Lecture Notes

Last updated: Saturday, January 19th at 6:30PM

These lecture notes are intended to recapitulate class lectures. Several notational considerations exist. Namely, appendices to lectures generally provide information not found in the lecture, but are nonetheless helpful for producing a more rigorous understanding of the material; an * (asterisk) next to a proposition, definition, theorem, etc. indicates it was not explicitly written (or mentioned) in class, but aids the flow or completeness of the notes; an $\longrightarrow$ to the left of a statement indicates this was given in class as a homework problem or personal exercise; finally, a footnote marked as ☠ will be provided (to the far left of the page if it refers to mathematical notation) if an error was found in the lecture.

Lecture 1 (01/14/08) [pp. 10-13]

In this course, the goal is to describe the universe of sets, $\mathbb{V}$. In this universe, two definitions hold: (1) everything is a **set**, and (2) if x is a set and $y \in x$, then y is a set. In order to begin our study, we will need to lay down the axioms of the set theoretical approach we will be using, namely, ZFC. We will initially examine the first five axioms. The language we need at first is merely $\mathcal{L} = \{ \in \}$.

Axiom 0 (nontriviality)

$\exists x\,(x = x)$, or $\exists x\,(x \in x \lor x \notin x)$

This axiom tells us there is at least one set in our universe. This is merely listed for clarity, as it can be proved from the other axioms.

However, note that $X = \{2, 3, 5\}$ and $Y = \{x : 1 \leq x \leq 5 \text{ with } x \text{ prime}\}$ are the same sets, despite being described in a separate fashion. In order to know these are the same sets, we need the next axiom.

Axiom 1 (extensionality)

$\forall x \forall y\,(x = y \Leftrightarrow \forall z(z \in x \Leftrightarrow z \in y))$

This axiom states that a set is uniquely described by its members: if two sets have the same members, they are the same set.

Now that we have established the nontriviality of our universe and a fundamental property of set equality, we need some tools to create sets. Since we have no knowledge of how the sets in our universe can look like, we will use a more general approach. Let $\phi(v)$ be a formula with free variable v. Ideally, we would like to be able to assert the existence of a set $\{x : \phi(x)\}$, or more formally, $\exists y\,\forall x\,(x \in y \Leftrightarrow \phi(x))$. However, a logical problem exists with such an attempt. Consider $\phi(v)$ to be the statement $v \notin v$. Then our previous statement would assert $\exists y\,\forall x\,(x \in y \Leftrightarrow x \notin x)$. This holds for all x, and hence specifically y, or $\exists y\,(y \in y \Leftrightarrow y \notin y)$. However, since $y \notin y$ is $\neg(y \in y)$, that is a contradiction. Hence, we cannot merely assert $\{x : \phi(x)\}$ is a set (this is known as Russell's Paradox). However, we *can* assert a more restricted version.

Axiom 3 (Comprehension-scheme)†

For each formula $\phi(v, \overline{w})$‡, $\forall \overline{w}\,\forall x\,\exists y\,\forall z\,(z \in y \Leftrightarrow (z \in x \land \phi(z, \overline{w})))$

Here, we say that y is a set with $y = \{z \in x : \phi(z, \overline{w})\}$. Note this avoids Russell's Paradox, because $z \in x$ ensures z is an existent set, and $\phi(z, \overline{w})$ is reduced to a mere Boolean condition. Effectively, y is a set that contains elements z of x which satisfy $\phi(z, \overline{w})$ (loosely speaking, y is a subset of x). Initially, this axiom may seem useless for bootstrapping ourselves a set, as we would still need an initial set x to draw elements from.

---

† Note this axiom is numbered 3. These notes will follow the book's notation, namely that axiom 2 is the impractical Foundation axiom, studied in chapter 3. For clarity, axioms will generally be referred to by name and not number, as to avoid confusion.

‡ Henceforth, $\overline{w}$ will represent a tuple of free variables, $(w_1, w_2, ..., w_n)$.

However, note we do not need to be explicitly aware of the contents of x, merely its existence. From this, we can prove the existence and structure of at least one set.

