Lecture 1 (January 26, 2009) - Diamond Chapter 1.1-1.3

§1.1 in Diamond

Modular group

Start with

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : ad - bc = 1 \right\}.$$

Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Then we define an action on $\hat{\mathbb{C}}$ by

$$\gamma \cdot \tau = \tfrac{a\tau + b}{c\tau + d}.$$

If $c \neq 0$, then $\tau = \frac{-d}{c} \mapsto \infty$ and $\infty \mapsto \frac{a}{c}$. If $c = 0$, then $\infty \mapsto \infty$. We then let

$$\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}.$$

If $\tau \in \mathbb{H}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then $\gamma\tau \in \mathbb{H}$, because

$$\mathrm{Im}(\gamma\tau) = \tfrac{\mathrm{Im}(\tau)}{|c\tau + d|^2}$$

(so that if $\mathrm{Im}(\gamma\tau) > 0$ then $\mathrm{Im}(\tau) > 0$). Note here that $\pm\gamma$ give the same action. It's simple to check that if $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, then $I \cdot \tau = \tau$, and if $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$, then $(\gamma\gamma')\tau = \gamma(\gamma'\tau)$ (so this is indeed an action).

Definition. *Let $k \in \mathbb{Z}$. A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is weakly modular of weight k if $f(\gamma\tau) = (c\tau + d)^k f(\tau)$ $\forall \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}), \tau \in \mathbb{H}$.*

From the first exercise in Diamond, we can check $\mathrm{SL}_2(\mathbb{Z})$ is generated by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then it turns out $\tau \mapsto \tau + 1$ and $\tau \mapsto -\frac{1}{\tau}$.

To check that a meromorphic function is weakly modular of weight k, one must only verify that $f(\tau + 1) = f(\tau)$ and $f(-1/\tau) = \tau^k f(\tau)$.

To proceed further, we first have to define the notion of a function being holomorphic at ∞. If f is weakly modular of weight k with $f(\tau + 1) = f(\tau)$, let

$$D = \{q \in \mathbb{C} : |q| < 1\} \text{ be the open unit disc}$$

and $D' = D - \{0\}$ the punctured open unit disc. Then the map $\tau \mapsto e^{2\pi i\tau}$ is define on $\mathbb{H} \to D'$ and is holomorphic and $\mathbb{Z}$-periodic. Define $g : D' \to \mathbb{C}$ by

$$g(q) = f(\log(q)/(2\pi i)).$$

Note that $f(\tau) = g(e^{2\pi i\tau})$. If f is holomorphic on $\mathbb{H}$, then g is holomorphic on D'. So

$$g(q) = \sum_{n \in \mathbb{Z}} a_n q^n \text{ (for } q \in D').$$

Definition. *We say* $\mathbf{0}$ *is holomorphic at* ∞ *if the corresponding function* $\mathbf{1}$ *can be extended to a holomorphic function on* $\mathbf{H}$, $\mathbf{0}(\tau)$ œ $\sum_{\mathbf{8}-\mathbb{Z}} \mathbf{+_8} \mathbf{/}^{\#\pi 38\tau}$.

Note that to show a weakly holomorphic function of weight $\mathbf{5}$, call it $\mathbf{0}$, is holomorphic at ∞ is equivalent to showing that

$$\lim_{\mathrm{Im}(\tau)\,\ddot{\mathrm{A}}\,\infty} \mathbf{0}(\tau)$$

is finite or bounded. Holly points out this is along *any path* that goes up on the imaginary axis (so we just need to find an upper bound).

Definition. *Let* $\mathbf{5} - \mathbb{Z}$*. A function* $\mathbf{0}$ À $\mathbb{H}$ Ä $\mathbb{C}$ *is a modular form of weight* $\mathbf{5}$ *if*

(1) $\mathbf{0}$ *is holomorphic on* $\mathbb{H}$,

(2) $\mathbf{0}$ *is weakly modular of weight* $\mathbf{5}$,

(3) $\mathbf{0}$ *is holomorphic at* ∞.

The set of modular forms of weight $\mathbf{5}$ *is actually a* $\mathbb{C}$*-vector space, written* $\mathcal{M}_{\mathbf{5}}(\mathbf{WP_\#}(\mathbb{Z}))$.

Define

$$\mathcal{M}_{\mathbf{5}}(\mathbf{WP_\#}(\mathbb{Z})) \text{ œ } \bigoplus_{\mathbf{5}\,-\,\mathbb{Z}} \mathcal{M}_{\mathbf{5}}(\mathbf{WP_\#}(\mathbb{Z}))$$

which is a graded ring.

Examples. (1) The zero function " Ÿ a modular form for all weights.

(2) Constant functions are modular forms for weight œ !.

Definition. *Let* $\mathbf{5}$ ☐ # *be even. The Eisenstein series*

$$\mathbf{K_5}(\tau) \text{ œ } \sum'_{(-\text{ß}.)-\mathbb{Z}^\#} \frac{"}{(-\tau\square.)^{\mathbf{5}}},$$

where the prime denotes summing over (-ß .) $-$ $\mathbb{Z}^{\#}$Ï({!ß!}).

Naturally, $\mathbf{K_5}(\tau)$ is holomorphic on τ (Exercise 1.1.4(c)). We can compute that it is indeed weakly modular of weight $\mathbf{5}$. If γ $-$ $\left(\begin{smallmatrix} + & , \\ - & . \end{smallmatrix}\right)$ $-$ $\mathbf{WP_\#}(\mathbb{Z})$, then

$$\mathbf{K_5}(\gamma\tau) \text{ œ } \sum'_{(-"ß.")} \frac{"}{\left(-"\left(\frac{+\tau\square,}{-\tau\square.}\right)\square."\right)^{\mathbf{5}}} \text{ œ } (-\tau \square .)^{\mathbf{5}} \sum'_{(-"ß.")} \frac{"}{[(-"+\square-.")\tau\square(-",\square..")]^{\mathbf{5}}} \text{ œ } (-\tau\square .)^{\mathbf{5}} \mathbf{K_5}(\tau),$$

since

$$(-"ß .")\begin{pmatrix} + & , \\ - & . \end{pmatrix} \text{ œ } (-"+\square-."ß-",\square..").$$

Finally, $\mathbf{K_5}(\tau)$ is holomorphic at ∞ since it is bounded as $\mathrm{Im}(\tau)$ Ä ∞ (duh, because the terms are $\frac{"}{(-\tau\square.)^{\mathbf{5}}}$). Then the Fourier series of $\mathbf{K_5}(\tau)$ will be [page 5 of the book]

$$\mathbf{K_5}(\tau) \text{ œ } \#\zeta(\mathbf{5}) \square \frac{\#(\#\pi 3)^{\mathbf{5}}}{(\mathbf{5}\square")\text{x}} \sum_{\mathbf{8}œ"}^{\infty} \sigma_{\mathbf{5}\square"}(\mathbf{8})\,;^{\mathbf{8}},$$

where $\sigma_{5□"}(8)$ œ $\sum_{7\,|\,8ß\,7□!} 7^{5□"}$. We then have the normalized Eisenstein series

$$\frac{K_5(\tau)}{\#\zeta(5)} \text{ œ } I_5(>).$$

Then $\mathcal{M}_{)}(WP_{\#}(\mathbb{Z}))$ has dimension ", and $I_{\%}(\tau)^{\#}$ß $I_{)}(\tau)$ − $\mathcal{M}_5(WP_{\#}(\mathbb{Z}))$. Further,

$$\sigma_{(}(8) \text{ œ } \sigma_{\$}(8) \;□\; "\#>\sum_{3œ"}^{8□"} \sigma_{\$}(3)\sigma_{\$}(8 \;□\; 3).$$

Definition. *A cusp form of weight* 5 *is a modular form of weight* 5 *if in the Fourier expansion its leading coefficient is* $+_{!}$ œ !. *The set of cusp forms are denoted* $\mathcal{S}_5(WP_{\#}(\mathbb{Z}))$, *and* $\mathcal{S}(WP_{\#}(\mathbb{Z}))$ œ $\bigoplus_{5-\mathbb{Z}} \mathcal{S}_5(WP_{\#}(\mathbb{Z}))$.

