Differentiable Manifolds

Lecture 11

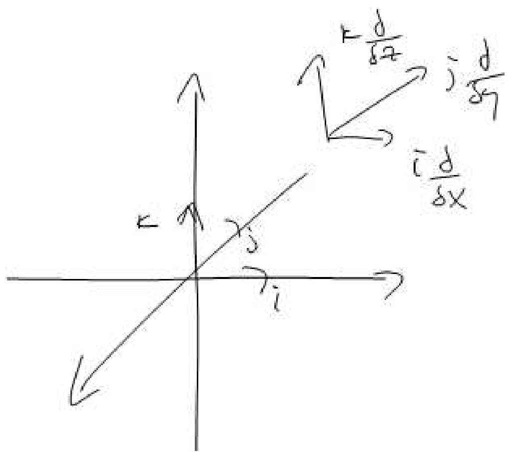
Let $p \in M$ be a smooth manifolds. Then $\mathfrak{F}_p$ is the algebra of germs of C^∞ functions near p. Then $\mathfrak{F}_p$ is an ideal of germs that vanish at p, with $\mathfrak{F}_p^2 = \{\sum f_i g_i \mid f_i g_i \in \mathfrak{F}_p\}$. Furthermore, T_pM the vector space of linear derivations $\tilde{\mathfrak{F}}_p \to \mathbb{R}$ is given by

$$L(af + bf) = aL(f) + bL(g), \text{ and}$$
$$L(fg) = f(p)g + g(p)f.$$

Remember we proved that $T_pM \cong \left(\mathfrak{F}_p/\mathfrak{F}_p^2\right)^*$ as a vector space.

Now look at $p \in \mathbb{R}^n$.

<u>Example</u> (of an element of $T_p\mathbb{R}^m$) Let $L(f) = \frac{\partial f}{\partial r_i}(p) \in \mathbb{R}$. The notation for L_i is $\left.\frac{\partial}{\partial r_i}\right|_p \in T_p$.



Check that $\left.\frac{\partial}{\partial r_1}\right|_p, \ldots, \left.\frac{\partial}{\partial r_n}\right|_p$ forms a basis for $T_p\mathbb{R}^n$.

<u>Lemma</u> If $f : \mathbb{R}^n \to \mathbb{R}$ is smooth near p, then its Taylor approximation

$$\underbrace{f(p+v) - f(p)}_{\in \mathfrak{F}_p} = \sum_{i=1}^{n} \frac{\partial f}{\partial r_i}(p)\, v_i + \overbrace{\underbrace{\sum a_{ij}(p+v)\, v_i v_j}_{\varepsilon(p+v)\|v\| \text{ where } \varepsilon(p+v) \to 0 \text{ as } v \to 0}}^{\in \mathfrak{F}_p^2}$$

where a_{ij} are smooth.

Proof We have

$$f(p+\boldsymbol{v}) = f(p) + \int_0^1 \frac{d}{dt} f(p+t\boldsymbol{v})\,dt = f(p) + \int_0^1 \sum_{i=1}^{n} \frac{\partial}{\partial r_i}(p)\,\boldsymbol{v_i} + \sum \frac{\partial^2 f}{\partial x_j \partial x_i}(p+t\boldsymbol{v})\,\boldsymbol{v_i}\,\boldsymbol{v_j}\,dt = f(p) + \sum_{i=1}^{n} \frac{\partial f}{\partial r_i}(p)\boldsymbol{v_i} + \sum \int_0^1 \frac{\partial^2 f}{\partial x_j \partial x_i}(p+t\boldsymbol{v})\,\boldsymbol{v_i}\,\boldsymbol{v_j}\,dt.$$

So our lemma implies if $L : \mathfrak{F}_p \to \mathbb{R}$ is linear and vanishes on $\mathfrak{F}_p^2$, then

$$L(f) = \sum \frac{\partial f}{\partial r_i}(p)\,\boldsymbol{v_i}$$

i.e., L is in the span of $\langle \frac{\partial}{\partial r_1}, \ldots, \frac{\partial}{\partial r_n} \rangle$. So now by independence, suppose