Proposition 1 *There exists a unique set with no elements.*

Proof. Applying $\phi(v) \equiv v \neq v$ to Comprehension, we know $\exists y\, \forall z(z \in y \Leftrightarrow z \in x \wedge z \neq z)$. Note $z \neq z$ is always false, so that $\neg \exists z\,(z \in y)$; in other words, y has no elements. Now, let x and y be sets with no elements. Then $\forall z(z \in x \Leftrightarrow z \in y)$, since the precedent $z \in x$ is always false (x has no elements) as is the antecedent $z \in y$. Hence, by extensionality, $x = y$. □

Definition 1* *The unique set with no elements is called the* ***empty set*** *and is denoted* $\emptyset$.

We have now explicitly constructed a set. Note we can not construct any more sets using just these three axioms (see the appendix of Lecture 1 for a proof). To construct more, we will need to use the empty set to create new sets. The next axiom allows us to do this.

Axiom 4 (pairing)

$\forall x\, \forall y\, \exists z\,(x \in z \wedge y \in z)$

In other words, for any two sets x and y there is a set z which contains them. A stronger version of this axiom would specifically indicate the existence of the smallest such set: $\{x, y\}$. However, note this can be proven.

Proposition 2* *If x and y are sets, there is a unique set z which contains only elements of x and y.*

Proof. To prove the proposition, we must show $\forall x\, \forall y\, \exists! z\, \forall w\,(w \in z \Leftrightarrow w = x \vee w = y)$. By non-triviality, let x and y be sets. By pairing, $\exists z'(x \in z' \wedge y \in z')$. By Comprehension with $\phi(v) \equiv v = x \vee v = y$,

$$\exists z\, \forall w\,(w \in z \Leftrightarrow w \in z' \wedge (w = x \vee w = y)).$$

However, note that $w = x \Rightarrow w \in z'$ and $w = y \Rightarrow w \in z'$, so

$$(w \in z' \wedge (w = x \vee w = y)) \Leftrightarrow (w = x \vee w = y), \text{ and hence,}$$

$$\forall x\, \forall y\, \exists z\, \forall w\,(w \in z \Leftrightarrow w = x \vee w = y).$$

To prove z is unique, let z' be another such set. If w is a set, $w \in z \Leftrightarrow w = x \vee w = y$ and $w \in z' \Leftrightarrow w = x \vee w = y$ so therefore $w \in z \Leftrightarrow w \in z'$. Then by extensionality, $z = z'$, and hence z is unique. □

Definition 2* *If x and y are sets, let $\{x, y\}$ represent z as in Proposition 2. If $x = y$, $\{x, x\}$ is represented as $\{x\}$.*

Example 1 By Proposition 2, $\{\phi, \phi\} = \{\phi\}$ exists and is unique. We have now constructed a non-empty set! Similarly, $\{\phi, \{\phi\}\}$, $\{\{\phi\}\}$, and $\{\phi, \{\phi, \{\phi\}\}\}$ exist and are unique.

⟶ Using only axioms 0–4, show there exists a set z that has at least 3 elements.

With the aid of Definition 2, we can define ordered pairs (and are assured of their existence by Prop. 2):

Definition 3 *If x and y are sets, let $\langle x, y\rangle$ represent $\{\{x\}, \{x, y\}\}$, called an* ***ordered pair***.

⟶ Using only axioms 0–4, prove $\forall x\, \forall y\, \forall u\, \forall v\,(\langle x, y\rangle = \langle u, v\rangle \Leftrightarrow x = u \wedge y = v)$.

Axiom 5 (union)

☠[1] $\forall x\,\exists y\,\forall z\,\forall w\,((z \in x \wedge w \in z) \Rightarrow w \in y)$

Informally, this axiom asserts there is a set which contains the elements of each set in x. Intuitively, the smallest such set would be the union of all sets in x. Although we could simply assert a stronger form of the axiom which asserts the existence of the union, it can be proven from the earlier axioms.

Proposition 3* *If x is a set, there exists a unique set y which has only elements of sets of x .*

Proof. The proposition is proved if we can show $\forall x\,\exists! y\,\forall w\,(\exists z(w \in z \wedge z \in x) \Leftrightarrow w \in y)$. Ignoring uniqueness, the first implication is assured by the union axiom; that is, since the union axiom holds for all z, there of course exists a z such that:

$$\forall x\,\exists y'\,\forall w\,(\exists z\,(z \in x \wedge w \in z) \Rightarrow w \in y').$$

By Comprehension with $\phi(v, w_1, w_2) \equiv v \in w_1 \wedge w_1 \in w_2$, we know

$$\forall u\,\forall v\,\forall x\,\exists y\,\forall w\,(w \in y \Rightarrow w \in x \wedge (w \in v \wedge v \in u)).$$