It's easy to see that $\mathcal{S}_5(WP_{\#}(\mathbb{Z}))$ is a vector subspace of $\mathcal{M}_5(WP_{\#}(\mathbb{Z}))$.

Example. *Let* $1_{\#}(\tau)$ œ '!$K_{\%}(\tau)$ß $1_{\$}(\tau)$ œ "%!$K_{'}(\tau)$, *and* $\Delta(\tau)$ œ $1_{\#}^{\$}(\tau)$ □ #($1_{\$}^{\#}(\tau)$ *with* $\Delta(\tau)$ − $\mathcal{S}_{"\#}(WP_{\#}(\mathbb{Z}))$.

§1.2 Congruent Subgroups

Definition. *A principal congruence subgroup of level* R − $\mathbb{N}$ *is given by*

$$\Gamma(R) \text{ œ } \left\{ \begin{pmatrix} + & , \\ - & . \end{pmatrix} - WP_{\#}(\mathbb{Z}) \text{ À } \begin{pmatrix} + & , \\ - & . \end{pmatrix} \text{ ´ } \begin{pmatrix} " & ! \\ ! & " \end{pmatrix} \bmod R \right\},$$

where the reduction mod R *is coefficient-wise in the matrix.*

Note that $\Gamma(")$ œ $WP_{\#}(\mathbb{Z})$ and $\Gamma(R)$ – $WP_{\#}(\mathbb{Z})$ (with $WP_{\#}(\mathbb{Z})$ Ä $WP_{\#}(\mathbb{Z}\,Î\,R\,\mathbb{Z})$. It turns out $WP_{\#}(\mathbb{Z}\,Î\,R\,\mathbb{Z})$ z $WP_{\#}(\mathbb{Z})\,Î\,\Gamma(R)$. Furthermore,

$$[WP_{\#}(\mathbb{Z}) \text{ À } \Gamma(R)] \text{ œ } R^{\$} \dagger \prod_{:|R} \left(" \;□\; \frac{"}{:^{\#}}\right) \;□\; \infty.$$

Definition. Γ © $WP_{\#}(\mathbb{Z})$ *is a congruence subgroup of level* R *if* $\Gamma(R)$ © Γ.

Definition. $\Gamma_{!}(R)$ œ $\left\{ \begin{pmatrix} + & , \\ - & . \end{pmatrix} - WP_{\#}(\mathbb{Z}) \right\}$ *with*

$$\begin{pmatrix} + & , \\ - & . \end{pmatrix} \text{ ´ } \begin{pmatrix} ‡ & ‡ \\ ! & ‡ \end{pmatrix} (\bmod R).$$

$\Gamma_{"}(R)$ œ $\left\{ \begin{pmatrix} + & , \\ - & . \end{pmatrix} - WP_{\#}(\mathbb{Z}) \right\}$ *with*

$$\begin{pmatrix} + & , \\ - & . \end{pmatrix} \text{ ´ } \begin{pmatrix} " & ‡ \\ ! & " \end{pmatrix} (\bmod R).$$

Further,

$$\Gamma(R) \text{ © } \Gamma_{"}(R) \text{ © } \Gamma_{!}(R) \text{ © } WP_{\#}(\mathbb{Z}).$$

$$\Gamma(R) \trianglelefteq \Gamma_{"}(R) \text{ and } \Gamma_{"}(R) \trianglelefteq \Gamma_{!}(R).$$

Lecture 2 (February 2, 2009) - Diamond Chapter 1.3-1.5

Definition. *For* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$*, define the factor of automorphy* $j(\gamma, \cdot)$ *for* $\tau \in \mathbb{H}$ *by* $j(\gamma, \tau) = c\tau + d$.

Definition. *For* $k \in \mathbb{Z}$*, the weight-k operator* $[\gamma]_k$ *on* $f : \mathbb{H} \to \mathbb{C}_j$ *is*

$$f([\gamma]_k)(\tau) = j(\gamma, \tau)^{-k} f(\gamma\tau).$$

Definition. Let Γ be a congruent subgroup. We say $f : \mathbb{H} \to \mathbb{C}$ is weakly modular of weight k for Γ if

(a) f is meromorphic, and

(b) $f[\gamma]_k = f \quad \forall \gamma \in \Gamma$

Lemma. $\forall \gamma, \gamma' \in SL_2(\mathbb{Z}), \tau \in \mathbb{H}$,

(a) $j(\gamma\gamma', \tau) = j(\gamma, \gamma'\tau) \cdot j(\gamma', \tau)$

(b) $(\gamma\gamma')(\tau) = \gamma(\gamma'\tau)$

(c) $[\gamma\gamma']_k = [\gamma]_k[\gamma']_k$

(d) $\mathrm{Im}(\gamma\tau) = \mathrm{Im}(\tau)/|j(\gamma, \tau)|^2$.

Definition. If Γ is a congruence subgroup, $k \in \mathbb{Z}, f : \mathbb{H} \to \mathbb{C}$, then f is a modular form for Γ if (1) f is holomorphic, (2) f is weight-k invariant under Γ, and (3) $f[\gamma]_k$ is holomorphic at ∞ for all $\gamma \in SL_2(\mathbb{Z})$.

If $a_0 = 0$ in all the Fourier expansions of (3), then we say f is a cusp form for Γ. Recall we defined

$$\mathcal{M}_k(SL_2(\mathbb{Z})) = \bigoplus_{k \in \mathbb{Z}} \mathcal{M}_k(SL_2(\mathbb{Z})) \text{ and } \mathcal{S}(SL_2(\mathbb{Z})) = \bigoplus_{k \in \mathbb{Z}} \mathcal{S}_k(SL_2(\mathbb{Z})).$$

We can write $SL_2(\mathbb{Z}) = \bigcup_j \Gamma\alpha_j$ (a finite union), and $f[\gamma\alpha_j] = f[\alpha_j]$. Also, $f[\gamma]_k = f$.

§1.3 Complex Tori

Definition. *A lattice is a subgroup of the form* $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z} \subseteq \mathbb{C}$ *such that* $\{\omega_1, \omega_2\}$ *are linearly independent over* $\mathbb{R}$*, and we requires* $\omega_1/\omega_2 \in \mathbb{H}$.

Definition. *A complex torus is a quotient of* $\mathbb{C}$ *by a lattice, that is,* $\mathbb{C}/\Lambda$.

Proposition. *Let* $\phi : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ *be a holomorphic map. Then* $\exists m, b \in \mathbb{C}$ *with* $m\Lambda \subseteq \Lambda$*, and* $\psi(z + \Lambda) = mz + b + \Lambda'$*. The map is invertible if and only if* $m\Lambda = \Lambda'$.

Corollary. *If* $\phi : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ *is a holomorphic map between complex tori with*

$$\phi(z + \Lambda) = mz + b + \Lambda',$$

and $m\Lambda \subseteq \Lambda$*, then the following are equivalent*

(a) ϕ *is a group homomorphism.*

(b) , − Λ‼, *so* ϕ(D □ Λ) œ 7D □ Λ‼.

(c) ϕ(!) œ !.

In particular, there exists a holomorphic group isomorphism between ℂÎΛ *and* ℂÎΛ‼ *if and only if there exists* 7 − ℂ *such that* 7Λ œ Λ‼.

