccc} G \times G \times G & \to & G \times G \\ \downarrow & & \downarrow \\ G \times G & \to & G \end{array}$$

Recall the identity will be a map $\text{Spec}\, k \to G$.

Lecture 18 (February 23, 2009) - Homework questions

Homework questions

For (2b): A k-derivation is essentially

$$\alpha_2 \times \text{Spec}(A) \to \text{Spec}(A).$$

The former is essentially $\text{Spec}(k[\varepsilon] \otimes_k A) \cong \text{Spec}(A[\varepsilon])$, so all we need to do is give a ring homomorphism $A \to A[\varepsilon]$. So, look at

$$\begin{gathered} \alpha_2 \times \alpha_2 \to \alpha_2 \\ k[\varepsilon', \varepsilon''] \leftarrow k[\varepsilon] \\ \varepsilon \mapsto \varepsilon' + \varepsilon'' \end{gathered}$$

$$\begin{gathered} \text{Spec}(k) \to \alpha_2 \\ k \leftarrow k[\varepsilon] \\ \varepsilon \mapsto 0 \end{gathered}$$

Then we can take

$$\begin{gathered} \qquad\qquad \varepsilon \mapsto 0 \\ A \to A[\varepsilon] \to A \\ a \mapsto a + \varphi(a)\varepsilon, \end{gathered}$$

where φ is a ring homomorphism and thus additive. Further, notice

$$ab \mapsto ab + \varphi(ab)\varepsilon = (a + \varphi(a)\varepsilon)(b + \varphi(b)\varepsilon) = ab + (\varphi(a)b + a\varphi(b))\varepsilon$$

so that $\varphi(ab) = \varphi(a)b + a\varphi(b)$ and hence φ is a derivation.

$$\begin{array}{ccccc} & & k & & \\ & \swarrow & \downarrow & \searrow & \\ A & \leftarrow & A[\varepsilon] & \leftarrow & A, \end{array}$$

with $a \mapsto a + 0\varepsilon$ for the $\downarrow$ map, and so $a \in k \implies \varphi(a) = 0$. Then we have to check

$$\begin{array}{ccc} \alpha_2 \times \alpha_2 \times \text{Spec}(A) & \to & \alpha_2 \times \text{Spec}(A) \\ \downarrow & & \downarrow \\ \alpha_2 \times \text{Spec}(A) & \to & \text{Spec}(A) \end{array}$$

commutes, which is simply a computation. Say the maps are

$$\begin{array}{ccc} a + \delta(a)\varepsilon' + \delta(a)\varepsilon'' & \leftarrow & a + \delta(a)\varepsilon \\ \uparrow & & \uparrow \\ a + \delta(a)\varepsilon & \longleftarrow & a. \end{array}$$

Then $a + \delta(a)\varepsilon' + \delta(a)\varepsilon''$ gets mapped to $a + \delta(a)\varepsilon'' + (\delta(a) + \delta(\delta(a))\varepsilon'') = \varepsilon$ (or is it?).

Digression In char $p > 0$, $\alpha_p = \text{Spec}(k[x]/x^p)$ with $x \mapsto x \otimes 1 + 1 \otimes x$. This is a ring homomorphism by the binomial theorem. In *any* characteristic, there is a correspondence between k-algebra homomorphisms $\varphi : A \to A[\varepsilon]$ such that $\varphi(a) = a + \delta(a)\varepsilon$, and derivations. In characteristic zero, there is a $1{-}1$ correspondence between k-derivations $\delta : A \to A$, and actions of $\mathcal{G}_a$ on $X = \text{Spec}(A)$, where

$\mathcal{G}_a$ is defined as follows. Recall $\mathbb{G}_a = \text{Spec}(k[x])$ with $x \mapsto x \otimes 1 + 1 \otimes x$. Notice that this makes sense on power series: $k[[x]] \to k[[x \otimes 1, 1 \otimes x]]$ sends a power series $\sum_{n=0}^{\infty} a_n x^n \mapsto \sum_{n=0}^{\infty} a_n (x \otimes 1 + 1 \otimes x)^n$. Then rather than take the tensor products we just take power series in the elements (call this diagram ★)

$$\begin{array}{ccc} A[[x']] \mathbin{\widehat{\otimes}} A[[x'']] & & \\ \| & & \\ A[[x', x'']] & \leftarrow & A[[x]] \\ \uparrow & & \uparrow \\ A[[x]] & \leftarrow & A \end{array}$$

Remark. $k[[x']] \otimes k[[x'']] \ncong k[[x, x'']]$

Then if $a \mapsto \varphi(a) := \sum \varphi^i(a)\, x^i$ in the right $\uparrow$ in the above diagram, the result on the top line will be

$$\varphi(a) := \sum \varphi^i(a)\, x^i \mapsto \sum_{i,j=0}^{\infty} \varphi^i \varphi^j(a)(x')^i (x'')^j = \sum \varphi^i(a)(x' + x'')^i, \quad (1)$$

with the condition $\varphi^0(a) = a$. **Here, we are using $\boldsymbol{\varphi^i \neq \varphi \circ \ldots \circ \varphi}$ as $\boldsymbol{i}$-times.** We will explain in what sense we use the notation "φ^i" momentarily. So, coming back to $\mathcal{G}_a$, this is similarly defined as for $\mathbb{G}_a$, but with the diagram above. Now, notice in (1) above, this is equivalent to giving

$$\varphi^0 = \text{id},\ \varphi^1, ..., \varphi^i : A \to A, ...$$

such that $\forall n \geq 0$, $\forall a$, $\varphi^n(a)(x')^i(x'')^j\binom{n}{i} = \varphi^i\varphi^{n-1}(a)$ with $i + j = a$. For example, $\varphi^2 \cdot 2 = \varphi^1 \cdot \varphi^1$, that is, $\varphi^2 = \frac{1}{2}(\varphi^1)^2$ (binomial theorem). **This is the sense in which we use the notation $\boldsymbol{\varphi^i}$**. Then notice that $\varphi^n = \frac{1}{n!}\varphi^{\circ n}$, where $\varphi^{\circ n} = \varphi \circ ... \circ \varphi$ composed n times.

Then the claim is as follows.

Proposition. To give a ring homomorphism $\varphi : A \to A[[x]]$ such that $\varphi(a) = \sum_{i=0}^{\infty}\varphi^i(a)x^i$ with (i) $\varphi^0(a) = a$ for all a, (ii) the diagram (★) commutes, i..e,

$$\sum_{i=0}^{\infty} \varphi^i(x' + x'')^i = \sum_i\sum_j \varphi^i\varphi^j(x')^i(x'')^j$$

is, in characteristic 0, equivalent to giving a derivation on A.

Hence, going back to the idea of a group scheme, recall this is just verifying that the diagram below commutes,

$$\begin{array}{ccc} G \times G \times X & \to & G \times X \\ \downarrow & & \downarrow \\ G \times X & \longrightarrow & X \end{array}$$

with the maps

$$\begin{array}{ccc} (g, h, x) & \mapsto & (gh, x) \\ \downarrow & & \downarrow \\ (g, hx) & \mapsto & ghx \end{array}$$

induced by multiplication.

Example. Consider $C^\infty(\mathbb{R})$ with derivation $\frac{d}{dt}$. Then if you evaluate

$$C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})[[x]]$$
$$f \mapsto \sum_{n=0}^{\infty} \frac{1}{n!}\, \frac{d^n f}{dt^n}\, x^n$$

at a point $t = t_0$ in $\mathbb{R}$ (i.e., coeffs of a formal power series in x), then this gives the Taylor series of f at t_0. ☐

Previous homework problem

Recall the problem dealing with $X \coloneqq \mathbb{A}_k^n \setminus \{0\} \subset \mathbb{A}_k^n$ (in the homework it was $n = 2$). Of course, $\{0\} = (x_1, ..., x_n) \triangleleft k[x_1, ..., x_n]$ is the point corresponding to / consisting of the maximal ideal generated by $x_1, ..., x_n$. So, a point $\mathfrak{p}$ in $\mathbb{A}_k^n$ is in X if and only if $\mathfrak{p} \neq (x_1, ..., x_n)$ if and only if $\exists\, x_i$ such that $x_i \notin \mathfrak{p}$ (since $(x_1, ..., x_n)$ is maximal). Then

$$X = \bigcup_{i=1}^n D(x_i) = \text{Spec}\Big(k[x_1, ..., x_n]\Big[\tfrac{1}{x_i}\Big]\Big)$$