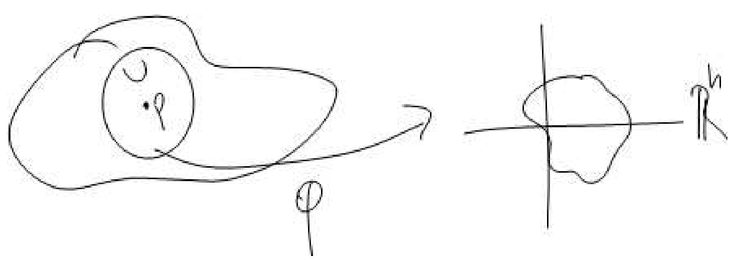
$$\sum a_i \frac{\partial}{\partial r_i}(f) = 0$$

for all functions $f \in \mathfrak{F}_p$. Take $f = r_i$, then $\frac{\partial r_i}{\partial r_j} = \delta_{ij}$, so $a_i = 0\,\forall i$.
In conclusion, $T_p\mathbb{R}^n$ is an n-dimensional vector space with basis

$$\frac{\partial}{\partial r_1}\Big|_p, \ldots, \frac{\partial}{\partial r_n}\Big|_p$$

on $\left(\mathfrak{F}_p/\mathfrak{F}_p^2\right)^*$. □

Now consider a manifold M.



Then

$$(\varphi_* f : U \to \mathbb{R}) \rightsquigarrow (f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}) \text{ and}$$
$$(\varphi^* g : \varphi(U) \to \mathbb{R}) \rightsquigarrow (g \circ \varphi : U \to \mathbb{R})$$

so we have

$$\varphi^* : \mathfrak{F}_p(M) \overset{\cong}{\rightrightarrows} \mathfrak{F}_{\varphi(p)}(\mathbb{R}^n) \text{ and } \varphi_* : \mathfrak{F}_{\varphi(p)}(\mathbb{R}^n) \overset{\cong}{\rightrightarrows} \mathfrak{F}_p(M),$$

as well as

$$\varphi^* : \mathfrak{F}_p^2(M) \overset{\cong}{\rightrightarrows} \mathfrak{F}_{\varphi(p)}^2(\mathbb{R}^n) \text{ and } \varphi_* : \mathfrak{F}_{\varphi(p)}^2(\mathbb{R}^n) \overset{\cong}{\rightrightarrows} \mathfrak{F}_p^2(M).$$

So in conclusion, $T_pM \cong T_{\varphi(p)}\mathbb{R}^n$ for any chart φ defined near p, and dim $T_pM = n$. To see this, let (U, φ) be a chart near p and define X_i on M, so that

$$X_i : (z) = \#_i\,(\varphi(z))$$

and so then $\varphi \to$ the "coordinate functions" $x_1, \ldots, x_n \in \tilde{\mathfrak{F}}_p(m)$. Then we can define

$$\frac{\partial}{\partial x_i}(f) = \frac{\partial}{\partial r_i}(f \circ \varphi^{-1}).$$

So we have a basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$, but note **this basis depends on choosing a chart!** So any time we pick a chart, that determines a basis for T_pM for us.

<u>Problem</u> Suppose we take a smooth path $\sigma : \mathbb{R} \to \mathbb{R}^n$ with $t \overset{\sigma}{\mapsto} (r_1(t), \ldots, r_n(t))$. We'd like to confirm the velocity vector of this path is what we've determined. So we need to show that we can think of $\sigma'(0)$ as $\sum_i r'(0) \frac{\partial}{\partial r_i}\Big|_{\sigma(0)}$.

<u>Definition</u> Suppose M and N are manifolds and $f : M \to N$ is a smooth function. Then f determines a linear transformation $df : T_pM \to T_{f(p)}N$.

Notice then $\left(df\big|_p(L)\right)(g) = L(gf)$ where $g : N \to \mathbb{R}$.

<u>Problem</u> Check that df_p and the old definition of df agree in the case of a map $f : \mathbb{R}^n \to \mathbb{R}^m$. In fact, the matrices with respect to bases $e_1, \ldots, e_n$ are the same.