Take 7 œ $\frac{"}{\omega_{\#}}$. Then $\frac{"}{\omega_{\#}}$Λ œ $\frac{\omega_{"}}{\omega_{\#}}$ℤ Š ℤ œ Λ_{τ}, where we let Λ_{τ} be the lattice for $\frac{\omega_{"}}{\omega_{\#}}$ℤ Š ℤ. Is it possible to get Λ_{τ} œ $\Lambda_{\tau''}$? The answer will be yes: through an element of WP$_{\#}$(ℤ).

Definition. *A nonzero holomorphic homomorphism between complex tori is called an isogeny.*

Example. We define a multiplication-by-[R] map to be the isogeny:

[R] À ℂÎΛ Ä ℂÎΛ with D □ Λ È RD □ Λ.

Note RΛ § Λ so that it is indeed an isogeny. Then ker([R]) z (ℤÎ8ℤ)$^{\#}$. [Ramin says Alina Cojocaru, along with many other people, have made a career of researching this kernel and the information it provides!]

Example. Let R − ℕ and let G § I [R] be such that G z ℤÎRℤ, so G ¨ Λ as lattices. Then we have a map

□jx−7 ´ R)−(3i , 1 7 : '3%: 'o1x) 7 : R−) (z 3 t 1 (7 $ ´ z ' (3n1x%)) ' : $−3m1x$ 7 ' : $3a1x) 7 : R ´ '%−3r1(7 $

(remember this comes from the Eisenstein series, G_4), and $g_3(\Lambda) = 140\sum'_{\omega\in\Lambda}\frac{1}{\omega^6}$.

If we let $\Delta = g_2^3 - 27g_3^2 \neq 0$, then recall for elliptic curves of characteristic 2, we have the Weierstrass normal form, $y^2 = 4x^3 - ax - b$, $(a^3 - 27b^2 \neq 0)$.

§1.5 - Modular curves and moduli spaces

We will call two complex tori equivalent, $\mathbb{C}/\Lambda \sim \mathbb{C}/\Lambda'$ if $\exists m$ such that $m\Lambda = \Lambda'$. Furthermore, we will call $\tau \sim \tau'$ equivalent ($\tau, \tau' \in \mathbb{H}$) if there is a $\gamma \in SL_2(\mathbb{Z})$ such that $\tau = \gamma\tau'$. Our goal is to form an equivalence between the tori $\sim$ and the τ $\sim$.

Definition. *Let $N \in \mathbb{N}$. An enhanced elliptic curve for $\Gamma_0(N)$ is an orded pair (E, C) where E is a complex torus and C is a cyclic subgroup of order N.*

We say $(E, C) \sim (E', C')$ (an equivalence relation) if there is an isomorphism such that $E \cong E'$ and $C \cong C'$. Denote $S_0(N) = \{\text{enhanced elliptic curves for } \Gamma_1(N)\}/\sim$.

Definition. *Let $N \in \mathbb{N}$. An enhanced elliptic curve for $\Gamma_1(N)$ is an orded pair (E, P) where E is a complex torus and P is a point of order N.*

We say $(E, P) \sim (E', P')$ if there exists an isomorphism $E \cong E'$ and $P \cong P'$. Denote $S_1(N) = \{\text{enhanced elliptic curves for } \Gamma_1(N)\}/\sim$.

Definition. *Let $N \in \mathbb{N}$. An enhanced elliptic curve for $\Gamma(N)$ is an orded pair $(E, (P, Q))$ where E is a complex torus and $P, Q \in E[N]$ where $e_N(P, Q) = e^{2\pi i/N}$.*

We say $(E, (P, Q)) \sim (E', (P', Q'))$ if there exists an isomorphism $E \cong E'$, $P \cong P'$, and $Q \cong Q'$. Denote $S(N) = \{\text{enhanced elliptic curves for } \Gamma(N)\}/\sim$.

These critters, S_0, S_1, and S are called **moduli spaces**.

Let $\Gamma \subseteq SL_2(\mathbb{Z})$ be a congruence subgroup:

$$Y(\Gamma) := \Gamma\backslash\mathbb{H} = \{\Gamma_\tau : \tau \in \mathbb{H}\} \text{ orbits.}$$

$$Y_0(N) := \Gamma_0(N)\backslash\mathbb{H}, \quad Y_1(N) := \Gamma_1(N)\backslash\mathbb{H}, \quad \text{and } Y(N) = \Gamma(N)\backslash\mathbb{H}.$$

Notation. We use brackets instead of parentheses in $[E, C]$, $[E, P]$, and $[E, (P, Q)]$ to represent the equivalence classes under the appropriate relation.

Theorem. (a) Let $S_0(N) = \left\{\left[E_\tau, \langle \frac{1}{N} + \Lambda_\tau\rangle\right] : \tau \in \mathbb{H}\right\}$. Then

(b) *The moduli space for $\Gamma_1(N)$ is*

$$S_1(N) = \{[E_\tau, 1/N + \Lambda_\tau] : \tau \in \mathcal{H}\}.$$

Two points $[E_\tau, 1/N + \Lambda_\tau]$ and $[E_{\tau'}, 1/N + \Lambda_{\tau'}]$ are equal if and only if $\Gamma_1(N)\tau = \Gamma_1(N)\tau'$. Thus there is a bijection

$$\psi_1 : S_1(N) \xrightarrow{\sim} Y_1(N), \qquad [\mathbf{C}/\Lambda_\tau, 1/N + \Lambda_\tau] \mapsto \Gamma_1(N)\tau.$$

(c) *The moduli space for $\Gamma(N)$ is*

$$S(N) = \{[\mathbf{C}/\Lambda_\tau, (\tau/N + \Lambda_\tau, 1/N + \Lambda_\tau)] : \tau \in \mathcal{H}\}.$$

Two points $[\mathbf{C}/\Lambda_\tau, (\tau/N+\Lambda_\tau, 1/N+\Lambda_\tau)]$, $[\mathbf{C}/\Lambda_{\tau'}, (\tau'/N+\Lambda_{\tau'}, 1/N+\Lambda_{\tau'})]$ are equal if and only if $\Gamma(N)\tau = \Gamma(N)\tau'$. Thus there is a bijection

$$\psi : S(N) \xrightarrow{\sim} Y(N), \qquad [\mathbf{C}/\Lambda_\tau, (\tau/N + \Lambda_\tau, 1/N + \Lambda_\tau)] \mapsto \Gamma(N)\tau.$$

Lecture 5 (March 2, 2009) - Diamond Chapter 3.2

Let **Z §** $\mathbb{C}$. Recall **0 À Z Ä ©** meromorphic on **Z** means it has a Laurent expansion

$$\mathbf{0}(\mathbf{D}) \; \mathbf{œ} \sum_{\mathbf{8œ7}}^{\infty} \mathbf{+_8}(\mathbf{>} \,\square\, \tau)^{\mathbf{8}}$$

for **. l>** in some disk about τ, where $\mathbf{+_8}$ **−** ℂß **7 −** $\mathbb{Z}$.

Definition. *The order of* **0** *at* τ *is*

$$\nu_\tau(\mathbf{0}) \; \mathbf{œ} \; \mathbf{7}$$

and when **0 ´ !**, *we say* $@_\tau(\mathbf{0})$ **œ ∞**.

Definition. *A function* **0 À** $\mathbb{H}$ **Ä** $\mathbb{C}$ *is an automorphic form of weight with respect to* Γ *if*

(1) **0** *is meromorphic on* $\mathbb{H}$.

(2) $\mathbf{0}[\gamma]_{\mathbf{5}}$ **œ 0** *for all* γ **−** Γ.