(so $D(x_i)$ is the line where we deleted the entire line $x_i = 0$, so for $\mathbb{A}^2$ it would be deletion of the y-axis if $i = 2$). Furthermore, notice the primes in the Spec above are in one-to-one correspondence with primes $\not\ni x_i$. Anyway, if $\varphi : \mathrm{Spec}(R) \to X$ is a morphism composed with $X \subset \mathbb{A}^n_k$, we get $\overline{\varphi} : \mathrm{Spec}(R) \to \mathbb{A}^n_k$, or equivalently,

$$k[x_1, ..., x_n] \to R \text{ with } x_i \mapsto r_i.$$

That is (and we saw this earlier), giving a homomorphism of $\mathrm{Spec}(R)$ to $\mathbb{A}^n_k$ is equivalent to giving a ring homomorphism as above. Now, the above map will factor through the open subset X if and only if $\{0\} \notin$ image of $X \Leftrightarrow X = \bigcup_{i=1}^n (\overline{\varphi})^{-1}(D(x_i))$. Now we just need to know that $\overline{\varphi}$ is. Call $(\overline{\varphi})^{-1}(D(x_i)) = x_{r_i}$. This is just the open subset of X where r_i are units. We can see this by looking at

$$\begin{array}{ccc} R\left[\frac{1}{r_i}\right] & \leftarrow & k\left[x_1, ..., x_n, \frac{1}{x_i}\right] \\ \uparrow & & \uparrow \\ R & \leftarrow & k[x_1, ..., x_n] \end{array}$$

where we notice $R \otimes_{k\left[x_1, ..., x_n, \frac{1}{x_i}\right]} R[x_1, ..., x_n] = R\left[\frac{1}{r_i}\right]$, and this diagram corresponds to

$$\begin{array}{ccc} (\overline{\varphi})^{-1}(D(x_i)) & \to & D(x_i) \\ \downarrow & & \downarrow \\ X & \underset{\overline{\varphi}}{\longrightarrow} & \mathbb{A}^n. \end{array}$$

Continuing our $\Leftrightarrow$ we get if and only if $(r_1, ..., r_n) \lhd R$ is the unit ideal $\Leftrightarrow \exists s_1, ..., s_n$ in R with $\sum r_i s_i = 1$ (that is, $\forall \mathfrak{p} \lhd R$ at least one r_i maps to units in $R_{\mathfrak{p}}$, which is the same as saying subschemes $V(r_i) \subset X$ are disjoint).

Lecture 19 (February 25, 2009) - Quasi-coherent, locally free of rank 1 sheaves

Recall if R is a ring, a map $\mathrm{Spec}(R) \to \mathbb{A}^n \backslash \{0\}$ is the same as an n-tuple $(r_1, ..., r_n)$ such that the ideal $(r_1, ..., r_n) = R$. This is equivalent to the map

$$R^n \to R \quad (x_1, ..., x_n) \mapsto \sum r_i x_i$$

being surjective. Recall also that $\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[t, t^{-1}])$ acts on $\mathbb{A}^{n+1} \backslash \{0\} \subset \mathbb{A}^{n+1} = \mathrm{Spec}(\mathbb{Z}[x_0, ..., x_n])$ and the quotient is $\mathbb{P}^n$. And $\mathbb{P}^n$ is the union of the affine open sets

$$U_i = \mathrm{Spec}\left(\mathbb{Z}\left[\frac{x_0}{x_1}, ..., \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, ..., \frac{x_n}{x_i}\right]\right).$$

Then the degree zero part of the $\mathbb{Z}$-graded ring $\bigstar = \mathbb{Z}\left[x_0, ..., x_i, \frac{1}{x_i}, ..., x_n\right]$. Then

$$\mathbb{A}^{n+1} \backslash \{0\} = \bigcup_{i=0}^n V_i = \mathrm{Spec}[\bigstar].$$

Also, $\mathbb{G}_m$ acts on each V_i with the quotient equal to U_i and furthermore there is a one-to-one correspondence between prime ideals in $\mathbb{Z}\left[\frac{x_0}{x_1}, ..., \frac{x_n}{x_1}\right]$, and homogeneous prime ideals in $\bigstar$ which do not contain the ideal $(x_0, ..., x_n)$.

Here, $\mathbb{P}^n$ as a set is the set of homogeneous prime ideals in $\mathbb{Z}[x_0, ..., x_n]$ not containing all of $x_0, ..., x_n$. In other words, U_i is the set of homogeneous primes $\mathfrak{p}$ such that $x_i \notin \mathfrak{p}$. If R is a ring, a map $\phi : \text{Spec}(R) \to \mathbb{P}^n_{\mathbb{Z}} = \bigcup U_i$ is given by maps $\phi_i : \phi^{-1}(U_i) \to U_i$ where $\phi^{-1}(U_i)$ is a Zariski open set of Spec(R) such that on $W_i \cap W_j$ the maps $\phi_i|_{W_i \cap W_j}, \phi_j|_{W_i \cap W_j} : W_i \cap W_j \to U_i \cap U_j$ agree.

Notice that each ϕ_i is determined by elements $\xi^i_0, \ldots, \xi^i_n$ (with $\zeta^i_i = 1$) in $\mathcal{O}_X(U_i)$ (where $X = \text{Spec}(R)$). On

$$U_i \cap U_j = \text{Spec}\,\mathbb{Z}\left[\tfrac{x_0}{x_i}, \ldots, \tfrac{x_i}{x_i}, \tfrac{x_i}{x_j}, \ldots, \tfrac{x_n}{x_i}\right] \text{ (degree 0 part of}$$

$$\mathbb{Z}\left[x_0, \ldots, x_i, \tfrac{1}{x_i}, \ldots, x_j, \tfrac{1}{x_j}, \ldots, x_n\right]),$$

we know $x_k/x_i = x_k/x_j \cdot x_j/x_i$. To say that $\phi_i = \phi_j$ on $W_i \cap W_j$ means $\xi^i_k = \xi^j_k \xi^i_j$ with ξ^i_j a unit and $\xi^i_j \cdot \xi^j_i = 1$. In other words, giving ϕ is the same as giving $X = U_0 \cup U_1$ and giving a function $\xi^0_1 \in \mathcal{O}_X(U_0)$, and a function $\xi^1_0 \in \mathcal{O}_X(U_1)$ such that on $U_0 \cap U_1$, we have $\xi^1_0 \xi^1_0 = 1$. So, $\xi^1_0 \in \mathcal{O}_X(U_0 \cap U_1)^*$(invertible function, so it's a unit in this open set). Now, out of this unit, we can construct a locally free sheaf of rank 1, $\mathcal{L}$, as follows: on U_0 take the (locally) free sheaf $\mathcal{O}_{U_0} \cong \mathcal{O}_X|_{U_0}$. On U_1, take $\mathcal{O}_{U_1} \cong \mathcal{O}_X|_{U_1}$. Glue these together on $U_0 \cap U_1$ by the map:

$$\mathcal{O}_{U_0}|_{U_0 \cap U_1} \to \mathcal{O}_{U_1}|_{U_0 \cap U_1}, \quad \alpha \mapsto \alpha \xi^0_1.$$

It is a remark here to notice if $X = U_0 \cup U_1$ and $\mathcal{M}_0, \mathcal{M}_1$ are quasi-coherent sheaves on U_0 and U_1 respectively, and $\phi : \mathcal{M}_0|_{U_0 \cap U_1} \to \mathcal{M}_1|_{U_0 \cap U_1}$ is an isomorphism, we get a sheaf $\mathcal{N}$ such that $\mathcal{N}|_{U_0} \cong \mathcal{M}_0$ and $\mathcal{N}|_{U_1} \cong \mathcal{M}_1$. If $V \subset X$ is open, then

$$\mathcal{N}(V) = \{(s,t) \mid s \in \mathcal{M}_0(U_0 \cap V), t \in \mathcal{M}_1(U_1 \cap V), \phi(s) = t\}.$$

More generally, given $X = \bigcup_{i \in I} U_i$ and $\mathcal{M}_i$ sheaves on U_i, then together with isomorphisms $\phi_{ij} : \mathcal{M}_j|_{U_i \cap U_j} \to \mathcal{M}_i|_{U_i \cap U_j}$ such that $\forall i, j, k$,

$$\phi_{ij}\,|_{U_i \cap U_j \cap U_k} \cdot \phi_{jk}|_{U_i \cap U_j \cap U_k} = \phi_{ik}|_{U_i \cap U_j \cap U_k},$$

then there is a sheaf $\mathcal{N}$ such that $\mathcal{N}|_{U_i} \cong \mathcal{M}_i$ and sections are given by a similar formula. In particular, if each $\mathcal{M}_i \cong \mathcal{O}_{U_i}$, then an isomorphism $\phi_{ij} : \mathcal{O}_{U_j}|_{U_i \cap U_j} \to \mathcal{O}_{U_i}|_{U_i \cap U_j}$ is just $\phi_{ij} : \mathcal{O}_{U_i \cap U_j} \to \mathcal{O}_{U_i \cap U_j}$ given by

$$1 \in \mathcal{O}_{U_i \cap U_j}(U_i \cap U_j) \mapsto \alpha \in \mathcal{O}_{U_i \cap U_j}(U_i \cap U_j).$$

Since this is a map of modules, any $v \mapsto \alpha u$, and since ϕ is an isomorphism,

$$\alpha \in \mathcal{O}_{U_i \cap U_j}(U_i \cap U_j)^*,$$

i.e., $\phi_{ij} =$ multiplication by a unit which we also denote $\phi_{ij} \in \mathcal{O}_X(U_i \cap U_j)^*$. Hence, a collection $\phi_{ij} \in \mathcal{O}_X(U_i \cap U_j)^*$ such that $\phi_{ii} = 1,\ \phi_{ij} \cdot \phi_{jk} = \phi_{ik} \in \mathcal{O}_X(U_i \cap U_j \cap U_k)$ $\forall i, j, k$ $(*)$ (this is called the "co-cycle condition") determines, by gluing, a quasi-coherent sheaf $\mathcal{L}$ on X such that $\mathcal{L}\,|_{U_i} \cong \mathcal{O}_X|_{U_i}$, i.e., $\mathcal{L}$ is a "**locally free of rank 1**".