Choose $x = y'$, and rename v to z and u to x. Then

$$\forall x\,\forall z\,\exists y\,\forall w\,(w \in\ y \Rightarrow w \in y' \wedge (w \in z \wedge z \in x)).$$

However, note $(w \in z \wedge z \in x) \Rightarrow w \in y'$, so that $(w \in y' \wedge (w \in z \wedge z \in x)) \Leftrightarrow (w \in z \wedge z \in x)$. Applying this to the previous statement, we obtain

$$\forall x\,\forall z\,\exists y\,\forall w\,(w \in\ y \Rightarrow (w \in z \wedge z \in x)).$$

Finally, since this holds for all z, we know

$$\forall x\,\exists y\,\forall w\,(w \in\ y \Rightarrow \exists z(w \in z \wedge z \in x)).$$

To prove uniqueness, assume for a set x there are two such y and y'. Then $w \in y \Leftrightarrow \exists z(w \in z \wedge z \in x)$ and similarly $w \in y' \Leftrightarrow \exists z(w \in z \wedge z \in x)$. Therefore, $w \in y \Leftrightarrow w \in y'$. By extensionality, $y = y'$. □

Notice the approach for proving uniqueness in the previous proposition was very similar to the proof of uniqueness in Proposition 2. Henceforth, any such relatively trivial exposition of uniqueness will be simply referred to as "by extensionality." We can now define set union.

Definition 4* *If a and b are sets, let $a \cup b$ represent the set y in Proposition 3 for $x = \{a, b\}$. This is called the **union** of a and b. For any set of sets $x = \{x_i\}$, let $\bigcup x_i$ represent the set y in Proposition 3.*

Example 2 Take $x = \{\phi, \{\phi\}\}$ and $y = \{\phi, \{\phi, \{\phi\}\}\}$. Then $x \cup y = \{\phi, \{\phi\}, \{\phi, \{\phi\}\}\}$.

Axiom 6 (Replacement-scheme)

☠[2] For any formula[†] $\phi(x, y, A, \overline{w})$, $\forall A\,\forall \overline{w}[\forall x \in A\,\exists! y\,\phi(x, y, A, \overline{w}) \Rightarrow \exists Y\,\forall x \in A\,\exists y \in Y\,\phi(x, y, A, \overline{w})]$.

Informally, this axiom states if the domain of a function is a set, so is its range. This is yet another set construction tool. The careful explication (and hence why the axiom seems "long") of the axiom is due to the goal of avoiding paradoxes. We can now explicitly state the existence of another family of sets.

Definition 5 *If A and B are sets, define $A \times B \equiv \{\langle a, b\rangle : a \in A, b \in B\}$, the **cartesian product**.*

☠[1] The union axiom begins with a universality not existentiality quantifier ($\forall x$, not $\exists x$); in the lectures, the latter was used.
☠[2] Technically, A is a free variable of ϕ, although this was not indicated in the lecture.
[†] Note that $\exists! x\,\phi(x, \overline{w}) \Leftrightarrow ((\exists x\,\phi(x, \overline{w})) \wedge (\forall x \forall y\,((\phi(x, \overline{w}) \wedge \phi(y, \overline{w})) \Rightarrow x = y)))$. The latter was used in the lecture.

Proposition 4 *The product $A \times B$ exists and is unique.*

Proof. First, note if y is a set, $\forall A\, \forall x \in A\, \exists! z\, (z = \langle x, y\rangle)$ by Definition 3. Then by Replacement,

$$\forall A\, \exists C\, \forall x \in A\, \exists z \in C\, (z = \langle x, y\rangle).$$

By Comprehension, choose the C such that C only contains elements of the form $\langle x, y\rangle$ (i.e., the image of $f_x(y) \equiv \langle x, y\rangle$). By extensionality, such a C is unique. This holds for all y, so

$$\forall B\, \forall y \in B\, \exists! C\, (\forall A\, \forall z \in C\, \exists! y \in B\, \exists! x \in A(z = \langle x, y\rangle)).$$

Again, we can apply Replacement to obtain

$$\forall B\, \exists D\, \forall y \in B\, \exists C \in D\, (\forall A\, \forall z \in C\, \exists! y \in B\, \exists! x \in A(z = \langle x, y\rangle))$$