(3) **0** *is meromorphic at the cusps of* **∞** *(i.e.,* $\mathbf{0}[\alpha]_{\mathbf{5}}$ *is meromorphic at* **∞** *for all* α **− WP**$_{\#}(\mathbb{Z})$*).*

Let **=** be a cusp of Γ, with **= −** $\mathbb{Q} \,\square\, \{$**∞**$\}$. Let α **− WP**$_{\#}(\mathbb{R})$ with $\alpha($**∞**$)$ **œ =** (with $\alpha^{\square"}(\mathbf{=})$ **œ ∞**). Then $\alpha^{\square"}\Gamma_{\mathbf{=}}\alpha$ gixes **∞** and so it is generated by **„** $\begin{pmatrix} " & 2 \\ ! & " \end{pmatrix}$ for some positive integer **2**.

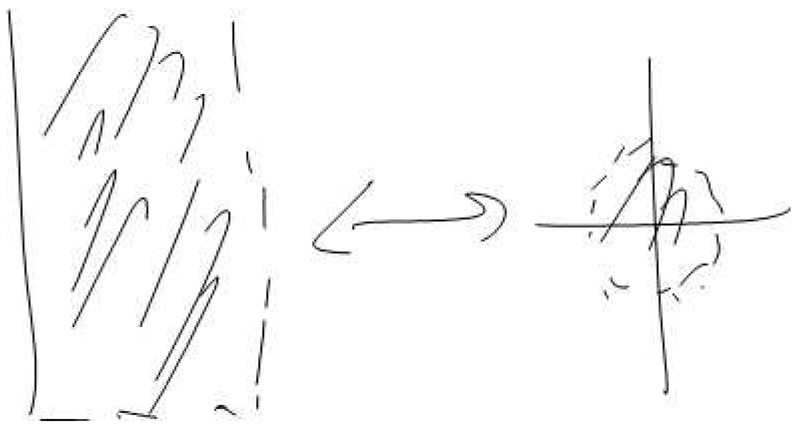
We claim $\mathbf{0}[\alpha]_{\mathbf{5}}[\sigma]_{\mathbf{5}}$ **œ** $\mathbf{0}[\alpha]_{\mathbf{5}}$ for any σ **−** $\alpha^{\square"}\Gamma_{\mathbf{=}}\alpha$. Indeed,

$$\mathbf{0}[\alpha]_{\mathbf{5}}[\sigma]_{\mathbf{5}} \text{ œ } \mathbf{0}[\alpha\sigma]_{\mathbf{5}}(\tau) \text{ œ } \mathbf{0}[\gamma\alpha]_{\mathbf{5}}(\tau) \text{ œ } \mathbf{4}(\gamma\alpha\text{ß}\tau)^{\square\mathbf{5}}\mathbf{0}(\gamma\alpha(\tau))$$
$$\text{ œ } \mathbf{4}(\gamma\text{ß}\alpha(\tau))^{\square\mathbf{5}}\mathbf{4}(\alpha\text{ß}\tau)^{\square\mathbf{5}}\mathbf{0}(\gamma\alpha(\tau)) \text{ œ } \mathbf{0}[\gamma]_{\mathbf{P}}(\alpha(\tau))\mathbf{4}(\alpha\text{ß}\tau)^{\square\mathbf{5}} \text{ œ } \mathbf{0}[\alpha]_{\mathbf{5}}(\tau).$$

Hence, it is invariant under any element in that subgroup. For σ œ „ $\left(\begin{smallmatrix} \text{"} & \mathbf{2} \\ ! & \text{"} \end{smallmatrix}\right)$ß we get

$$\mathbf{0}[\alpha]_{\mathbf{5}}(\tau \square \mathbf{2}) \text{ œ } \mathbf{0}[\alpha]_{\mathbf{5}}(\text{ „ } \sigma(\tau)) \text{ œ } \mathbf{0}[\alpha]_{\mathbf{5}}(\tau),$$

where $\mathbf{0}[\alpha]_{\mathbf{5}}$ has period $\mathbf{2}$ when $\mathbf{5}$ is even.



Define $\varphi(\mathbf{1})$ œ $(\mathbf{1}$ ‰ $\varepsilon^{\square\text{"}})(\mathbf{;})$ for $\mathbf{1}$ œ $\mathbf{0}[\alpha]_{\mathbf{5}}$.

Theorem. $\varphi(\mathbf{1})$ *is meromorphic if and only if* $\mathbf{1}(\mathbf{D})$ *is meromorphic.*

Proof. We know there exists a Laurent expansion for **;** in the punctured disk,

$$\varphi(\mathbf{;}) \text{ œ } \sum_{\mathbf{8}\text{œ}\square\infty}^{\infty} \mathbf{+}_{\mathbf{8}}\mathbf{;}^{\mathbf{8}}.$$

Now, $\mathbf{0}[\alpha]_{\mathbf{5}}$ meromorphic at ∞ means $\varphi(\mathbf{;})$ is meromorphic at **!**. If **5** is odd, and $\square$M − Γ then $\mathbf{0}(\mathbf{D})$ œ $\square\mathbf{0}(\mathbf{D})$. If $\square$M Â Γ, then $\mathbf{0}[\alpha]_{\text{"}}$ has period **#2** and **;** œ $\mathbf{/}^{\pi\mathbf{3D}\hat{\mathbf{I}}\mathbf{2}}$. [etc look at pg 74]. ☐

Recall **E** œ $\bigoplus_{\mathbf{5}\,-\,\mathbb{Z}}$ $\mathbf{E}_{\mathbf{5}}(\Gamma)$. Now, **0** − $\mathbf{E}_{\mathbf{5}}(\Gamma)$ is not well-defined on **** (Γ) if for $\gamma\tau$ßγ"τ in $\Gamma\tau$, $\mathbf{0}(\gamma\tau)$ œ $\mathbf{4}(\gamma\text{ß}\tau)^{\mathbf{5}}\mathbf{0}(\tau)$ Á $\mathbf{4}(\gamma\text{"ß}\tau)^{\mathbf{5}}\mathbf{0}(\tau)$ œ $\mathbf{0}(\gamma\text{"}\tau)$. If **5** œ **!**, then **0** is γ-invariant and so is well-defined on **** ÐΓ).

$\mathbf{E}_{\mathbf{!}}(\Gamma)$ is the field of meromorphic functions on **** (Γ), denoted by $\mathbb{C}($ **** $(\Gamma))$.

Example. Let **4** œ **"(#)** $\frac{\mathbf{1\$}}{\Delta}$ where the numerator is in $\mathcal{M}_{\text{"}\mathbf{\#}}(\mathbf{WP}_{\mathbf{\#}}(\mathbb{R}))$ (a modular form) and Δ is in $\mathcal{S}_{\text{"}\mathbf{\#}}(\mathbf{WP}_{\mathbf{\#}}(\mathbb{R}))$ (a cusp form). Then if **4** − $\mathbf{E}_{\mathbf{!}}(\mathbf{WP}_{\mathbf{\#}}(\mathbb{R}))$ and **4** has a pole at ∞, it makes sense to think of **4** À **** (**"**) Ä ©.

Fact. $\mathbb{C}(\mathbf{4})$ œ $\mathbf{E}_{\mathbf{!}}(\mathbf{WP}_{\mathbf{\#}}(\mathbb{Z}))$.

Fact. If **0** − $\mathbf{E}_{\mathbf{5}}(\Gamma)$, then if **0** Á **!**, then $\mathbf{E}_{\mathbf{5}}(\Gamma)$ œ $\mathbf{E}_{\mathbf{!}}(\Gamma)$.