<u>Lecture 12</u>

<u>Lemma</u> Suppose $g : [0, a] \to \mathbb{R}$ is smooth and $g(0) = g'(0) = \ldots = g^{(n)}(0) = 0$ then $g(x) = \frac{x^n}{n!} \int_0^1 (1-t)^n g^{(n+1)} xt \, dt$. In particular, $g(x) = f(x)x^n$ for some smooth f (by differentiating under the $\int$ sign).

$$\frac{x^{n+1}}{n!} \int_0^1 (1-t)^n g^{(n+1)}(xt) dt = \frac{x^{n+1}}{n!} (1-t)^n \frac{1}{x} g^{(n)}(xt) \Big|_0^1 + \frac{x^n}{(n-1)!} \int_0^1 (1-t)^{n-1} g^{(n)}(x) \, dt$$

($u = (1-t)^n$, $du = -n(1-t)^{n-1}$, $dv = g^{(n+1)}(x)$...?)

$f - P$ (Taylor poly) is a degree $k+1$-th form (homogeneous polynomial) with smooth coefficients.

$f - P = \varepsilon(x)\|x\|^n$ where $\varepsilon(x) \to 0$ as $x \to 0$

If P linear, $f \equiv P$ mod $\mathfrak{F}_o^2$.

$T_pM = \left(\mathfrak{F}_p / \mathfrak{F}_p^2\right)^*$.

Equivalence classes of paths through p, $\sigma_1 \sim \sigma_2$ if they have same velocity.

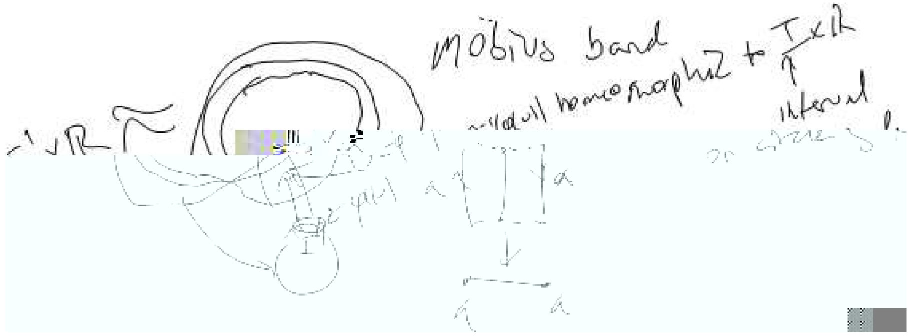
Equivalnce classes of $\mathbb{R}^n$'s "glue by charts".

<u>Tangent bundle</u>

We are taking a <u>set</u> (not top space), $\bigcup_{p \in M} T_P M = M \times \mathbb{R}^n$.

Tangent bundle of $\mathbb{R}^n$: use the product topology on $M \times \mathbb{R}^n \equiv \bigcup_{p \in M} T$, $\{p\} \times \mathbb{R}^n \equiv T_p$.

Charts: For each chart (U, φ) on M, define $\widehat{U} = \bigsqcup_{u \in U} T_p$ to be a domain of a chart (so they are open), define $\widehat{\varphi} : \widehat{U} \to \mathbb{R}^n \times \mathbb{R}^n \equiv T\mathbb{R}^n$, $\widehat{\varphi}(p, v) \to (\varphi(p), d\varphi_p(v))$.



For the tangent bundles, any smooth map $f : M \to N$ determines $df : TM \to TN$ given by $df|_{T_p} = df_p : \mathrm{Hom}(T_p M \to T_{f(p)} N)$.

Lecture 14

Last time, looked at p a polynomial as a map $p(z) : \mathbb{C} \to \mathbb{C}$. Then $p'(z) = a + ib$ so $dp_z = \begin{pmatrix} a & -b \\ -b & a \end{pmatrix}$. Singular iff $a + ib = 0$; critical points of p are the zeroes of p', so there are finitely many.

$p \rightsquigarrow f : S^2 - N \to S^2 - N$ (N is north pole), then f extends to a smooth map $\widehat{f} : S^2 \to S^2$ s.t. $\widehat{f}(N) = N$. Then f has finitely many critical points.

<u>Definition</u> $x \in S^2$ is a <u>regular value</u> of f if there are no critical points in $f^{-1}(x)$ (if $f^{-1}(x) = \emptyset$, x is a regular value).