Conversely, if we are given a quasi-coherent sheaf $\mathcal{L}$ on X such that there exists an open cover U_i and an isomorphism $\sigma_i : \mathcal{L}|_{U_i} \cong \mathcal{O}_X|_{U_i}$, i.e., $\mathcal{L}$ is locally free of rank 1. Then if we set $\phi_{ij} = \sigma_i \cdot \sigma_j^{-1} \cdot \mathcal{O}_X|_{U_i \cap U_j} \to \mathcal{O}_X|_{U_i \cap U_j}$, then the ϕ_{ij} satisfy $(*)$.

Now, we want to go back and relate this to projective space.

Proposition. *Suppose we are given a scheme X, and a locally free rank 1 sheaf $\mathcal{L}$ on X determined by a cocycle $\{\phi_{ij}\}$ with respect to an affine open cover U_i of X, and a homomorphism $\theta : \mathcal{O}_X^{n+1} \to \mathcal{L}$ which is surjective.*

Remark. For $\mathcal{M},\ \mathcal{N}$ quasi-coherent sheaves, $\theta : \mathcal{M} \to \mathcal{N}$ is surjective if and only if for all affine opens, $\mathcal{M}(U) \to \mathcal{N}(U)$.

Next time, we will show that θ gives a map to $\mathbb{P}^n$.

Lecture 20 (February 27, 2009) - Constructing maps to $\mathbb{P}^n$

Recall from last time that the idea is the following. If X is a scheme and $\mathcal{L}$ is a sheaf of locally free $\mathcal{O}_X$-modules of rank 1, and we are given a surjective homomorphism of sheaves of modules $\varphi : \mathcal{O}^{n+1} \to \mathcal{L}$, then we get a well-defined map

$$f_\varphi : X \to \mathbb{P}_{\mathbb{Z}}^n$$

as follows. Recall that by definition, there exists an open cover U_i of X and an isomorphism $\sigma_i : \mathcal{O}_X|_{U_i} \xrightarrow{\simeq} \mathcal{L}|_{U_i}$. This is called a "**local trivialization**". Notice that to give σ_i is equivalent to giving the section $\sigma_i(1) \in \mathcal{L}(U_i)$ with σ_i an isomorphism; in other words, $\forall x \in U_i,\ \sigma_i(1) \in \mathcal{L}_{X,x}$, where $\mathcal{L}_{X,x}$ is an $\mathcal{O}_X$-module that is free or rank 1--that is, $\sigma_i(1)$ vanishes nowhere in U_i.

Reminder. If R is a local ring and L is free of rank 1, then $\ell \in L$ generates L if and only if $\ell \notin mL$ (for m a maximal ideal of R) if and only if a non-zero $\overline{\ell} \in R/m \otimes_R L$.

Given such a local trivialization, $\sigma_i^{-1} \circ \varphi : (a_0, ..., a_n) \mapsto \sum_{i=0}^n a_i\, r_i \in \mathcal{O}_X(U_i)$ for some $(r_0, ..., r_n)$ because φ is surjective. Hence, there exists an $\underline{a}$ such that $\varphi(\underline{a}) = 1$, that is, $(r_0, ..., r_n) \subset \mathcal{O}_X(U_i)$ is the unit ideal, or equivalently, $\psi_i = \sigma_i^{-1} \circ \varphi$ corresponds to the map $(r_0, ..., r_n) : U_i \to \mathbb{A}^{n+1} \backslash \{0\}$. On $U_i \cap U_j$, we have

$$\sigma_j = \varphi_{ji} \sigma_i,$$

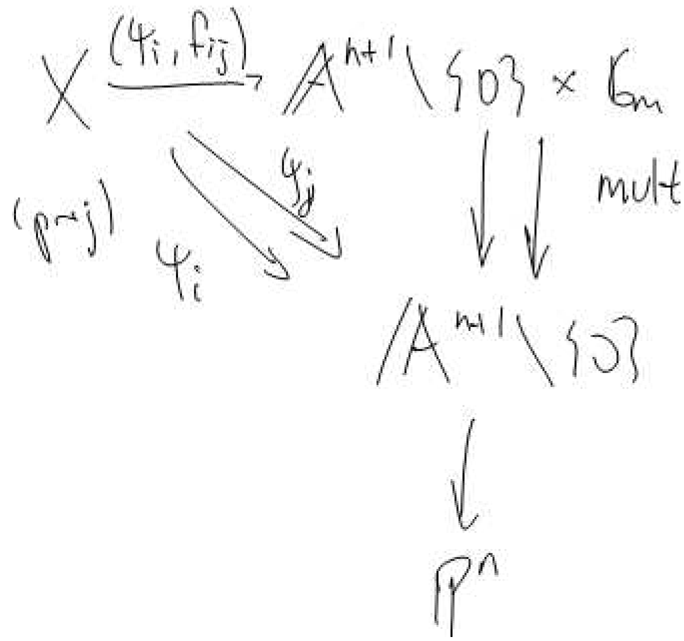
with $\varphi_{ji} \in \mathcal{O}_X(U_i \cap U_j)^*$. Hence, we have two maps

$$U_i \cap U_j \underset{\psi_j}{\overset{\psi_i}{\rightarrow}} \mathbb{A}^{n+1} \backslash \{0\} \times \mathbb{G}_m,$$

where ψ_i corresponds to $\left(r_0^{(i)}, ..., r_n^{(i)}\right)$ and ψ_j corresponds to $\left(r_0^{(j)}, ..., r_n^{(j)}\right)$. Let's look at the coordinates of the pull-backs of these maps (where elements are mapped to).

$$\text{For } (x_0, ..., x_n, t),\ \text{each } x_k \overset{\psi_i^*}{\mapsto} r_k^{(i)} \text{ and } x_k \overset{\psi_j^*}{\mapsto} r_k^{(j)},$$
$$\text{and under both maps } t \text{ maps to } \varphi_{ji}.$$

Observe that $\psi_j^*(x_k) = \psi_i^*(x_k) f_{ij}^*(t)$, where we let $f_{ij} : U_i \cap U_j \to \mathbb{G}_m$ (remember it makes sense to talk about this map). Hence, ψ_i and ψ_j are the composition



Since $\mathbb{P}^n$ is the quotient of $\mathbb{A}^{n+1}\backslash\{0\}$ by the action of $\mathbb{G}_m$, the composition of these maps with proj^n to $\mathbb{P}^n$ are the same.

Now we can ask the question: what happens if we had made a different choice of the (U_i, σ_i)? In other words, consider $\{(V_j, \tau_j)\}$. Then by what we have just seen, these will have to differ by multiplication by a unit, and hence on $U_i \cap V_j$, the maps to $\mathbb{A}^{n+1}\backslash\{0\}$ associated to σ_i and τ_j induce the same map to $\mathbb{P}^n$. So the map $f_\varphi : X \to \mathbb{P}^n$ is not dependent on these choices (see beginning of lecture for f_φ).

Now that we know about our map f_φ, we claim that given a $f : X \to \mathbb{P}^n$, we can construct a surjective homomorphism $\varphi : \mathcal{O}_X^{n+1} \to \mathcal{L}$ such that $f_\varphi = f$ (where $\mathcal{L}$ is locally free of rank 1). Here is the construction.