Consider

$$\mathbf{Y} \overset{\pi}{\text{Ä}} \pi(\mathbf{Y}) \overset{\varphi}{\text{Ä}} \mathbf{Z}$$

where Y § ℍ‡ß π(Y) § \ (Γ)ß and Z § ℂ. Let $\pi(\tau)$ − \ (Γ) be not a cusp. Recalling that 0(>) œ $\sum_{8œ7}^{\infty}$ +8(> □ τ)8, we can think of local coordinates as

$$> □ \tau \; Ê \; (> □ \tau)^2 \; œ \; ;^2$$

so we can write 0(>) œ $\sum_{8œ7}^{\infty}$ +8;8î2. Then

$$\nu_{\pi(\tau)}(0) \; œ \; \frac{7}{2} \; œ \; \frac{\nu_\tau(0)}{2}.$$

If π(=) is a cusp, consider the cases 5 even/odd. If □M − Γ, then

$$\alpha^{□"}\Gamma_{=}\alpha \; œ \; Ø \; „ \begin{pmatrix} " & 2 \\ ! & " \end{pmatrix} Ùß \; \alpha(\infty) \; œ \; =, \text{ and}$$

$$\nu_{\pi(=)}(0) \; œ \; \frac{7}{\#} \; œ \begin{cases} \frac{\nu_=(0)}{\#} & \text{if } \alpha^{□"}\Gamma_{=}\alpha \; œ \; Ø□\begin{pmatrix} " & 2 \\ ! & " \end{pmatrix}Ù \text{ and } 5 \text{ is odd.} \\ \nu_=(0) & \text{otherwise.} \end{cases}$$

Also, define

$$\eta(\tau) \; œ \; ;_{\#\%} \prod_{8œ"}^{\infty} (" □ ;^8)$$

where ;#% œ /#π3Dî#%.

Proposition. *Let* 5ß R − ℤ□ *such that* 5(R □ ") œ #%. *Define* $\varphi_5(\tau)$ œ $\eta(\tau)^5\eta(R\tau)^5$. *If* W5($\Gamma_3$(R)) Á ! *for* 3 œ !ß" *then* W5(Γ_3(R)) œ ℂ φ_5. *If* 5 œ "# *and* R œ ", *then* W"#(WP#(ℤ)) œ ℂ Δ, *where* $\Delta(\tau)$ œ (#π)"#$\eta(\tau)$#%.

Differentials

Let Z § ℂ with Z open. We define the meromorphic differentials of degree 8 on Z to be

$$\Omega^{Œ8}(Z) \; œ \; \{0(;)(.;)^8 \; l \; 0 \text{ is meromorphic on } Z\},$$

where ; is the local variable on Z. Let

$$\Omega(Z) \; œ \bigoplus_{8 \, − \, \mathbb{N}} \Omega^{Œ8}(Z)$$

Let (.;)8(.;)7 œ (.;)8□7. Let φ À Z" Ä Z# be such that φ is holomorphic

$$\varphi^{‡} \; À \; \Omega^{Œ8}(Z_\#) \; Ä \; \Omega^{Œ8}(Z_")$$

defined by

$$0(;D)(.;D)^8 \; È \; 0(\varphi(;"))(.\varphi(;"))^8 \; œ \; 0(\varphi(;"))(\varphi^{w}(;"))^8(.;")^8.$$

Let \ be a Riemann surface, and let {Y4}4−N be neighborhoods of \, and {Z4}4−τ neighborhoods of ℂ. Let the φ_4 be the coordinate charts. Define a differential ω on \ to be a tuple ω œ $(\omega_4)_{4−N}$ − ∏ ΩŒ8(Z4) that is compatible with respect to the transition maps.

Now, we want ω − ΩŒ8(\ (Γ)) to pullback to a differential on ℍß

$$\mathbb{H}^{\ddagger} \text{ Ä } \setminus(\Gamma) \overset{\varphi}{\text{Ä}} \mathbb{C}.$$

Let $\mathbf{Y}_4^{\text{"}}$ **œ** $\mathbf{Y}_4 \cap \mathbb{H}$ and $\mathbf{Z}_4^{\text{"}}$ **œ** $(\varphi \text{ ‰ } \pi)\left(\mathbf{Y}_4^{\text{"}}\right)$. Recall $\omega\, \mathbf{l}\, \mathbf{Z}_{4^{\text{"}}}$ **−** $\Omega^{\text{Œ}8}\left(\mathbf{Z}_4^{\text{"}}\right)$. Define

$$\pi^{\ddagger}(\omega)\mathbf{l}_{\mathbf{Y}_4^{\text{"}}} \;{}^{3}\; (\varphi \text{ ‰ } \pi)^{\ddagger}\left(\omega\, \mathbf{l}_{\mathbf{Z}_4^{\text{"}}}\right) \text{ œ } \mathbf{0}(\tau)(\mathbf{.>})^{\mathbf{8}}$$

on $\mathbf{Y}_4^{\text{"}}$. We claim these local patches glue together because of compatibility.

We define a global differential on $\mathbb{H}$ to be $\mathbf{0}(\tau)(\mathbf{.}\tau)^{\mathbf{8}}$. We then **claim** that $\mathbf{0}$ is an automorphic form of weight $\mathbf{\#8}$.

$$\mathbf{0}(\tau)(\mathbf{.}\tau)^{\mathbf{8}} \text{ œ } \mathbf{0}(\gamma(\mathbf{>}))(\mathbf{.}(\gamma\tau))^{\mathbf{8}} \text{ œ } \mathbf{0}(\gamma(\tau))(\gamma^{\text{"}}(\tau))^{\mathbf{8}}(\mathbf{.}\tau)^{\mathbf{8}}.$$

We saw last time that $\gamma^{\text{"}}(\tau)$ **œ** $\mathbf{4}(\gamma\text{ß}\,\tau)^{\square\#}$. Hence, the above equals

$$\mathbf{0}(\gamma(\tau))\left(\mathbf{4}(\gamma\text{ß}\,\tau)^{\square\#\mathbf{8}}\right)(\mathbf{.}\tau)^{\mathbf{8}} \text{ œ } \mathbf{0}(\tau)(\mathbf{.}\tau)^{\mathbf{8}}$$

so it is weakly modular of weight $\mathbf{\#8}$. Next, we need to show that $\mathbf{0}[\alpha]_{\#\mathbf{8}}$ is meromorphic at $\boldsymbol{\infty}$ for all α **−** $\mathbf{WP}_{\#}(\mathbb{Z})$. As before, let **= œ** $\alpha(\boldsymbol{\infty})$. Let $\rho(\mathbf{D})$ **œ** $\mathbf{/}^{\#\pi 3\mathrm{D}\hat{\imath}2}$ **œ ;**. Since ω **−** $\Omega^{\text{Œ}8}(\setminus(\Gamma))$ is meromorphic on $\setminus(\Gamma)$, when we restrict to $\mathbf{Z}$, we can $\omega\mathbf{l}_{\mathbf{Z}}$ **œ** $\mathbf{1}(\mathbf{;})(\mathbf{.;})^{\mathbf{8}}$, where **1** is meromorphic (particularly, at **!**). Then

$$\begin{aligned}
\pi^{\ddagger}(\omega)\mathbf{l}_{\mathbf{Y}^{\text{"}}} &\text{ œ } (\rho \text{ ‰ } \delta)^{\ddagger}(\mathbf{1}(\mathbf{;})(\mathbf{.;})^{\mathbf{8}}\mathbf{l}_{\mathbf{Z}}) \text{ œ } \mathbf{\$}(\rho \text{ ‰ } \delta(\tau))\left((\varphi \text{ ‰ } \delta)^{\text{"}}(\tau)\right)^{\mathbf{8}}(\mathbf{.}\tau)^{\mathbf{8}} \\
&\text{ œ } \mathbf{\$}\left(\mathbf{/}^{\#\pi 3\delta(\tau)\hat{\imath}2}\right)\left(\mathbf{/}^{\#\pi 3\delta(>)\hat{\imath}2}\right)^{\mathbf{8}}\left(\tfrac{\#\pi 3}{2}\right)^{\mathbf{8}}(\delta^{\text{"}}(\tau))^{\mathbf{8}}(\mathbf{.}\tau)^{\mathbf{8}} \\
&\text{ œ } \mathbf{\$}\left(\mathbf{/}^{\#\pi 3\delta(\tau)\hat{\imath}2}\right)\left(\mathbf{/}^{\#\pi 3\delta(>)\hat{\imath}2}\right)^{\mathbf{8}}\left(\tfrac{\#\pi 3}{2}\right)^{\mathbf{8}}\mathbf{4}(\delta\text{ß}\,\tau)^{\square\#\mathbf{8}}(\mathbf{.}\tau)^{\mathbf{8}} \\
&\text{ œ } \mathbf{0}(\tau)\,(\mathbf{.}\tau)^{\mathbf{8}}
\end{aligned}$$