Then $\bigcup_{C \in D} C$ exists and is unique by Definition 4. By Comprehension, there is a $\mathfrak{C} \subseteq \bigcup_{C \in D} C$ such that all elements in $\mathfrak{C}$ are of the form $z = \langle x, y\rangle$ and is unique by extensionality. In other words,

$$\forall B\, \forall A\, \exists! \mathfrak{C}\, (\forall z \in \mathfrak{C}\, \exists! y \in B\, \exists! x \in A\, (z = \langle x, y\rangle)).\ \square$$

In conclusion, the entirety of most mathematics can be constructed from several logical rules and axioms. In the most common system of axioms, Zermello-Fraenkel set theory, the first two axioms--nontriviliality and extensionality--specify the existence and nature of equality of sets, respectively. The fourth axiom, Comprehension, is used to specify the existence of sets that are a subset of some larger set and have property ϕ. Finally, the pair, union, and Replacement axioms are used to construct larger sets.

Appendix

The material here was not discussed in class, but may aid in an understanding of the material or the completeness of the notes.

When we use set builder notation, we are doing so *a priori*, and hence it is informal. However, we can prove the functionality of set builder notation.

Proposition A1 *If A is a set and $\phi(x, A, \overline{w})$ is a formula, there is a unique set B such that $x \in B$ if and only if $x \in A$ and $\phi(x, A, \overline{w})$.*

Proof. By Comprehension, there is at least one such set. Let B and B' be such sets. Then $x \in B \Leftrightarrow x \in A \wedge \phi(x, A, \overline{w})$ and $x \in B' \Leftrightarrow x \in A \wedge \phi(x, A, \overline{w})$, so that $x \in B \Leftrightarrow x \in B'$. By extensionality, $B = B'$. Hence, B is unique. $\square$

Definition A1 *If A is a set and $\phi(x, A, \overline{w})$ is a formula, $\{x \in A : \phi(x, A, \overline{w})\}$ represents the unique set B in Proposition A1.*

Note when we discussed the existence and uniqueness of elements such as $\{\phi, \{\phi\}\}$, $\{\{\phi\}\}$ in Example 1, we ignored the possibility that $\{\phi, \{\phi\}\} = \{\{\phi\}\}$ and that the set notation is masking their true equality. However, this is easily overcome by noting that negating each side of the bijection in extensionality means we have to find an element in one set that is not in the other; in this case, ϕ for example.

Lecture 2 (01/16/08) [pp. 13-15]

Last time, we were introduced to five fundamental axioms of ZFC. Using these axioms, we can define some "basic" concepts like functions and well-orders. The careful and formal notation of the earlier lecture will be loosened in all future proofs and discussion. We will begin by proving a certain property of our university:

Proposition 1 *There is no set of all sets.*

Proof. By way of contradiction, assume there is such a universal set $\mathbb{V}$. By Comprehension, we can take a set $\mathbb{V}' \equiv \{x \in \mathbb{V} : x \notin x\}$. Equivalently, for $x \in \mathbb{V}$, $x \in \mathbb{V}' \Leftrightarrow x \notin x$. However, for $x = \mathbb{V}'$, this implies $\mathbb{V}' \in \mathbb{V}' \Leftrightarrow \mathbb{V}' \notin \mathbb{V}'$. This is a logical contradiction, so our original assumption about the existence of a universal set was invalid. □

If there is no set of all sets, then what is our universe $\mathbb{V}$? We can not call $\mathbb{V}$ a set, because it is "too big", and would not satisfy our axioms due to paradoxes such as containing itself. However, we can give a name and definition to objects like $\mathbb{V}$ that are too big to be sets.

Definition 1 *If $\phi(v, \overline{w})$ is a formula and $\overline{a} \in \mathbb{V}$, then $\{x : \phi(x, \overline{a})\}$ is a **class**.*

Applying Definition 1 with $\phi \equiv x = x$, we know $\mathbb{V}$ is a class. Returning to our discussion of properties of sets, we can define the concept of a linear order and well-order.

Definition 2 *Let A be a set. A **linear order** of A, $<$, is a binary relation*† *$R \subseteq A \times A$. If $(a, b) \in R$, we write $a < b$. Linear orders have several properties:*

1) $a \nless a$; (*anti-reflexive*)
2) $a < b \wedge b < c \Rightarrow a < c$; (*transitive*)
3) $a < b \vee b < a \vee a = b$.