Step 1. Consider the sheaves $\mathcal{O}_{\mathbb{P}^n}(k)$. Recall that

$$\mathbb{P}^n = \bigcup_{i=0}^n \text{Spec}\left(\mathbb{Z}\left[x_0, ..., x_i, \tfrac{1}{x_i}, ..., x_n\right]_{\text{deg } 0}\right)$$

with homogeneous coordinates $(x_0 : ... : x_n)$ (where deg 0 means "take the degree 0 part of this guy").Notice that $\mathbb{Z}\left[x_0, ..., x_i, \frac{1}{x_i}, ..., x_n\right]$ is $\mathbb{Z}$-graded, and it has a unit in degree 1 (namely, x_i). In general, if $S = \bigoplus_{n\in\mathbb{Z}} S_n$ is such a ring and $u \in S_1$ is the unit, we can multiply by u to get $S_k \to S_{k+1}$, which will be an isomorphism of S_0-modules. So, S_k (for any $k \in \mathbb{Z}$) is a free rank 1 S_0-module generated by u^k.

Lemma. *For each $k \in \mathbb{Z}$, the modules M_i of homogeneous elements of degree k in*

$$\mathbb{Z}\left[x_0, ..., x_i, \tfrac{1}{x_i}, ..., x_n\right]$$

patch together to give a locally free rank 1 sheaf on $\mathbb{P}^n_{\mathbb{Z}}$.

Proof. Let $k \in \mathbb{Z}$. Our goal is to give an isomorphism

$$M_i = \mathbb{Z}\left[x_0, ..., x_i, \tfrac{1}{x_i}, ..., x_n\right]_{\text{deg } k} \longrightarrow \mathbb{Z}\left[x_0, ..., x_j, \tfrac{1}{x_j}, ..., x_n\right]_{\text{deg } k} = M_j.$$

That is, $M_i = \mathcal{O}_X(k)(U_i)$ is the elements of $\mathbb{Z}\left[x_0, ..., x_i, \frac{1}{x_i}, ..., x_n\right]$ of degree k. This is free of rank 1 over the homogeneous components of degree zero, and so generated by the $(x_i)^k$, that is,

$$f = \underbrace{\tfrac{a(x_0,...,x_n)}{x_i^{\ell+k}}}_{\text{degree } 0} \cdot x_i^k$$

on $U_i \cap U_j$ with $f \cdot \left(\frac{x_j}{x_i}\right)^k =$ (something of degree zero).

Now, we look at the localization of M_i with respect to x_j and M_j with respect to x_i, and the intersection with respect to both. Let's look at specifics here.

What does M_i look like? Take $f \in M_i$. Then $f = \frac{a(x_0,...,x_n)}{x_i^\ell}$ where a is of degree $l + k$. Similarly, given $g \in M_j$, it will look like $g = \frac{b(x_0,...,x_n)}{x_i{}^m}$ with b of degree $m + k$. Now, we want to make ourselves a nice little map

$$M_i\left[\tfrac{1}{x_j}\right] \to M_j\left[\tfrac{1}{x_i}\right].$$

Using the fact x_i/x_j is a unit on $U_i \cap U_j$,

$$\tfrac{a(x_0,...,x_n)}{x_i^\ell} \mapsto \tfrac{a_0(x_0,...,x_n)}{x_j^\ell} \cdot \left(\tfrac{x_j}{x_i}\right)^k,$$

with a_0 *a* function. Now, recall the co-cycle condition $(*)$ for a locally free sheaf of rank 1 (see previous lecture). Then the co-cycle defining $\mathcal{O}_{\mathbb{P}^n}(k)$ is $\left(\frac{x_i}{x_j}\right)^k$ on $U_i \cap U_j$.

Refer back to our discussion at the beginning of this proof, and we can extend the result for f to say that since on $U_i \cap U_j \cap U_m$

$$\left(\tfrac{x_i}{x_j}\right)^k \left(\tfrac{x_j}{x_m}\right)^k = \left(\tfrac{x_i}{x_m}\right)^k,$$

we have a locally free rank 1 sheaf. [Wait, what? Go back and look at this.] ☐

Notice if $k \geq 1$, then any homogeneous polynomial of degree k in $x_0, ..., x_n$ defines a global section in $\mathcal{O}_{\mathbb{P}^n}(k)(\mathbb{P}^n)$. In other words, if

$$f(x_0, ..., x_n) \in \mathbb{Z}[x_0, ..., x_n]$$

is homogeneous of degree k, then

$$f = \tfrac{f}{x_i^k} \cdot x_i^k,$$

where the first term is homogeneous of degree 0. So these "glue together" as co-cycles. Notice

$$\varphi_i = \left(\tfrac{x_i}{x_j}\right)^k \varphi_j$$

so consider the (φ_i). Then in particular, $\mathcal{O}(1)$ has $n+1$ global sections $x_0, ..., x_n$. Hence, we have constructed

$$\mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{O}_{\mathbb{P}^n}(1) \text{ given by } (a_0, ..., a_n) \mapsto \sum a_i\, x_i.$$

Notice further that this map is surjective. On U_i, $\mathcal{O}_{\mathbb{P}^n}(1)\,|_{U_i} \cong \mathcal{O}_{U_i}$ is trivial, generated by $x_i = \sigma(0, ..., 1, ...0)$.

Step 2. Now given any $f : X \to \mathbb{P}^n$, just take $\mathcal{L} = f^*(\mathcal{O}_{\mathbb{P}^n}(1))$. We will see this done next time.

Lecture 22 (March 4, 2009) - Quasicoherent Sheaves

Let $X = \text{Spec}(A)$, M an A-module, and $\tilde{M}$ a quasi-coherent sheaf on X associated to M. Then

$$\tilde{M}\left(X_f = \text{Spec}\left(A\left[\tfrac{1}{f}\right]\right)\right) = M\left[\tfrac{1}{f}\right] \cong A_f \otimes_A M.$$

If $f : X \to Y$ is a map of schemes $\mathcal{M}$, there exists a quasi-coherent sheaf of $\mathcal{O}_Y$-modules. Define $f^*\mathcal{M} = \mathcal{O}_X \mathcal{O}_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{M}$.

Then there is an equivalence of categories:

$$\text{quasi-coherent sheaves on spec } A \;\longleftrightarrow\; A\text{-modules.}$$

For X a general scheme, a quasicoherent sheaf on X, is equivalent to giving the following:

For every affine $U = \text{Spec}(A) \subset X$ an A-module $M(U)$ s.t. if $V = \text{Spec}\left(A\left[\frac{1}{f}\right]\right) \subset \text{Spec}(A) = U$, with $m_i \in M\left[\frac{1}{f_i}\right]$, then the map

$$M(U)\left[\tfrac{1}{f}\right] \to M(V)$$

is an isomorphism.

Also, if $X = \bigcup_i U_i$, and given quasi-coherent sheaves $\mathcal{M}_i$ on U_i,

$$\theta_{ji} = \mathcal{M}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{M}_j|_{U_i \cap U_j}$$

satisfying $\theta_{ki} = \theta_{kj}\theta_{ji} \Longrightarrow$ get a quasi-coherent $\mathcal{M}$ on X.

Given $\mathcal{M}$ a quasi-coherent sheaf of $\mathcal{O}_Y$-modules, define the quasi-coherent sheaf $f^*\mathcal{M}$. We only need to find an open cover $X = \bigcup U_i$, and to define $f^*\mathcal{M}\,|_{U_i}$ such that these patch together. Consider $f : X \to Y = \bigcup_i U_i = \text{Spec}(A_i)$. Of course we can write $X = \bigcup_j V_j = \text{Spec}(B_j)$. Further $f(V_j) \subset U_i$ for some i,$\varphi(j)$ i.e., $f|_{V_j}$ is determined by a map $f_j^* : A_{\varphi(1)} \to B_i$ such that these are compatible.

So, let's say we have

$$V = \text{Spec}(B) \xrightarrow{f_{U,V}} U = \text{Spec}(A)$$

with $V \subset X$ and $U \subset Y$. Then giving f is equivalent to giving (i) a map of topological spaces $X \to Y$, (ii) for all pairs U, V of affine opens such that $f(V) \subset U$, a ring homomorphism $f^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(V)$ such that $\text{Spec}(f^*) = f|_U$, and this is natural with respect to inclusion of affine opens.

We can also do this in a category-theoretic say. Say we can have a quasi-coherent sheaf $\mathcal{N}$ on X, and a quasi-coherent sheaf $\mathcal{M}$ on Y.

Definition. *A homomorphism of quasi-coherent sheaves $\varphi : \mathcal{M} \to \mathcal{N}$ over f consists of giving for all pairs $V = \text{Spec}(B)$, $U = \text{Spec}(A)$ as above, that is, such that $f(V) \subset U$, a homomorphism of abelian groups,*

$$\varphi_{U,V} : \mathcal{M}(U) \to \mathcal{N}(V)$$

which is $\mathcal{O}_X(U)$-linear, where $\mathcal{N}(V)$ is an $\mathcal{O}_X(U)$ *module via*

$$f_{UV} : \mathcal{O}_Y(U) \to \mathcal{O}_X(V),$$

and these homomorphisms are natural with respect to inclusions $U' \subset U$ and $V' \subset V$.