where we defined $\mathbf{0}(\tau)$ in the last equality. Now we just need to show $\mathbf{0}(\tau)$ is meromorphic at $\boldsymbol{\infty}$:

$$\mathbf{0}(\tau) \text{ œ } \mathbf{1}[\delta]_{\#\mathbf{8}}(\mathbf{;})\left(\tfrac{\#\pi 3}{2}\right)^{\mathbf{8}}\mathbf{;}^{\mathbf{8}}, \text{ where } \mathbf{;} \text{ œ } \mathbf{/}^{\#\pi 3\tau\hat{\imath}2}.$$

Then $\mathbf{0}[\alpha]_{\#\mathbf{8}}(\tau)$ **œ** $\mathbf{1}[\alpha^{\square\text{"}}]_{\#\mathbf{8}}[\alpha]_{\#\mathbf{8}}(\mathbf{;})\dagger\left(\tfrac{\#\pi 3}{2}\right)^{\mathbf{8}}\mathbf{;}^{\mathbf{8}}$ **œ** $\mathbf{1}(\mathbf{;})\left(\tfrac{\#\pi 3}{2}\right)^{\mathbf{8}}\mathbf{;}^{\mathbf{8}}$, which is mero-morphic at **; œ !** because of this statement.

Hence, given ω **−** $\Omega^{\text{Œ}8}(\setminus(\Gamma))$, the function $\mathbf{0}$ defining the pullback is an automorphic form of weight $\mathbf{\#8}$. The converse is also true: given an automorphic form of weight $\mathbf{\#8}$, we can construct a meromorphic differential on $\setminus(\Gamma)$ of degree $\mathbf{8}$.

Theorem 3.3.1. *Let* $\mathbf{5}$ **−** $\mathbb{N}$ *be even and let* Γ *be a congruence subgroup of* $\mathbf{WP}_{\#}(\mathbb{Z})$. *The map*

$$\omega \text{ À } \mathcal{A}_{\mathbf{5}}(\Gamma) \text{ Ä } \Omega^{\text{Œ}5\hat{\imath}\#}(\setminus(\Gamma)) \;\; \textit{with} \;\; \mathbf{0} \text{ È } (\omega_{\mathbf{4}})_{\mathbf{4-N}}$$

where $(\omega_{\mathbf{4}})$ *pulls back to* $\mathbf{0}(\tau)(\mathbf{.}\tau)^{\mathbf{5}\hat{\imath}\#}$ **−** $\Omega^{\text{Œ}5\hat{\imath}\#}(\mathbb{H})$ *is an isomorphism of complex vector spaces.*

Lecture 6 (March 9, 2009) - Diamond Chapter 3.4-3.6

Riemann-Roch Theorem

Let $\backslash$ be a compact Riemann surface.

Definition. *A divisor on* $\backslash$ *is a finite sum* $\sum_{B-\backslash} 8_B B$ *with* $8_B - \mathbb{Z}$ *where all but finitely many are* $!$.

We have a homomorphism $\deg$ À $\mathrm{Div}(\backslash)$ Ä $\mathbb{Z}$ with $\deg(\sum 8_B B)$ œ $\sum 8_B$. This gives a partial order: $\sum 8_B B \quad \sum 8''_B B$ if $8_B \quad 8''_B$ aB. Denote $\mathbb{C}(\backslash)$ the meromorphic functions on $\backslash$. Then $0 - \mathbb{C}(\backslash)^{‡}$, so define $\mathrm{div}(0)$ œ $\sum \nu_B(0)\,\chi$. Denote $\{\mathrm{div}(0) \,|\, 0 - \mathbb{C}(\backslash)^{‡}\}$ by Div^{j}. Notice

(1) $\mathrm{div}(0_" 0_\#)$ œ $\mathrm{div}(0_")$ □ $\mathrm{div}(0_\#)$ and (2) $\deg(\mathrm{div}(0))$ œ $!$.

(2) follows because $\deg(0)$ œ $\sum_{B-0^{□"}(C)} \mathrm{mult}_B(0)$, so $\deg(0)$ œ $\sum_{B-0^{□"}(!)} \mathrm{mult}_B(0)$, and $\deg(0)$ œ $\sum_{B-0^{□"}(\infty)} \mathrm{mult}_B(0)$. So

$$\mathrm{div}(0) \text{ œ } \sum_{B-0^{□"}(!)} \mathrm{mult}_B(0) \; □ \sum_{B-0^{□"}(\infty)} \mathrm{mult}_B(0) \text{ œ } !.$$

Define $\mathrm{Div}^{!}$ to be the divisors ($H - \mathrm{Div}(\backslash)$) of degree H œ $!$. Because of what we just showed, Div^{j} © $\mathrm{Div}^{!}$, so then want to look at $\mathrm{Div}^{!}$ ÎDiv^{j}.

Definition. *The linear space of a divisor is*

$$P(H) \text{ œ } ! \; □ \{0 - \mathbb{C}(\backslash)^{‡} \,|\, \mathrm{div}(0) \; □ \; H \quad !\}.$$

The dimension of this space is denoted $j(H)$. It is a fact that dim $j(H)$ □ ∞.

Given $\omega - \Omega^{Œ8}(\backslash)$ a non-zero differential 8-form on $\backslash$, then for all $B - \backslash$, we have a local representation ω_B œ $0_B(;)(.;)^{8}$, where $;$ is the local coordinate about $\backslash$. We will define $\mathrm{div}(\omega)$ ³ $\sum \nu_!(0_B)B$ (with $\nu_B(\omega)$).

Exercise. Why is $\nu_!$ cofinite of nonzeros?

Notice $\mathrm{div}(\omega_" \omega_\#)$ œ $\mathrm{div}(\omega_")$ □ $\mathrm{div}(\omega_\#)$.

Definition. *If* $\lambda - \Omega^{"}(\backslash)$, *then* $\mathrm{div}(\lambda)$ *is a canonical divisor.*

Theorem. *Let* $\backslash$ *be a compact Riemann surface of genus* 1. *Let* $\mathrm{div}(\lambda)$ *be a canonical divisor on* $\backslash$. *Then for any divisor* $H - \mathrm{Div}^{!}(\backslash)$,

$$j(H) \text{ œ } \deg(H) \; □ \; 1 \; □ \; " \; □ \; j(\mathrm{div}(\lambda) \; □ \; H).$$

Corollary 3.4.2 [in Diamond].

Note if $0 - \Delta_\#(\Gamma)$ is nonzero, then the associated $\omega(0) - \Gamma^{"}(\backslash(\Gamma))$ will have canonical divisor $\mathrm{div}(\omega)$, so has degree $\#1$ □ $\#$. For 5 even, $\omega^{5Î\#}$ will have a divisor of degree $5(1$ □ $")$. Since $\mathcal{A}_5(\Gamma)$ is $\mathbb{C}(\backslash)0$ for any nonzero 0 of weight 5. The same holds for $\Omega^{Œ5Î\#}(\backslash(\Gamma))$. So all $\omega - \Omega^{Œ5Î\#}(\backslash(\Gamma))$ has degree $5(1$ □ $")$.