Definition 3 *A linear order $<$ on A is a **well-order** of A if for any non-empty set $B \subseteq A$, there is a least element $b \in B$ according to $<$ (i.e., $\forall c \in B\,(c \nless b)$).*

Example 1 Consider $(\mathbb{N}, <)$. Then $<$ is a linear order. If we "stack" $\mathbb{N}$ next to $\mathbb{N}$ for every element in $\mathbb{N}$, we can also order that, and it is $(\mathbb{N} \times \mathbb{N}, <)$ ordered lexicographically:

$$\forall n, n', m, n' \in \mathbb{N}\; (n, m) < (n', m') \Leftrightarrow (n \in n' \vee (n = n' \wedge m < n')).$$

Example 2 The set $\mathbb{C}$ is not a linear order. The sets $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{Z}$ are linear orders, but not well-orders. For example, for $\mathbb{Z}$, consider the subset of negative integers: there is no least element.

We have defined well-orders before anything else because they have some very convenient and beautiful properties, such as in Propositions 2 and 3 below.

Definition 4* *The notation $f : A \to B$ signifies f is a **function** and a subset of $A \times B$ with $f(x) = y$ denoting $(x, y) \in f$, and the property $\forall x \in A\, \exists! y \in B\,(y = f(x))$. A function is **strictly increasing** if and only if for a well-order $(A, <)$, $x < y \Rightarrow f(x) < f(y)$.*

Proposition 2 *If $(W, <)$ is a well-order and $f : W \to W$ is strictly increasing, then $\forall x\, f(x) \geq x$.*

† Note $A \subseteq B$ simply implies $A = \{x \in B : \phi\}$ for some formula ϕ.

Proof. By way of contradiction, take $C = \{x \in W : f(x) < x\}$ to be non-empty. Then $C \subseteq W$ implies C has a least element, say x_0, so that $f(x_0) < x_0$. But $f(x_0) \in W$ and if $f(x_0) < x_0$ then $f(x_0) \notin C$ since x_0 is its least element. Therefore, $f(f(x_0)) \geq f(x_0)$. However, this contradicts the fact f is strictly increasing. Therefore, C is empty. □

Definition 5* *If $(W, <)$ is a well-order, a subset $I \subseteq W$ is an **initial segment** of W if and only if $\forall a \in I\,((b \in W \wedge b < a) \Rightarrow b \in I)$. Such a segment is **proper** if $I \neq W$. If $x \in W$, define $W(x)$ to be the set $\{z \in W : z < x\}$.*

Corollary 1 *No well-order is isomorphic[†] to a proper initial segment of itself.*

Proof. Let $f : W \to I$ be an isomorphism, where I is a proper initial segment. Then $\exists x\,(x \in W \backslash I)$. But $f(x) \geq x$ by the previous proposition, so $f(x) \notin I$. However, that is a contradiction, so there is no such isomorphism. □

Proposition 3 *Suppose $(W_1, <)$ and $(W_2, <)$ are well-orders. Exactly one of the following is true:*

1) $W_1 \cong W_2$
2) *There is a proper initial segment $I \subsetneq W_1$ such that $W_2 \cong I$.*
3) *There is a proper initial segment $I \subsetneq W_2$ such that $W_1 \cong I$.*

Lemma 1 *If W_1 and W_2 are isomorphic well-orders, there is a unique isomorphism between them.*

Proof. By way of contradiction, let $f, g : W_1 \to W_2$ be such isomorphisms. Furthermore, assume the set $C = \{x \in W_1 : f(x) \neq g(x)\}$ is non-empty. By definition, C has a least element, say x_0. Without loss of generality, assume $g(x_0) > f(x_0)$. Then there is no x with $g(x) = f(x_0)$, since x is certainly not x_0, and any other choice violates the assumption that x_0 is the least element of C (if there is such an x_1, then $x_1 > x_0$ and $g(x_1) > f(x_0)$ or g would not be strictly increasing). Hence, $x_0 \notin \operatorname{img} g$, so g is not surjective. Therefore, g is not an isomorphism, so C must be empty and there is only one such isomorphism. Note this lemma implies an automorphism $f : W \to W$ is unique: the identity. □