Now, backtracking a little, if $f : R \to S$ is a ring homomorphism, and M is an R-module (left if these are noncommutative), then what is $S \otimes_R M$? We have two categories, Mod_S modules over S and Mod_R modules over R, and there is the forgetful functor (think universal algebra):

$$\text{Mod}_S \overset{\text{forget}}{\longrightarrow} \text{Mod}_R$$
$$N \mapsto F(N)$$

where we view N as R-modules.

Exercise. $\text{Hom}_R(M, N) \cong \text{Hom}_S(S \otimes M, N)$

Then we can define

$$f^*\mathcal{M}(V) := \mathcal{O}_X(V) \otimes_{\mathcal{O}_Y(U)} \mathcal{M}(U)$$

for all pairs V, U with $f(V) \subset U$, and the maps for $V' \subset U$ where the restriction map is simply induced by $\mathcal{O}_X(V) \to \mathcal{O}_X(V')$:

$$\mathcal{O}_X(V) \otimes \mathcal{M}(U) \to \mathcal{O}_X(V') \otimes \mathcal{M}(V).$$

Notice if $V = \text{Spec}(B), V' = \text{Spec}\left(B\left[\frac{1}{g}\right]\right)$, $U = \text{Spec}(A)$, then

$$B\left[\tfrac{1}{g}\right] \otimes_B (B \otimes_A \mathcal{M}(U)) \cong B\left[\tfrac{1}{g}\right] \otimes_A \mathcal{M}(U) \cong B \otimes_A \mathcal{M}(U)\left[\tfrac{1}{g}\right].$$

In particular, if $\mathcal{E}$ is a locally free of rank n sheaf on Y (so $\exists$ an open cover $Y = \bigcup_i U_i$ such that $\sigma_i \mathcal{E}|_{U_i} \overset{\sim}{\to} \mathcal{O}_Y^n|_{U_i}$), then for every affine $V \subset X$ such that $f(V) \subset U_i$,

$$f^*(\mathcal{E})|_V \cong \mathcal{O}_X^n|_V.$$

That is, $\mathcal{O}_X(V) \otimes_{\mathcal{O}_Y(U_i)} \mathcal{O}_Y(U_i)^n \cong \mathcal{O}_X(U)^n$. Then

$$\theta_{ji} : \sigma_j \sigma_i^{-1} : \mathcal{O}^n|_{U_i \cap U_j} \to \mathcal{O}^n|_{U_i \cap U_j} \text{ (an iso given by an element of } GL_n(\mathcal{O}_{U_i \cap U_j}))$$

If $f(V) \subset U_i$ and $f(W) \subset U_j$, we have

$$f^*(\mathcal{E})|_V \overset{f^*(\sigma_i)}{\longrightarrow} \mathcal{O}_X^n|_V$$

with $f^*(\sigma_j) f^*(\sigma_i)^{-1} = f^*(\theta_{ji})$. In other words, you can think of a locally free sheaf as being determined by the patching data given by the θ_{ji}, and then we can think of the pullback f^* as:

Proposition. *$f^*(\mathcal{E})$ is determined by the cocycle*

$$f^*(\theta_{ij}) \in GL_n(\mathcal{O}_X(f^{-1}(U_i) \cap f^{-1}(U_j)))$$

$(X = \bigcup_i f^{-1}(U_i))$.

Geometrically, suppose we have a vector bundle

$$\begin{array}{c} E \\ \downarrow \pi \\ Y = \bigcup_i U_i = \text{Spec}(A_i) \end{array}$$

$$\begin{array}{ccccc} \mathbb{A}^n_{U_i} & \overset{\tau_i}{\leftarrow} & \pi^{-1}(U_i) & \subset & E \\ & \searrow & \downarrow & & \downarrow \\ & & U_i & \subset & Y \end{array}$$

with $\tau_i : \pi^{-1}(U_i) \cong A^n_{U_i} \cong \text{Spec} A_i[t_1, ..., t_n]$. Over $U_i \cap U_j$, we require that

$$\tau_j \cdot \tau_i^{-1} : \mathbb{A}^n_{U_i \cap U_j} \to \mathbb{A}^n_{U_i \cap U_j}$$

is a matrix

$$\sigma_{ji} \in GL_n(\mathcal{O}_Y(U_i \cap U_j))$$

with

$$(\tau_j \cdot \tau_i)^*(t_k) = \sum_{\ell=1}^{n} \theta_{ji}^{k\ell} t_\ell$$

with θ_{ji}^{**} entries in the matrix θ_{ji}. We will get to the completely geometric interpretation of locally free sheaves and pullback.

Lecture 24 (March 9, 2009) -

[I walked in late, missing a few notes.]

Standard example: $X = \mathbb{A}^{n+1}_A = \text{Spec}\, A[x_0, .., x_n]$, and $\underline{s} = (x_0, ..., x_n) : \mathcal{O}_X^{n+1} \to \mathcal{O}_X$. The image of s is the ideal of origin on $\mathbb{A}^{n+1} \backslash \{0\}$, $\underline{s}$ is surjective, so we have the standard map

$$\mathbb{A}^{n+1}_A \backslash \{0\} \to \mathbb{P}^n_A.$$

Our map $\Sigma : \mathcal{O}_X^{n+1} \to \mathcal{L}$ has image $\mathcal{M} \subset \mathcal{L}$ which is a subsheaf. Now, $\mathcal{M}_x \subset \mathcal{L}_x$ is the image $\mathcal{O}^{n+1}_{X,x}$, and since $\mathcal{L}_x = \mathcal{O}_{X,x}\ell,\ \mathcal{M}_x = g_x \mathcal{L}_x$, where g is an ideal. This is true in any affine neighborhood of x such that $\mathcal{L}|_U \cong \mathcal{O}_X|_U \ell$. Globally, $\mathcal{M} = \mathcal{G}\mathcal{L}$ where $\mathcal{G}$ is a sheaf of ideals in $\mathcal{O}_X$. If U is affine open, then im $\underline{s}|_U = \mathcal{G}(U) \cdot \mathcal{L}(U)$. So, we get a closed subscheme $Y \subset X$, i.e., $Y \subset X$ closed and $\forall U \subset X$ affine open,

$$Y \cap U = \text{Spec}(\mathcal{O}_X(U)/\mathcal{G}(U)),$$

that is, $i : Y \hookrightarrow X$ with $i_* \mathcal{O}_Y \cong \mathcal{O}_X/\mathcal{G}$.

Let $\underline{s} : \mathcal{O}_X^{n+1} \to \mathcal{L}$. Then image $\mathcal{M} \subset \mathcal{L}$ and is isomorphic to $\mathcal{G}\mathcal{L}$, where $\mathcal{G}$ is the sheaf of ideals. Let $Y \subset X$. Then $\mathcal{O}_Y \cong \mathcal{O}_X/\mathcal{G}$ is the subscheme where $\underline{s}$ vanishes on $X \backslash Y$, and this is the open subscheme on which $\underline{s}$ is surjective. We then get an induced map $X \backslash Y \to \mathbb{P}^n_A$.

$$\begin{array}{ccc} U & & \\ \cap & \searrow & \\ X & \underset{?}{\rightarrow} & P. \end{array}$$

$\Gamma_{f|_U} \subset X \times P$. Define $\tilde{X} =$ closure $\Gamma_{f|_U}$. Then

$$\begin{array}{c} U \subset \tilde{X} \rightarrow P \\ \searrow \ \swarrow \\ X. \end{array}$$

We can do this with schemes. If $X = \bigcup_i U_i = \text{Spec}(R_i)$, and $P = \bigcup_j V_j = \text{Spec}(S_j)$, then $X \times P = \bigcup_{i,j} U_i \times V_j = \text{Spec}(R_i \otimes S_j)$. Suppose that $U \subset X$ is an open subscheme (i.e., an open subset, $\mathcal{O}_U = \mathcal{O}_X|_U$). If given a map of schemes, $f : U \rightarrow P,$ then we know there exists an affine open cover of U, namely $W_k = \text{Spec}(T_k) \subset U$ such that $\forall k\ \exists j$ such that $f(W_k) \subset V_j$, and a map $f^*_{kj} : S_j \rightarrow T_k$.

Glueing together the graphs of the maps $f|_{W_k} : W_k \rightarrow V_j,\ W_k \rightarrow W_k \times V_j$ gives a map of schemes $\Gamma_f : U \rightarrow U \times_A P$. This makes sense for any morphisn of schemes $U \rightarrow P$.