Dimension formulas

If 5 is even, and $0 - \mathcal{A}_5(\Gamma)$ is nonzero, we have

$$\nu_{\pi(\tau)}(\;) \qquad \nu_\tau(\;)$$

for τ a noncusp of period **2**. Further, $\nu_{\pi(=)}(0)$ ³ $\nu_{=}(0)$ for **=** a cusp.

Define (formally)

$$\mathrm{div}(0) \text{ œ } \sum \nu_{B}(0)\, B.$$

What does it mean to be holomorphic? This exactly means div(**1**) **!**. Then

$$\mathcal{M}_5(\Gamma) \text{ œ } \{1 - \mathcal{A}_5(\Gamma) \,\mathsf{l}\, \mathrm{div}(1) \;\; !\} \text{ œ } \{0_!0 - \mathcal{A}_5 \,\mathsf{l}\, \mathrm{div}(0_!0) \;\; !\} \text{ z }$$
$$\{0_! - \mathbb{C}(\,\backslash\,(\Gamma)) \,\mathsf{l}\, \mathrm{div}(0_!) \,□\, \mathrm{div}(0) \;\; !\}.$$

Definition. $\lfloor \mathrm{div}\, 0 \rfloor$ œ $\sum \lfloor \nu_B(0) \rfloor B$.

We know

$$\mathrm{div}(0_!) \,□\, \mathrm{div}(0) \;\; ! \;\; \acute{\mathrm{I}} \;\; \mathrm{div}(0_!) \,□\, \lfloor \mathrm{div}(0) \rfloor \;\; !.$$

So

$$\mathcal{M}_5(\Gamma) \text{ z } P(\lfloor \mathrm{div}\, 0 \rfloor).$$

Hence, $\dim(\mathcal{M}_5(\Gamma))$ œ $j(\lfloor \mathrm{div}\, 0 \rfloor)$.

Claim. Let $\omega - \Omega^{\text{Œ}5\hat{\mathrm{I}}\#}(\,\backslash\,(\Gamma))$ whose pullback is $0(\tau)(\,.\,\tau)^{5\hat{\mathrm{I}}\#}$. Write $\{B_{\#ß3}\}ß\ \{B_{\$ß3}\}ß\ \{B_3\}$ of period **#ß $**, and cusps, respectively, with sizes $\varepsilon_{\#}ß\,\varepsilon_{\$}ß\,\varepsilon_{\infty}$, respectively. Define

$$\mathrm{div}(\,.\,\tau) \text{ œ } \sum \tfrac{"}{\#} B_{\#ß3} \,□\, \sum \tfrac{\#}{\$} B_{\$ß3} \,□\, \sum B_3.$$

From 3.3, recall $H_!(\omega)$ œ $\nu_{\pi(\tau)}(0) \,□\, \frac{5}{\#}\left(" \,□\, \frac{"}{2}\right)$ with $\tau - \mathbb{H}$ and ω is associated to **0**. Then

$$\lfloor \mathrm{div}(0) \rfloor \text{ œ } \mathrm{div}(\omega) \,□\, \sum \lfloor \tfrac{5}{\#} \rfloor B_{\#ß3} \,□\, \sum \lfloor \tfrac{5}{\$} \rfloor B_{\$ß3} \,□\, \sum \tfrac{2}{\#} B_3.$$

So

$$\deg(\lfloor \mathrm{div}\, 0 \rfloor) \text{ œ } 5(1 \,□\, ") \,□\, \lfloor \tfrac{5}{\%} \rfloor \varepsilon_{\#} \,□\, \lfloor \tfrac{5}{\$} \rfloor \varepsilon_{\$} \,□\, \lfloor \tfrac{5}{\#} \rfloor \varepsilon_{\infty}$$
$$\tfrac{5}{\#}(\#1 \,□\, \#) \,□\, \tfrac{5□\#}{\%}\varepsilon_{\#} \,□\, \tfrac{5□\#}{\$}\varepsilon_{\$} \,□\, \tfrac{5}{\#}\varepsilon_{\infty}$$
$$\text{œ } \#1 \,□\, \# \,□\, \tfrac{(5□\#)}{\#}(\#1 \,□\, \# \,□\, \varepsilon_{\#}\hat{\mathrm{I}}\# \,□\, \#\varepsilon_{\$}\hat{\mathrm{I}}\$ \,□\, \varepsilon_{\infty}) \,□\, \varepsilon_{\infty} \quad \#1 \,□\, \# \,□\, /_{\infty}$$
$$□\, \#1 \,□\, \#.$$

For $\mathcal{S}_5(\Gamma)$, we have the same things, but we use $\lfloor \mathrm{div}\, 0 \,□\, \sum B_3 \rfloor$. Then

$$\mathrm{div}(0_!) \,□\, \mathrm{div}(0) \,□\, \sum B_3 \quad !$$

yields $\deg(\lfloor \mathrm{div}(0) \,□\, \sum B_3 \rfloor)$ œ $\deg(\lfloor \mathrm{div}\, 0 \rfloor) \,□\, \varepsilon_{\infty}$. So for **5** **%**,

$$\dim(\mathcal{S}_5(\Gamma)) \text{ œ } j(\lfloor \mathrm{div}\, 0 \rfloor) \,□\, \varepsilon_{\infty}.$$

If **5** is nonpositive, we want $\mathcal{M}_!(\Gamma)$.

Lecture 7 (March 30, 2009) - Diamond Chapter 4

We define the Eisenstein space of weight **5**

$$\Sigma_5(\Gamma) \text{ œ } \mathcal{M}_5(\Gamma)\hat{\mathrm{I}}\mathcal{S}_5(\Gamma).$$

We will be computing the bases of these Eisenstein spaces, which are Eisenstein series. In this talk, we will only consider $k \geq 3$. Recall

$$G_k(\tau) = \sum\nolimits'_{(c,d)\in\mathbb{Z}^2\setminus\{0\}} \frac{1}{(c\tau+d)^k},$$

and the normalized Eisenstein series

$$E_k(\tau) = G_k(\tau)/2\zeta(k).$$

Now notice we can write

$$\begin{aligned} G_k(\tau) &= \sum\nolimits'_{(c,d)\in\mathbb{Z}^2\setminus\{0\}} \frac{1}{(c\tau+d)^k} = \sum_{n=1}^{\infty} \sum_{\substack{(c,d)\\ \gcd(c,d)=n}} \frac{1}{(c\tau+d)^k} = \sum_{n=1}^{\infty} \frac{1}{n^k} \sum_{\substack{(c,d)\\ \gcd(c,d)=1}} \frac{1}{(c\tau+d)^k} \\ &= \zeta(k) \sum_{\substack{(c,d)\\ \gcd(c,d)=1}} \frac{1}{(c\tau+d)^k}. \end{aligned}$$

Hence,

$$E_k(\tau) = \frac{1}{2} \sum_{\substack{(c,d)\\ \gcd(c,d)=1}} \frac{1}{(c\tau+d)^k}.$$

Define

$$P_+ = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

We claim that we can rewrite the above as

$$E_k(\tau) = \frac{1}{2} \sum_{\gamma\in P_+\setminus SL_2(\mathbb{Z})} j(\gamma,\tau)^{-k}.$$

Then

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix}.$$

It is easy to show that $E_k(\tau)$ is a weakly modular form of weight k.