$x \in S^2$ is a critical value if $f^{-1}(x)$ contains a critical point. f has finitely many critical values, so the set of regular values is connected. Observation: The function $x \to \# f^{-1}(x)$ is a locally constant function on the set of regular values.

$\# f^{-1}(x)$ is finite for any regular value x.

$\# f^{-1}(x) = 0$ for all regular x. Then f is constant so P is constant.
<u>Cor</u> If P is not constant, then $P^{-1}(0)$ is non-empty

<u>Theorem (Sard's Thm)</u> If $f : M^n \to M^m$ is smooth then the set of critical values has measure 0.

Lecture 16

Last time:

Def A smooth manifold-with-boundary is a Hausdorff second countable space with an atlas of charts (u, φ) where $\varphi : U \to H^n = \{(x_1, ..., x_n) \mid x_n \geq 0\} \subset \mathbb{R}^n$. i.e., $\bigcup U_\alpha = M$. $\varphi_\alpha \circ \varphi_b^{-1}$ is a diffeomorphism from $\varphi_\beta(U_\alpha \cap U_\beta)$ to $\varphi(U_\alpha \cap U_\beta)$. A boundary point of M is a point that maps to ∂H^k under some (every) chart)

Theorem Suppose M^m, N^n are manifolds-with-boundary and $f : M \to N$ is smooth. If y is a regular value for both f and for $f\,|_{\partial M}$ then $f^{-1}(y)$ a manifold-with-boundary and $\partial(f^{-1}(y)) = f^{-1}(y) \cap \partial M$.

Proof Omitted.

Brouwer Fixed Pt Theorem For any $f : D^n \to D^n$ continuous has a fixed point.

Pf Suffice to assume f is smooth. Use Weierstrass Thm.

HOMEWORK

Lecture 17

Homework. (1) Prove that any smooth n-manifold has a Riemannian metric.

(2) If M has a Riemannian metric, define the unit sphere bundle

$$U_M := \{v \in T_p M \mid p \in M,\ \|v\| = 1\}.$$

Prove that $U_{S^2} \approx \mathbb{P}^3$ (unit sphere bundle of 2-sphere diffeomorphic to $\mathbb{P}^3$).

(3) Prove that $\mathbb{RP}^1 \subset \mathbb{RP}^2$ ($\mathbb{R}^2 - \{0\} / \sim\ \subset \mathbb{R}^3 - \{0\} / \sim$) is not $f^{-1}(y)$ for a regular value y.

(4) Let $f : S^1 \to \mathbb{R}$ be smooth. Suppose y is a regular value.

- (a) Show that # $f^{-1}(y)$ is even.
- (b) Show the # critical points $\geq$ # $f^{-1}(y)$ for any $y \in \mathbb{R}$.

Last time, we showed the Brouwer fixed point theorem.

Definition. A (smooth) vector field on a manifold M is a section of TM, the tangent bundle, that is, a map $XM \to TM$ such that $X(p) \in T_p M$.

Definition. A Riemannian metric on a smooth manifold M is a family $\langle , \rangle_p$ where $\langle , \rangle_p$ is an inner prouct on T_p which is smooth in the following sense: $f : p \mapsto \langle X, Y \rangle_p$ is a smooth map from $M \to \mathbb{R}$ for any vector fields X, Y.

If (U, φ) is a chart on M, then U has a Riemannian metric, with $\langle v, w\rangle_p = d\varphi_p(v) \cdot d\varphi_p(w)$. Furthermore, if $M \subset \mathbb{R}^N$, then M has a Riemannian metric and $\langle v, w\rangle_p = \sigma(0) \cdot \tau(0)$. A corollary to this is that a compact manifold-with-boundary has a $R \subset M$.

Partition of Unity Lemma. Assume M is a smooth manifold. Then there exist open sets $U_1, \ldots$ and functions $f_1, ..., f_n : M \to \mathbb{R} - \mathbb{R}^-$ so that $\overline{U_n}$ is compact, and the support $(f_n) \subset U_n \ = f_n^{-1}(\mathbb{R} - \{0\})$. Finally, $\sum_{i=1}^{\infty} f_i = 1$ and U_n is a locally finite family. Also note each U_n is contained in the domain fo a chart.

Theorem. A connected 1-manifold-with-boundary is diffeomorphic to S^1, $[0, 1]$, $[0, 1)$, $(0, 1) \approx \mathbb{R}$.