Definition. *f is separated if Γ_f is the inclusion of a closed subscheme.*

In our situation, $U \xrightarrow{f} P$ (with $U \subset X$), we can define a closed subscheme $\tilde{X} \subset X \times P$ such that in each affine open $U_i \times V_j,\ U_i \subset X$ affine and $V_j \subset P$ opens. Then

$$\tilde{X} \cap (U_i \times V_j) = \text{Zariski closure of the graph of } f|_{U_i \cap U_j} \cap (U_i \times V_j).$$

We shall show this (next lecture) as an explicit ideal.

Suppose we are over a field k

$$\begin{array}{ccc} \mathbb{A}^2 \backslash \{0\} & & \\ \cap & \searrow & \\ \mathbb{A}^2_{x,y} & & \mathbb{P}^1_k = \text{Proj}(k[x,y]), \end{array}$$

with $\tilde{X} \subset \mathbb{A}^2_{x,y} \times \mathbb{P}^1_k = \text{Proj}(k[x', y'])$.

Exercise. $\tilde{X} \subset \mathbb{A}^2_{x,y} \times \text{Spec}\, k\left[\frac{x'}{y'}\right] \cup \mathbb{A}^2_{x,y} \times \text{Spec}\, k\left[\frac{y'}{x'}\right] := U_1 \cap U_2$. Hence, $\tilde{X} \cap U_1$ has equation $x = \left(\frac{x'}{y'}\right) \cdot y$. Similarly, $\tilde{X} \cap U_2$ has equation $y = \left(\frac{y'}{x'}\right) x$. Then

$$\tilde{X} \cap U_1 \cap U_2$$

we get $x/y = x'/y'$.

Lecture 28 (March 9, 2009) -

Valuation Rings

Recall if $\mathcal{O}$ is a valuation ring then any finitely generated ideal is principal. Since if $f_1 - f_k \in \mathcal{O}$ then choose i such that $v(f_i) = \min(v(f_1), ..., v(f_n))$, then $\forall j \neq i$, $v(f_j) \geq v(f_i)$.

(Valuation ring: $\mathcal{O}$ local domain s.t. if $k =$ fraction field of $\mathcal{O}$, $\exists$ valuation $v : k^* \to \Gamma$ totally ordered subgroup s.t. $v(xy) = v(x) + v(y)$ w/ $v(x+y) > \min(v(x), v(y))$ if $x + y \neq 0$, with $\mathcal{O} = \{x \in k^* \mid v(x) \geq 0\} \cup \{0\}$.)

That implies

$$v(f_j/f_i) \geq 0 \Longrightarrow f_j/f_i \in \mathcal{O} \Longrightarrow f_j \in (f_i) \Longrightarrow (f_1, ..., f_n) = (f_i).$$

Corollary. *Let $\mathcal{O}$ be a valuation ring with a fraction field k. Let $P \in \mathbb{P}^n(k)$. Then $\exists!$ point $\tilde{P} \in \mathbb{P}^n(\mathcal{O})$ which restrict to P.*

Proof. $\exists$ elements $a_0, ..., a_n \in k$ such that $P = (a_0 : ... : a_n) \in \mathbb{P}^n(k)$. Then $\exists$ $g \in \mathcal{O}$ such that $\forall i$, $ga_i \in \mathcal{O}$. Hence, we may assume $\forall i$, $a_i \in \mathcal{O}$. By remark above, the ideal $(a_0, ..., a_n) = (a_i)$ for some i. Thus

$$P = \left(\tfrac{a_0}{a_i}, ..., 1, ..., \tfrac{a_n}{a_i}\right),$$

(where 1 is in the ith place) for all $a_i/a_j \in \mathcal{O}$, and this represents an $\mathcal{O}$-valued point of $\mathbb{P}^n$.

Finally, if $(a_0, ..., a_n)$ and $(b_0, ..., b_n)$ are two sequences of elements of $\mathcal{O}$ generate the unit ideal in $\mathcal{O}$ such that viewed as points in $\mathbb{P}^n(k)$,

$$(a_0 : ... : a_n) = (b_0 : ... : b_n).$$

That implies $\exists c \in k$ such that $b_j = ca_j \, \forall j$.

We claim that $c \in \mathcal{O}^*$ equivalently $v(i) = 0$. Since $(a_0, ..., a_n) = (b_0, ..., b_n) = \mathcal{O}$, there exists i such that $v(a_i) = 0 \Longrightarrow v(c) = v(b_i) \geq 0$, and $\exists j$ such that $v(b_j) = 0 \Longrightarrow 0 = v(c) + v(a_j) \Longrightarrow v(c) \leq 0$.

Motivation from topology

$f : X \to Y$ is a map of metric spaces.

We say f is proper if $f^{-1}(k)$ is compact $\forall$ compact k. This implies f is a closed map, i.e., for $Z \subset X$ closed $\Longrightarrow$ $f(Z)$ is closed.

That implies that $\forall$ sequences $a_n \in Z$ converging in X, with $\lim\limits_{n\to\infty} a_n = a \in X$, that $a \in Z$.

Suppose that $\{b_n\}$ is a sequence in $f(Z)$ with limit $b \in Y$. $\exists a_n \in Z$ s.t. $f(a_n) = b_n$. Further, $\exists$ compact nbhd K of b and so we may assume that $b_n \in K \, \forall n$.

$\Longrightarrow a_n \in f^{-1}(K)$ which is compact

$\Longrightarrow a_n$ have an accum. point in $f^{-1}(K)$

$\Longrightarrow$ since b_n is convergent and f is continuous, $f(c) = b$ so $b \in f(Z)$.

Remark. A topological space is sequentially separated if Cauchy sequences have at most one limit.

If $\mathcal{O}$ is a valuation ring with fraction field k, think of map from Spec(k) to a scheme X as a sequence, and if map extends to Spec$(\mathcal{O}) \to X$ think of the induced map Spec$(k) \to X$ as the limit of the sequence.

Result above about $\mathbb{P}^n$ says that $\mathbb{P}^n$ is "compact".

In topology, a "nice" topological space X is compact if every open cover has a finite subcover. In algebraic geometry: consider for example $\mathbb{A}^1_{\mathbb{C}}$. The open sets here are complements of finitely many points ($\mathbb{A}^1 \backslash S$, with S finite). This implies any open cover has a finite subcover.

For example, the projection $p : \mathbb{A}^2 \xrightarrow{k} \mathbb{A}^1$ given by $k[x] \hookrightarrow k[x,y]$, projection onto x-axis, is not a closed map (closed sets $\to$ closed sets). For example, the hyperbola $xy = 1$ has open image $V(xy-1) = \mathbb{A}^1 \backslash \{0\}$.

Consider the discrete valuation $\mathcal{O} = k[t]_{(t)} \subset k(t) = K$ and Spec$(K) \to \mathbb{A}^2$ with $x \mapsto t$ and $y \mapsto t^{-1}$. Then Spec$(k[t,t^{-1}]) \cong C \subset \mathbb{A}^2$. The induced map $p \cdot f : \text{Spec}(K) \to \mathbb{A}^1_x$ extends to a map Spec$(\mathcal{O}_K) \to \mathbb{A}_x$ and $\to$ Spec$\left(k[x]_{(x)}\right)$. But $f : \text{Spec}(K) \to \mathbb{A}^2$ does not extend since $v(f^*(y)) = -1$.

Recall that a morphism $f : X \to Y$ is **separated** if $\Delta : X \to X \times_Y X$ is a closed immersion.

Definitions: (i) $f : X \to Y$ is of **finite type** if Y has an affine open cover $Y = \bigcup \text{Spec}(A_i)$, and $\forall i,\ f^{-1}(\text{Spec}(A_i))$ has a finite affine open cover $\bigcup_{j=1}^n \text{Spec}(B_{ij})$ with B_{ij} an A_i-algebra of finite type.

(ii) $f : X \to Y$ is **universally closed** if $\forall g : T \to Y$, the induced map $f_T : T \times_Y X \to T$ is closed, i.e., closed sets to closed sets.

Definition. $f : X \to Y$ is proper if it is separated, of finite type, and universally closed.

Theorem. *If Y is Noetherian (i.e., has a finite cover by affine opens* Spec(B_i) *with the B_i Noetherian), then a morphism $f : X \to Y$ is proper if for all valuation rings $\mathcal{O}$ and maps $\alpha :$* Spec$(k) \to X, \beta :$ Spec$(\mathcal{O}) \to Y$ (*with k fraction field of $\mathcal{O}$*),

$$\begin{array}{ccc} \text{Spec}(k) & \xrightarrow{\alpha} & X \\ \downarrow & & \downarrow \\ \text{Spec}(\mathcal{O}) & \underset{\beta}{\to} & Y \end{array}$$

*such that $f\alpha = \beta \,|\,$*Spec(k). *Then $\exists!\ \gamma :$* Spec$(\mathcal{O}) \to X$ *making the diagram commute.*

Corollary. $\mathbb{P}^n_{\mathbb{Z}}$ *is proper over* Spec$(\mathbb{Z})$.