We claim that

$$\dim(E_k(\Gamma)) = \begin{cases} \Sigma_\infty & k \geq 4 \text{ and even} \\ \Sigma_\infty^{\text{reg}} & k \geq 3 \text{ is odd and } -I \notin \Gamma \\ \Sigma_\infty - 1 & k = 2 \\ \Sigma_\infty^{\text{reg}}/2 & k = 1 \text{ and } -I \notin \Gamma \\ 1 & k = 0 \\ 0 & k < 0 \text{ or } (k > 0 \text{ is odd and } -I \in \Gamma). \end{cases}$$

Now let us look at the Eisenstein series for $\Gamma(N)$ $(k \geq 3)$. First, take $N \in \mathbb{Z}^+$ and let $\bar{v} \in (\mathbb{Z}/N\mathbb{Z})^2$ a row vector of order N. Let

$$\delta = \begin{pmatrix} a & b \\ c_{\bar v} & d_{\bar v} \end{pmatrix},$$

where $(c_{\bar v}, d_{\bar v})$ is a lift of $\bar v$ to $\mathbb{Z}^2$. We define ε_N to be $\frac{1}{2}$ if $N = 1, 2$ and 1 otherwise. Then define

$$G_k^{\bar v}(\tau) = \varepsilon_N \sum_{\substack{(c,d) \equiv \bar v \pmod N \\ \gcd(c,d) = 1}} (c\tau + d)^{-k}.$$

We claim that $G_k^{\bar v}(\tau) = \varepsilon_N \sum_{\gamma \in (P_+ \cap \Gamma(N)) \backslash \Gamma(N)\delta} j(\gamma, \tau)^{-k}$.

Proof. Let's write $\gamma \in \Gamma(N)$ as $\begin{pmatrix} Ne+1 & Nf \\ Ng & Nh+1 \end{pmatrix}$. Then

$$\gamma\delta = \begin{pmatrix} \ast & \ast \\ \underbrace{Nga + (Nh+1)c_{\bar v}}_{c'} & \underbrace{Ngb + (Nh+1)d_{\bar v}}_{d'} \end{pmatrix}.$$

Notice indeed $\gcd(c', d') = 1$.

Proposition. *For all* $\gamma \in SL_2(\mathbb{Z})$, $(G_k^{\bar v}[\gamma])(\tau) = G_k^{\overline{v\gamma}}(\gamma(\tau))$.

Proof. We have

$$\begin{aligned} j(\gamma,\tau)^{-k} G_k^{\bar v}(\gamma(\tau)) &= j(\gamma,\tau)^{-k} \varepsilon_N \sum_{\gamma' \in (P_+ \cap \Gamma(N)) \backslash \Gamma(N)\delta} j(\gamma', \gamma(\tau))^{-k} \\ &= \varepsilon_N \sum_{\gamma' \in (P_+ \cap \Gamma(N)) \backslash \Gamma(N)\delta} j(\gamma'\gamma, \tau)^{-k} \\ &= \varepsilon_N \sum_{\gamma'' \in (P_+ \cap \Gamma(N)) \backslash \Gamma(N)\delta\gamma} j(\gamma'', \tau)^{-k} \\ &= \varepsilon_N G_k^{\overline{v\gamma}}(\tau). \end{aligned}$$

(where we write

$$j(\gamma', \gamma(\tau)) = j(\gamma'\gamma, \tau)/j(\gamma, \tau).)$$

Corollary. $G_k^{\bar v}(\tau) \in \mathcal{M}_k(\Gamma(N))$.

Proof. It is holomorphic on $\mathbb{H}$, and for all $\gamma \in \Gamma(N)$, each γ reduces to I mod N. So by our proposition above, $\overline{v\gamma} = \bar v$. Hence, $G_k^{\bar v}$ is weight-k invariant with respect to $\Gamma(N)$. Fourier coefficients satisfy $|a_n| \le C n^r$ where C, r are positive constants. □

Now we can create modular forms for any congruence subgroup of level N, namely

$$G_{k,\Gamma}^{\bar v}(\tau) = \sum_{\gamma_j \in \Gamma(N) \backslash \Gamma} G_k^{\bar v}[\gamma_j]_k(\tau).$$

We can show that

$$\lim_{\mathrm{Im}(\tau)\,\text{Ä}\,\infty} \mathbf{I}^{@}_{\mathbf{5}}(\tau)\ \text{œ} \begin{cases} (\ \text{„}\ \text{"})^{\mathbf{5}} & \text{if } @\ \text{œ}\ \ \text{„}\ \overline{(\text{!ß"})}\text{, unless } \mathbf{5} \text{ is odd and } \mathbf{R} \text{ is " or \#.} \\ \text{!} & \text{otherwise.} \end{cases}$$

In this exceptional case, M − $\Gamma(\mathbf{R})$, so that $\dim(\Sigma_{\mathbf{5}}(\Gamma(\mathbf{R})))$ œ !. Hence, $\Sigma_{\mathbf{5}}(\Gamma(\mathbf{R}))$ has a trivial basis. Now, $\mathbf{I}^{@}_{\mathbf{5}}$ is nonvanishing at ∞ if @ œ „ $\overline{(\text{!ß"})}$, and vanishes at ∞ otherwise.

What about for any @ œ $\overline{(\text{-ß .})}$? Take any @ œ $\overline{(\text{-ß .})}$ − $(\mathbb{Z}\,\hat{\text{I}}\,\mathbf{R}\,\mathbb{Z})^{\#}$ of order $\mathbf{R}$ with its corresponding

$$\delta\ \text{œ} \begin{pmatrix} + & , \\ - & . \end{pmatrix}.$$

Take any cusp = œ +ʷÎ-ʷ − $\mathbb{Q}$ □ {∞} such that some matrix

$$\alpha\ \text{œ} \begin{pmatrix} +^{w} & ,^{w} \\ -^{w} & .^{w} \end{pmatrix}$$

takes ∞ to =. The Fourier series $\mathbf{I}^{@}_{\mathbf{5}}[\alpha]_{\mathbf{5}}$ describes the behavior at $\mathbf{I}^{@}_{\mathbf{5}}$. By our earlier proposition, $\mathbf{I}^{@}_{\mathbf{5}}[\alpha]_{\mathbf{5}}$ œ $\mathbf{I}^{@\alpha}_{\mathbf{5}}$ œ $\mathbf{I}^{\overline{(\text{!ß"})\delta\alpha}}_{\mathbf{5}}$ since

$$(\text{!ß"})\begin{pmatrix} + & , \\ - & . \end{pmatrix}\ \text{œ}\ (\text{-ß .})\ \text{œ}\ @.$$

So $\mathbf{I}^{@}_{\mathbf{5}}[\alpha]_{\mathbf{5}}$ is non-vanishing at ∞ only when $\overline{(\text{!ß"})\delta\alpha}$ œ „ $\overline{(\text{!ß"})}$ if and only if $\overline{(\text{!ß"})\delta}$ œ „ $\overline{(\text{!ß"})\alpha^{□"}}$ if and only if

$$\begin{pmatrix} +^{w} \\ -^{w} \end{pmatrix}\ ´\ \ \text{„} \begin{pmatrix} . \\ - \end{pmatrix} \pmod{\mathbf{R}}$$

if and only if $\Gamma(\mathbf{R})$= œ $\Gamma(\mathbf{R})(□.\hat{\text{I}}-)$. So $\mathbf{I}^{@}_{\mathbf{5}}$ is nonvanishing at $\Gamma(\mathbf{R})(□.\hat{\text{I}}-)$ and vanishes at all other cusps. If $\mathbf{5}$ is even and $\mathbf{R}$ □ #, pick a set of vectors

$$\{@\}\ \text{œ}\ \{\overline{(\text{-ß .})} \text{ s.t. the quotients } □.\hat{\text{I}}- \text{ represent all cusps at } \Gamma(\mathbf{R})\}.$$

By the above, the $\{\mathbf{I}^{@}_{\mathbf{5}}\}$ are linearly independent. This set has Σ_{∞} elements (which is the dimension of $\Sigma_{\mathbf{5}}(\Gamma(\mathbf{R}))$), so it is a basis.