Proof. Let M be a 1-manifold. If $I \subset \mathbb{R}$ is an interval, an $\sigma : I \to M$, $\sigma'(t) \neq 0$ for $t \in I$. Then there exists $\tau : J \to M$ so that $\tau = \sigma \circ s$. Then $\|\tau'(t)\| = 1$ for all t.

Lecture 18

Classify the smooth 1-manifolds:

Definition. Let M be a connected smooth 1-manifold. M has a Riemannian metric $\sigma : [a, b] \to M$ is unit speed if $|\sigma'(t)| = 1$ for all $t \in [a, b]$.

Properties of unit speed

• If $\sigma : [a, b] \to M$ is any path with $\sigma'(t) \neq 0$, then there is a reparametrization $\tau(t) = \sigma(f(t))$, so that τ is unit speed.

• If $\sigma : I \to M$ and $\tau : J \to M$ are unit speed and if there exists $t \in I \cap J$ so that $\sigma(t) = \tau(t)$, then $(\tau \circ \sigma^{-1}|_U)'(t) = 1$ (not -1) (for some subset U). Furthermore, then there is a path $\nu : I \cup J \to M$ so that $\nu|_I = \sigma$ and $\nu|_J = \tau$. Hence, $\tau \circ \sigma^{-1} = \text{id}$ where both are defined).

• If (U, φ) is a chart around $p \in M$, then φ^{-1} is a path and we can reparametrize it to be unit speed. Hence, U is the image of a unit speed path.

Construction. Let (U_1, φ_1), (U_2, φ_2), … be a sequence of charts, so that U_i is connected and $\bigcup_{i=1}^{\infty} U_i = M$. Construct a sequence of unit speed paths $\sigma_n : I_n \to M$ so that either σ_n is not injective for some n ($M \approx S^1$), or σ_n is surjective for some n ($M \approx I$ an interval in R), or $I_n \supsetneq I_{n-1}$. If the sequence is infinite, we need to show there is a diffeomorphism from $\bigcup I_n \to M$. Now consider $\sigma_n(I_n)$ a proper open set. Then there exists X which is a limit point of $\sigma_n(I_n)$, but $x \notin \sigma_n(I_n)$. Then choose (U_k, φ_k) with $x \in U_k$ so that $\varphi_k(x) = a_n$ or b_n. Define $I_{n+1} = I_n \cup \bigcup \{\varphi(U_i) \mid x \in U_i\}$. Then for each (U_i, φ_i) with $x \in U_i$, there is a path τ_i so that $\tau_i(a_n) = x$ and $(\tau_i \circ \sigma_n^{-1})'(a_n) = 1$. Translate φ_i, reflect if necessary to get $\widehat{\varphi}_i$. Set $\tau_i = \widehat{\varphi}_i^{-1}$. Now suppose σ_n is not injective.

If $\sigma_n(x) = \sigma_n(x')$, let $\alpha = \|x - x'\|$. Then $\sigma_n(x + \alpha) = \sigma_n(x)$. [too confusing to copy from the board!]

Lecture 19

Lemma 1. Suppose M is a smooth manifold. There exist open sets $V_1 \subset V_2 \subset \dots$ such that $M = \bigcup_{i=1}^{\infty} V_i$, and $\overline{V}_i \subset V_{i+1}, \overline{V}_i$ compact.

Proof. Let B be a countable basis, $B = \{B_i\}_{i \in \mathbb{Z}}$ with $\overline{B}_i$ compact. Construct V_i inductively. Take $V_1 = B_1$. Then $\overline{V}_1$ is compact, so $\overline{V}_1 \subset B_1 \cup \dots \cup B_n$ for some N. Let n_1 be the first integer so that $\overline{V}_i \subset \bigcup_{i=1}^{n_1} B_i$. Then set $V_2 = \bigcup_{i=1}^{n_1} B_i$. For the inductive step, let n_k be the first integer so $V_{k-1} \subset \bigcup_{i=1}^{n_k} B_i = V$. Then $\bigcup_{i=1}^{\infty} V_i = \bigcup_{i=1}^{\infty} B_i = M$ since $n_k \to \infty$. □

Lemma 2. Suppose M is a manifold and let B be a basis for the topology. Then there are elements $\mathrm{U}_1, \mathrm{U}_2, \dots \in B$ such that

(1) $M = \bigcup_{i=1}^{\infty} U_i$.