Lecture 30 (March 30, 2009) -

Valuative Criterion of Properness

Recall if k is a field, $\mathcal{O} \subset k$ is a valuation ring if equivalently, $\exists$ a valuation $v : k \backslash \{0\} \to \Gamma$ a totally ordered group such that $\mathcal{O} = \{x \,|\, v(x) \geq 0\} \cup \{0\}$, or, $\mathcal{O}$ is maximal among local ring $R \subset K$ with respect to $R < S$ if $\mathfrak{m}_S \cap R = \mathfrak{m}_R$

Valuative Criterion of Seperatedness

Theorem. *If $f : X \to Y$ with X Noetherian* (*given the union of a finite* Spec(A), *A Noetherian*), *then f is separated if and only if $\forall$ valuation rings $\mathcal{O}$, and all commutative squares*

$$\begin{array}{ccc} \text{Spec}(k) & \xrightarrow{h} & X \\ i \downarrow & & \downarrow f \\ \text{Spec}(\mathcal{O}) & \xrightarrow[g]{} & Y, \end{array}$$

$\exists$ at most one map $\tilde{g}$: Spec$(\mathcal{O}) \to X$ (lift in the diagram) making the square commute.

Theorem. *Let X be Noetherian. If f is of finite type and Y is Noetherian, then f is proper if and only if $\exists$ exactly one lift $\tilde{g}$.*

Proof. Assume f is proper. Then given

$$\begin{array}{ccc} X_{\mathcal{O}} & \to & X \\ f' \downarrow & & \downarrow f \\ \text{Spec}(\mathcal{O}) & \xrightarrow[g]{} & Y \end{array}$$

let $f' : X_{\mathcal{O}} \to$ Spec$(\mathcal{O})$ be the pullback of f along g.

Now just consider

$$\begin{array}{ccc} \text{Spec}(k) & \xrightarrow{j} & X_{\mathcal{O}} \\ & i \searrow & f' \downarrow \text{f.t., sep} \\ & & \text{Spec}(\mathcal{O}). \end{array}$$

We would like to show there is an extension of the section Spec$(\mathcal{O}) \to X_{\mathcal{O}}$. Let $U =$ Spec(K) and let $S =$ Spec$(\mathcal{O})$. Let V be the Zariski closure of $j(U) \subset X_{\mathcal{O}}$. Since U is Spec(a field), U is irreducible. So really, $U = \{u\}$, and so $j(\{u\}) \in$ affine open set Spec(B) with Kernel$(K \leftarrow B)$ closed subscheme defined by prime ideals. Then V is an integral subscheme (all its coordinate rings are integral domains). Hence $f'(V) \subset S$ is closed and non-empty since f is proper (in fact, it contains a generic point Spec(k) of S). So $f'(V) = \overline{\text{Spec}(k)} \subset S$ which is just S.

$$\begin{array}{ccc} U \hookrightarrow & V & \subset X_{\mathcal{O}} \\ \searrow & \pi \downarrow & \text{proper} \\ & S. & \end{array}$$

Since $\{s\} \in S$ is a closed point of S, $\pi^{-1}(s)$ is a closed subset of V, a scheme of f.t. over $s =$ Spec(k) with k fraction field of $\mathcal{O}$. By the Nullstelensatz, there exist closed points $v \in V$ such that $\pi(v) = s$. (Note: For residue fields, $[k(v) : k(s)] < \infty$).

Essentially, we now want to show π is an isomorphism. $\forall v \in \pi^{-1}(s)$,

$$\pi^* : \mathcal{O} = \mathcal{O}_{S,s} \to \mathcal{O}_{V,v} \subset k$$

is a local homomorphism, but they have the same fraction field. Hence, since $\mathcal{O}$ is a valuation ring, $\pi_{S,V}$ is an isomorphism, and since this is true for all points above S, it is easy to check that $\exists$ exactly one point above s (notice $\mathcal{O}_{V,v'} \supset \mathcal{O}_{V,v}$). Hence, $V \cong S$ and so we get a map $S \to X$ lifting g. □

Example. Consider the nodal cubic $x = t^2 - 1$ and $y = t(t^2 - 1)$ so that $y^2 = x^2(x+1)$ and hence we have $f : \mathbb{A}^1_t \to \mathbb{A}^2_{x,y}$ with $f^{-1}(\{0, 0\}) = 2$ points. Then $\mathcal{O}_{C,P}$ is not a valuation ring.

Amazing stuff (finally, some geometry!!)

Remarks. (1) If X is a curve over a field, then X is non-singular if and only if all the local ring $\mathcal{O}_{X,P}$ for $P \in X$ closed, are valuation rings and they are in fact d.v.r.'s.

(2) $\dim(X) > 1$, $P \in X$, X finite type over a field k, X irreducible, then if P is a closed point $\mathcal{O}_{X,P}$ is never a valuation ring.

e.g. If we take $(0,0) \in \mathbb{A}^2_{x,y} = x$, then

$$\mathcal{O}_{X,P} = \{f(x,y) \in k(x,y) \,|\, f = g/h,\; g,h \in k[x,y],\; h(0,0) \neq 0\}.$$

Exercise. This is not a valuation! [Hint: complement of valuation must be consistent with reciprocal of ideal.]

$$\text{Bl}_{p_2}(\text{Bl}_p(X)) \leftarrow \ldots$$

with $p_2 \in \text{Bl}_p(X) \;\to\;$ p $\in X$, then $\bigcup \mathcal{O}_{p_i}$ is a valuation ring.

Next time: "Zariski-Riemann Space"

Lecture 31 (April 1, 2009) -

Cohomology

Goal: (1) Describe the cohomology of sheaves of abelian groups.

(2) (i) Cohomology $H^i(X, \mathcal{F}) = 0$ for X a scheme of finite type over a Noetheria nring and $i \gg 0$.

(ii) $H^i(X, \mathcal{F}) = 0$ if X is affine and $\mathcal{F}$ is quasi-coherent.

(3) if X is projective over $S = \text{Spec}(A)$ and $\mathcal{F}$ is a coherent sheaf on X then $H^i(X;\mathcal{F})$ is a finitely A-module for all $i \gg 0$.

Cohomology of sheaves of abelian groups

Suppose X is a topological space, and consider the category $Ab_X =$ sheaves of abelian groups on X. We thus have a functor

$$\Gamma : \text{Ab}_X \to \text{Ab}$$
$$\mathcal{F} \mapsto \Gamma(X, \mathcal{F}) = \mathcal{F}(X).$$

This functor is not exact, i.e., if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence of sheaves of abelian groups, then we have

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G}) \to \Gamma(X, \mathcal{H})$$

is exact, but the last map need not be surjective.

Recall an exercise: $\Gamma(X, \Gamma) \to \Gamma(X, \mathcal{H})$ is surjective if $\mathcal{F}$ is flasque.

Remark. In general, we have the notion of derived functors.

Idea. Suppose $F : A \to B$ is a left exact functor between abelian categories. A δ-functor is a sequence of functors

$$F^i : A \to B,\; i \geq 0,\; F^0 = F$$

and for every exact sequence

$$0 \to A_1 \to A_2 > A_3 \to 0$$

in A, maps $\partial^i : F^i(A_3) \to F^{i+1}(A_1)$ such that we have a long exact sequence

$$0 \to F(A_1) \to F(A_2) \to F(A_3) \overset{\partial^0}{\to} F^1(A_1) \to F^1(A_2) \to F^1(A_3) \overset{\partial^1}{\to} F^2(A_1) \to \ldots$$

and the ∂'s are "natural" with respect to the maps of exact sequences.

Derived functors $(R^i F, \partial^i), i \gg 0$, are (if they exist) the universal ∂-functor extending F, i.e., for any ∂-functor $\exists$ a unique transformation of ∂-functors

$$(R^i F, \partial^i) \to (F^i, \partial).$$

Injective objects

If A is an abelian category, then an object $I \in A$ is injective if $\forall$ diagrams

$$\begin{array}{ccc} & & B \\ \tilde{f} & \swarrow & \uparrow i \\ I & \underset{f}{\leftarrow} & A \end{array}$$

with i a monomorphism, there exists an $\tilde{f}$ making the diagram commute.

Lemma. *I is injective iff every exact sequence*

$$0 \to I \overset{i}{\to} A \overset{\varepsilon}{\to} B \to 0$$

splits, i.e., $\exists\ s : A \to I$ such that $s \cdot i = \mathrm{Id}_A$ so $A \cong I \oplus B$.