[3] *Proof.* (of Proposition 3) By Corollary 1 with Lemma 1, the three statements of Proposition are mutually exclusive. Let $f = \{(x, y) \in W_1 \times W_2 : W_1(x) \cong W_2(y)\}$. Then in the first case, we need to prove[‡]

$$\forall x\,(\exists! y \vee \neg\exists y)\,((x, y) \in f).$$

Suppose that $(x, y), (x, z) \in f$. Then $W_1(x) \cong W_2(y)$ and $W_1(x) \cong W_2(z)$. Therefore, $W_2(y) \cong W_2(z)$. If $y \neq z$, assume without loss of generality that $y > z$. However, $W_2(z)$ is then a proper initial segment of $W_2(y)$, so this contradicts Corollary 1 [4]. Therefore, we cannot have both (x, y) and (x, z) in f for $y \neq z$. Similarly, $(y, x), (z, x) \in f$ implies $y = z$. For a continuation of the proof, see the next lecture.

Appendix

There is one basic but nonetheless important result whose careful consideration was ommitted in the lecture and above proofs, but is crucial to the proof of Proposition 3 and other developments.

Proposition A1 *If $(W, <)$ is a well-order and $X \subseteq W$, then $(X, <|_X)$ is a well-order.*

[†] Where isomorphism is the usual existence of a bijective function. See page 14 of Kunen for a more careful treatment.
[3] In class, $W_1(x) \cong W_2(x)$ was written; of course, this is not valid, as it is not necessarily true $x \in W_2$.
[‡] Where $\exists! y \vee \neg\exists y$ denotes there is at most one y.
[4] In the lectures, it was said this contradicts Lemma 1; however, it is the Corollary which applies.

Proof. By definition, $X \subseteq W$ has a $<$ least element, so it has a $<|_X$ least element, say x_0. Let $A \subseteq X$ be non-empty. Then $A \subseteq W$, and hence has a $<$ least element, call it x_1. Since $A \subseteq X$, $x_0 <|_X x_1$. Hence, A has a $<|_X$ least element. By definition, $(X, <|_X)$ is a well-order. $\square$

Lecture 3 (01/18/08) [pp. 15-17]

Previously, we introduced the definition and properties of well orders. We will continue our exposition on well orders, and next examine ordinals.

Proof. (of Proposition 2.3, cont.) For the second case, consider $x \in \text{dom } f$ and $x' < x$. Then we must prove $x' \in \text{dom } f$. By definition of f, we know there is an isomorphism h from $W_1(x)$ to $W_2(f(x))$. Then h restricted to $W_1(x')$ is an isomorphism with $W_2(h(x'))$. Hence, $(x', h'(x)) \in f$. This holds for all $x' < x$, so $h = f$. Similarly, we can show if $y \in \text{img } f$ and $y' < y$, then $y' \in \text{img } f$. Note dom $f \subseteq W_1$ is an initial segment, as is img $f \subseteq W_2$ (not necessarily proper).

To show either case 2 or 3 holds, we must prove dom $f = W_1$ or img $f = W_2$. By way of contradiction, assume not. Then there is a least element $x \in W_1 \backslash \text{dom } f$ and a least element $y \in W_2 \backslash \text{img } f$. Furthermore, $W_1(x) = \text{dom } f$ and $W_2(y) = \text{img } f$, and then f is an isomorphism from $W_1(x)$ to $W_2(y)$. In other words, $(x, y) \in f$, which implies $x \in \text{dom } f$. However, this contradicts our earlier assumption that $x \in W_1 \backslash \text{dom } f$. Therefore, dom $f = W_1$ or img $f = W_2$. □

The properties of well-orders illustrated in the Propositions in the previous lecture offer useful tools for construction of further concepts. We will now turn our attention to ordinals.

Definition 1 *A set x is* ***transitive*** *if $y \in x \wedge z \in y \Rightarrow z \in x$.*

Example 1 *The empty set $\emptyset$ is transitive. So is $\{\emptyset\}$; $\{\emptyset, \{\emptyset\}\}$; $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$; $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$. A non-transitive set is $\{\{\emptyset\}\}$; for example, $\emptyset \in \{\emptyset\}$ but $\emptyset \notin \{\{\emptyset\}\}$.*

Definition 2 *A set x is an* ***ordinal*** *if*

1) *x is transitive, and*
2) *x is well-ordered by $\in$.*

Example 2 *The empty set $\emptyset$ is an ordinal. So is $\{\emptyset, \{\emptyset\}\}$. On the other hand, $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$ is transitive but not an ordinal, since $\emptyset \notin \{\{\emptyset\}\}$.*

Informally, we know ordinals are numbers that are used for counting, such as "first", "second", etc. However, the above definition (due to Von Neumann) mirrors that concept in a set theoretic manner, as we will see soon. We can now prove several results about ordinals.