(2) Each U_i meets only finitely many U_J.

Proof. Let V_i be given by Lemma 1. Then choose $U_1, \dots, U_{n_1}$ so that they cover $\overline{V}_1$, and choose $U_{n_1+1}, \dots, U_{n_2}$ so that they cover $\overline{V}_2 - V_1$. For the inductive step, $U_{n_k+1}, \dots, U_{n_{k+1}}$ cover $\overline{V}_k - \bigcup_{i=1}^{n_k} \overline{U}_i$ and be disjoint from $\overline{\bigcup_{i=1}^{n_k-2} U_i}$ for $n_k < i \leq n_{k+1}$. Then U_i is disjoint from U_j ($j \leq n_k$). In particular, each point of M is in finitely many U_i. □

Corollary. (*Partition of unity lemma*) Choose B to consist of (open) chart domains U such that there exists a smooth $f_U : M \to \mathbb{R}$ with $f_U(x) > 0$ for $x \in U$, $f_U(x) = 0$ for $x \in M - U$. Then construct $U_1, U_2, \dots$ by Lemma 2 so that $f = \sum_{i=1}^{\infty} f_{U_i}$ is a positive smooth function. Define $g_i := f_{U_i}/f$ so that $\sum_{\text{supp } g_i} g_i = 1$ (and note supp $g_i = U_i$). □

Lemma. Suppose M is a connected manifold. Let B be a basis for the topology consisting of connected open sets. Then there exist $W_1, W_2, \dots \in B$ such that $\bigcup_{i=1}^{\infty} W_i = M$ with $W_n \cap \left(\bigcup_{i=1}^{n-1} W_i \right) \neq \emptyset$.

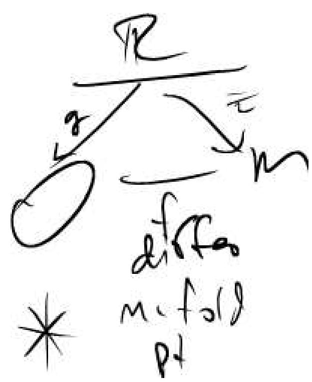
Proof. Let V_i be given as in Lemma 2. Then $W_1 = V_1$, $W_2, \dots, W_{n_2}$ are all elements of (V_i) that meet W_1. $W_{n_k+1}, \dots, W_{n_{k+1}}, \dots$ are elements that meet $\bigcup_{i=1}^{n_k} W_i$. Consider $\bigcup \{V_i \mid V_i \neq W_j \text{ for any } i, j\} = A$. Then A is open, $B = \bigcup_{i=1}^{\infty} W_i$ is open, and $A \cap B \neq \emptyset$. Also, M is connected. □

Let M be a connected 1-manifold. Then we can give M a Riemannian metric.

- If I is an interval in $\mathbb{R}$, then $\sigma : I \to M$ is unit speed if $\|\sigma'(t)\| = 1$ for all $t \in I$.
- Any path $\sigma : I \to M$ can be reparametrized as a unit speed path with the same image.

- Suppose $\sigma : I \to M$ and $\tau : J \to M$ are unit-speed. Then $I \cap J \ni t$, $\sigma(t) = \tau(t)$, and $(\sigma \circ \tau^{-1})'(t) = 1$ [★]. But this implies there exists a unit-speed path $\nu : I \cup J \to M$ with $\nu|_I = \sigma$ and $\nu|_J = \tau$. Also notice $\sigma \circ \tau^{-1}|_{I \cap J} = \text{id}$.
- If (U, φ) is a chart for M, then φ^{-1} is a path so there is a unit speed path $\sigma_U : I \to M$, and $\sigma_U(I) = M$ and $\sigma_U(I) = U$.
- Suppose $\sigma : I \to M$, $\tau : J \to M$ are unit speed and $x \in \sigma(I) \cap \tau(J)$. Consider then there is $a \in R$ such that $\tau(a \pm t)$ and σ satisfy the conditions in [★].