Corollary. *If $F : A \to B$ is a left exact additive functor and i is an injective object in A, and*

$$0 \to I \to A_2 \to A_3 \to 0$$

is exact in A, then

$$0 \to F(I) \to F(A_2) \to F(A_3) \to 0$$

is exact.

Proof. Follows from lemma and fact additive functors preserve direct sums. (exercise)☐

Suppose that A has "enough" injectives. i.e., $\forall$ objects $\boldsymbol{A} \in A$, $\exists I$ and a monomorphism $A \hookrightarrow I$. If so, every object has an injective resolution:

$$0 \to A \to I^0 \to I^1 \to \ldots$$

i.e., an exact sequence with I^j injective for $j \gg 0$.

Lemma. *Given two injective resolutions $A \overset{\varepsilon}{\to} I^0$ and $A \overset{\varepsilon^1}{\to} J^0$, $\exists$ a map of cochain cxs*

$$\varphi^0 : I^0 \to J^0$$

such that $\varphi \circ \varepsilon = \varepsilon'$, that is, $A \to I^0 \to J^1 \to \ldots$ and $A \hookrightarrow J^0 \to J^1 \to \ldots$ with φ^0, φ^1 between the I and J's, and $A \to I^0$ and $A \to J^0$.

If $\varphi, \varphi' : I^0 \to J^0$ are 2 such maps, then they are chain homotopic.

Corollary. *If J^0 is also an injective resolution of A, then $\exists\psi : J^0 \to I^0$ extending the identity on A, and $\psi \circ \varphi$ is chain homotopic to the identity of I, and $\varphi \circ \psi \sim \mathrm{Id}_J$.*

Suppose $F : A \to B$ is a left exact functor. Pick an injective resolution of $A \in \mathbb{Q}$, $A \to I^0$, and consider the complex $F(I^0)$ in B. Observe up to chain homotopy equivalence, this does not depend on I^1, and any two chains of φ or ψ induces chain homotopic maps $F(I) \to F(J)$ and $F(J) \to F(I)$. Hence, we get canonical isomorphisms $H^i(F(I)) \to H^i(F(J))$ for $i \gg 0$, and so we can define

$$R^i F(A) := H^i(F(I))$$

for any choice of injective resolution I of A.

Lecture 33 (April 6, 2009) -

Definition. $H^i(X, \mathcal{F}) = H^i(\Gamma(X, \mathcal{I}^*))$.

(By previous discussion, this is (up to canonical iso) indep of choice of $\mathcal{I}^*$.

$f_* : \mathcal{O}_X$-modules $\to \mathcal{O}_Y$-modules.

This is left exact, since

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(u)),$$

and in general not exact, e.g., if $Y = \text{pt}$, then f_* is just $\Gamma(X, \_\_)$.

We can define $R^i f_* : \mathcal{O}_X\text{-modules} \to \mathcal{O}_Y\text{-modules}$ given by $\mathcal{F} \mapsto H^i(f_*(\mathcal{I}^*))$ (a complex of sheaves of $\mathcal{O}_Y$-modules).

Remark. The object one should "really" consider is the whole complex $f_*(\mathcal{I}^*)$. This is well-defined up to chain-homotopy equivalence, and hence up to quasi-isomorphism. This is the map of complexes which induces an isomorphism on cohomology.

Derived Category

$D^{\square}(X, \mathcal{O}_X)$. Start with the category of cochain complexes of $\mathcal{O}_X$-modules, where $\square$ can be either "bounded", "bounded below" ($\mathcal{F}^i \neq 0$ only for finitely many i, and $H^i(\mathcal{F}^i) = 0$ for $i \ll 0$, respectively), and "qc sheaves $(X, \mathcal{O}_x)$. Then, invert all the quasi-isomorphisms (and you get $D^{\square}(X, \mathcal{O}_X)$). (That implies if $\mathcal{F} \to \mathcal{I}^*$ is a resolution, then $\mathcal{F}$ and $\mathcal{I}^*$ iso. in $D(X)$).

$$Rf_* : D(X) \to D(X)$$

with $\mathcal{F}^* \cong \mathcal{I}^* \implies Rf_*(\mathcal{F}) = f_*(\mathcal{I}^*)$.

Warning. A priori, $R^i f_*$ may depend on what category of modules you are working with.

If $A \to R$ is a ring hom with I an injective R-module, then what is injective over [diagram]. - May depend on choice of $\mathcal{O}_X$ and on particular category of modules.

Lemma. *Any injective sheaf $\mathcal{I}$ of $\mathcal{O}_X$-modules is flasque.*

Proof. If $U \subset X$ is open, consider $j : U \hookrightarrow X$, and let $j_!\mathcal{O}_U$ be the extension by zero of $\mathcal{O}_U$, i.e., if $V \subseteq U$, then $j_!\mathcal{O}_U(v) = \mathcal{O}_U(v) = \mathcal{O}_X(v)$ and if $V \nsubseteq U$, then $j_!\mathcal{O}_U(v) = 0$.

Exercise. $\mathrm{Hom}_{\mathcal{O}_X}(j_!\mathcal{O}_U, \mathcal{F}) = \mathcal{F}(U)$.

The natural map $j_!\mathcal{O}_U \to \mathcal{O}_X$ is injective. Hence, since $\mathcal{I}$ is injective,

$$\text{Hom}(\mathcal{O}_X, \mathcal{I}) \to \text{Hom}(j_! \mathcal{O}_U, \mathcal{I})$$

is surjective.

Proposition. *If $\mathcal{F}$ is a flasque sheaf of $\mathcal{O}_X$-modules on X, then $H^i(X, \mathcal{F}) = 0$ for $i > 0$.*

Proof. Choose $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0$ with $\mathcal{I}$ injective. Recall if $\mathcal{F}$ is flasque, then $\mathcal{I}(X)$ surjects into $\mathcal{G}(X)$. Since $\mathcal{I}$ is injective, $H^i(X, \mathcal{I}) = 0$ for $i > 0$ (it is its own injective resolution).

Now look at the long exact sequence:

$$0 \to \mathcal{F}(X) \to \mathcal{I}(X) \to \mathcal{G}(X) \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{I}) \to H^1(X, \mathcal{G}) \xrightarrow{\partial} H^2(X, \mathcal{F})..$$

Hence, $H^1(X, \mathcal{F}) = 0$ for $i > 1$. But $\mathcal{G}$ is flasque. Then

$$\begin{array}{ccccc} \mathcal{F}(X) & \to & \mathcal{I}(X) & \to & \mathcal{F}(X) \\ & & \downarrow & & \downarrow \\ & & \mathcal{F}(U) & \to & \mathcal{G}(U) \end{array}$$

where the bottom row is a surjection. Hence by induction, $H^i(X, \mathcal{F}) = 0$ for $i > 0$. i.e., flasque sheaves acylic as indeed $\mathcal{F}$ flasque implies $R^i f_* \mathcal{F} = 0$ for $i > 0$. □

Exercise. $R^i f_*(U) = H^i(f^{-1}(U), \mathcal{F})$.

Exercise. $R^i f_* =$ sheaf assoc to presheaf $U \mapsto H^i(f^{-1}(U), \mathcal{F})$.

Proposition. *Suppose $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, and $\mathcal{F} \xrightarrow{\varepsilon} \mathcal{A}^*$ is a resolution of $\mathcal{F}$ by acyclic sheaves. Then $H^i(X, \mathcal{F}) \cong H^i(\Gamma(X, \mathcal{A}^*))$.*

Proof. Look at

$$0 \to \mathcal{F} \to \mathcal{A}^0 \to \mathcal{G}^0 \to 0.$$

Then

$$H^0(X, \mathcal{F}) \to H^0(X, \mathcal{A}^0) \to H^0(X, \mathcal{G}^0) \xrightarrow{\partial} H^1(X, \mathcal{F}) \to H^1(X, \mathcal{A}^0) = 0 \to ...$$

And in general $H^i(X, A^0) = 0$ for $i \geq 1$,

(1) $H^1(X, \mathcal{F}) = \text{Coker}(H^i(X, A^0) \to H^0(X, \mathcal{G}^0))$ **(★)**

$H^{i+1}(X, \mathcal{F}) \cong H^i(X, \mathcal{G}^i)$ **(★★)**

Look at resolution

$$0 \to F \to \mathcal{A}^0 \to \mathcal{A}^1 \to \mathcal{A}^2 \to ...$$

So

$$0 \to \mathcal{G}^0 \to A^1 \to A^2 \to ...$$

is a resolution. Left exactness of H^0 implies

$$\bigstar = H^i(\Gamma(X, A^*)),$$

and ★★ and ★ imply by induction that

$$\text{H}^i(X, \mathcal{F}) = H^1(X, \mathcal{G}^{i-1}) = H^i(\Gamma(X, A^*)).$$

Hence, we can use any acyclic resolution to compute cohomology. □