Theorem 1 *Every well-order is isomorphic to a unique ordinal.*

With this result, note if α is an ordinal and $\beta \subseteq \alpha$ is transitive, β is an ordinal since subsets of well-orders are also well-orders. The theorem will be proved in the next lecture.

Lemma 1 *The following are all properties of ordinals.*

1) *The empty set is an ordinal.*
2) *If α is an ordinal, and $\beta \in \alpha$, β is an ordinal.*
3) *If α, β are ordinals and $\alpha \subsetneq \beta$ then $\alpha \in \beta$.*
4) *If α, β are ordinals, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$ ($\alpha = \beta$ or $\alpha \in \beta$ or $\beta \in \alpha$).*

Proof. (2) By transitivity of α, $\beta \subseteq \alpha$. Now, suppose $x \in y \in \beta$. Since α is transitive, $x \in \alpha$ and $y \in \alpha$. Now, $\in$ is a linear order of α, so that implies $x \in \beta$. Hence, β is transitive. Well-ordering of β follows from Proposition 2.A1 (subsets of well-orders are well-ordered).

(3) Set membership $\in$ is a well-ordering on α and β, so there exists a least member $\gamma \in \beta \backslash \alpha$. If $x \in \alpha$, then either $x \in \gamma$, $\gamma \in x$, or $x = \gamma$. If $y \in x$, then by transitivity of α, $\gamma \in \alpha$, and if $x = \gamma$, $\gamma \in \alpha$. However, this is a contradiction, so $x \in \gamma$. Conversely, if $x \in \gamma$, then since γ is the least element of $\beta \backslash \alpha$ and is well-ordered by $\in$, we necessarily have $x \in \alpha$. Hence, α and γ share the same elements and are thus the same set, so that $\alpha \in \beta$.

(4) Define $\gamma = \alpha \cap \beta$ (where intersection is defined on page 13 of Kunen). Since α and β are transitive, $\alpha \cap \beta \subseteq \alpha, \beta$ is transitive. Assume $\gamma \neq \alpha$ and $\gamma \neq \beta$. Then $\gamma \subsetneq \alpha$ and $\gamma \subsetneq \beta$. Therefore, γ is an ordinal. By the previous property, $\gamma \in \alpha$ and $\gamma \in \beta$. By definition of intersection, $\gamma \in \gamma$. However, $\in$ is a linear order of β and since $\gamma \in \beta$, $\gamma \notin \gamma$. Hence, we have a contradiction, so $\gamma = \alpha$ or $\gamma = \beta$. In other words, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. □

Corollary 1 *If $\alpha \neq \beta$ are ordinals, then $\alpha \subsetneq \beta$.*

Proof. First, note $\in$ is a well-order of the ordinals. By Lemma 1.4, without loss of generality we can say $\alpha \in \beta$. Now let C be a non-empty set of ordinals. We can show there is a least ordinal. Let $\alpha \in C$. If $\alpha \cap C = \emptyset$, then α is the least element of C. Otherwise, α is well-ordered and so it has a least element, which is also the least element of C. In other words, C is an ordinal. Hence, it is well-ordered by $\in$ and transitive, so $\alpha \neq \beta$ implies $\alpha \in \beta$ or $\beta \in \alpha$ with $\alpha \subsetneq \beta$. □

Finally, we can introduce a proposition about the set of all ordinals much similar to Proposition 2.1.

Proposition 1 *The collection $\{\alpha : \alpha \text{ is an ordinal}\} \equiv O_n$ is not a set.*

Proof. Assume it is. Then $O_n \in O_n$, since O_n would be an ordinal as in the proof of Corollary 1. However, O_n can not be an ordinal and a member of itself, as that would violate the linear order of $\in$. Hence, O_n is not a set. □

In summary, we finished the proof which intuitively said there is only one unique well-order of a given "size." Next, we introduced transitive and ordinal sets, and provided several properties. Ordinal sets are well-ordered by set membership. Finally, in another attempt to describe our universe, we precluded the existence of a collection of all ordinals, much like we earlier dismissed the existence of a set of all sets.

Lecture 4 (01/23/08)

These lecture notes will be finished Wednesday or Thursday.