Choose a connected chart neighborhoods for M as in Lemma 3. Then let $U_1, U_2, \ldots$ be so that $\bigcup_{i=1}^{n} U_i$ is connected and meets U_{n+1}; also $\bigcup_{i=1}^{\infty} U_i = M$. Then charts $\varphi_i : U_i \to \mathbb{R}$ with unit speed paths $\sigma_1, \sigma, \ldots$ so that $\sigma_n(I_n) = U_n$. Then we can construct by induction intervals $J_1 \subset J_2 \subset \ldots$, and unit speed paths $\tau_1, \tau_2, \ldots$ so that $\tau_i : J_i \to M$ with $\tau_i|_{J_{i+1}} = \tau_{i-1}$ and $\tau_i(J_i) = \bigcup_{j=1}^{i} U_i$. Then $J = \bigcup_{i=1}^{\infty} J_i$ and $\tau : J \to M$ can be constructed unit speed $t|_{J_i} = \tau_i$ so that $\tau(J) = M$. If τ is injective it is a diffeomorphism. Otherwise $\tau(a) = \tau(a + m)$ for some $a, a + m \in J$. Then construct $\overline{\tau} : \mathbb{R} \to M$ so that $\tau(x) = \tau(x + km)$ where k is chosen so $x + km \in [a, a + m]$. Then observe $\overline{\tau}(x + m) = \tau(x)$ is smooth, so we can take any point in M, take the pre-image of $\overline{\tau}$ to get back to the real line $\mathbb{R}$, and then map it to the circle with $x \xrightarrow{g} e^{2\pi i x/m}$, and we've constructed a diffeomorphism from f to the circle.



Lecture 20

Definition. A smooth function $f : M \to N$ is <u>smoothly homotopic</u> to smooth $g : M \to N$ if there is a smooth homotopy

$$H : M \times [0, 1] \to N \text{ with } H(x, 0) = f(x),\ H(x, 1) = g(x).$$

Definition. An isotopy for a smooth manifold M is when $f, g : M \to M$ are diffeomorphisms so then f is <u>smoothly isotopic</u> if there is a smooth $H : M \times [0, 1] \to M$ with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$, and $H(x, t)$ is a diffeomorphism for $t = 1, 2$.

Theorem. If N is connected n-manifold, and z, y are two points of N there there is a diffeomorphism $f : N \to N$ such that $f(y) = z$ and f is isotopic to id_N.

[Laptop lost power...]

Lecture 22

[Not understandable...]

Definition. An *orientable manifold* M^n has a differentiable structure such that $d\left(\varphi_\alpha \circ \varphi_\beta^{-1}\right)$ has positive determinant at each point.

Example. The annulus is orientable. The Mobius strip is not (if you take overlapping chart neighborhoods eventually they overlap through the Mobius band). An ordered basis of $\mathbb{R}^n$ $(e_1, \ldots, e_n)$ is positive/negative if f...

A manifold M is oriented if and only if there is a choice of ordered basis $(e_1, \ldots, e_n)$ for T_pM, $p \in M$, so that for any chart (U, φ) $(d\varphi_p(e_1), \ldots, d\varphi_p(e_n))$ is positive.

Define "deg" of $f : M \to N$ in $\mathbb{Z}$ by if y is a regular value of f, and $x \in f^{-1}(y)$, $\text{sign}_f(x) = +1$ if df_x maps a positive basis to a positive basis, and -1 if it maps a positive basis to a negative basis. Then deg f is the $\sum_{x \in f^{-1}(y)} \text{sign}_f(x)$ for any regular value y.

Lecture 23

An *orientable manifold* M^n has a differentiable structure such that $d\left(\varphi_\alpha \circ \varphi_\beta^{-1}\right)$ has positive determinant at each point, if and only if we can choose a (not necessarily continuous) basis for each T_pM such that if p and q are in the chart domain for φ then the bases map to compatible bases for $\mathbb{R}^n$ under $d\varphi$.

If $f : M \to N$ takes compact, oriented n-manifolds to y : regular value for f, for $x \in f^{-1}(y)$ define

$$\text{sign}_f(x) = \begin{cases} 1 & \text{if } df_x \text{ carries a positive basis to a positive basis} \\ -1 & \text{otherwise.} \end{cases}$$

Then "define" deg $f = \sum_{x \in f^{-1}(y)} \text{sign}_f(x)$.

Proposition. Suppose M^{n+1} is a compact orentied $n+1$-manifold-with-boundary and N is a compact, oriented n-manifold. If $f : M \to N$ is smooth then y is a regular value with $\deg_y(f|_{\partial M}) = 0$.

Lecture 24

As from last time, we have a map $f : M \to N$ with the former an $(n+1)$-manifold-with-boundary, and the latter an n-manifold (both oriented). Then $y \in N$ is regular for f, $f/\partial M$. Then deg $f|_{\partial M} = 0$.

Proof. Consider $f^{-1}(y)$. First, an orientation of M gives us an orientation of ∂M. Take a point $p \in \partial M$ with a basis $(e_1, ..., e_n)$ for $T_p\partial M$. Then V is an outward vector in T_pM. Then $\text{sign}(e_1, ..., e_n) = \text{sign}(v, e_1, ..., e_n)$. Let v be a non-zero vector in $T_p f^{-1}(y)$ and then extend v to a positive basis of T_pM. Then $df_p(e_1, ..., e_n)$ is a basis for $T_{f(p)}N$. We have to show $\text{sign}(v) = \text{sign } df_p(e_1, ..., e_n)$. Say $p \in \partial M$. Then $(v, e_1, ..., e_n)$ is positive for M which implies $(e_1, ..., e_n)$ is positive for ∂M is v points out and negative if v points in. If v is a positive tangent vector to $f^{-1}(y)$ then $\text{sign}_{f|_{\partial M}}(p)$ is positive or negative if v points out or in, respectively. Let $M = W \times [0, 1]$ where W is oriened. Then $W \times \{0\}$, $W \times \{1\}$ is oriented. How does this orientation of $W \times \{0, 1\}$ compare with the orientation inherited from M?

If $f_0 \sim f_1 \implies \deg_y f_0 = \deg_y f_1$. Assume y is regular for f, f_2. What if y is regular for f_0, f_1but not H. Choose nbhd U of y s.t. for every pt of u is regular for f_0, f_1 with $\deg_U f_0 = \deg_y f_0$ and same for f_1. Then choose $z \in U$ regular. Then $\deg_z f_0 = \deg_z f_1$. If y, z are regular for f then $\deg_y f = \deg_{H(y,z)} f$ for all t, so there exists an isotopy $H : M \times I \to M$ so that $H(x, t)$ is a differeomoprhism, $H(x, 0) = x$, and $H(y, 1) = 1$.

If $f : M \to M$ is an orientation-reversing diffeomorphism then f is not homotopic to φ_M with deg $f = -1 =$ -deg φ_M.

If $M = S^n$ with $f(x) = -x$, then deg $f = 1$ if n is odd and -1 if n is even, with

$$\det df = \begin{pmatrix} -1 & 0 & \cdots \\ 0 & \ddots & 0 \\ \cdots & 0 & -1 \end{pmatrix} = +1 \text{ if } n \text{ is even and } -1 \text{ if } n \text{ is odd.}$$

If n is even, then any vector field on S^n has a zero vector.

Lecture 25

We were looking at $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ with $\alpha(x) = -x$, a linear transformation. Have det $\alpha = -1$ if n is odd, else even. So α is orientation preserving if n is even.

Look at $\alpha : S^{n-1} \to S^{n-1}$. Then α is orentiation preserving on S^{n-1} if and only if it is on $\mathbb{R}^n$.

Lecture 26

Theorem. Let V_1, V_2 be vector fields on a compact smooth manifold M with isolated zeroes. Then ind $V_1 =$ ind V_2.

(1) Show that $\text{ind}_Z V$ is well-defined. [etc]

Lemma. Suppose U is a convex open set in $\mathbb{R}^n$ and V is a vector field on U with a zero at $z \in U$. Also suppose $f : U \to \mathbb{R}^n$ is a diffeomorphism from U to $f(U)$.

Theorem 2. If U is a convex open set in $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is a diffemorphism from U, then f is isotopic to id.

Proposition. If V is a vector field such that (1) V points out on ∂M $(V \cdot N > 0)$ and (2) V has isolated zeroes in $M - 2M$. Then ind $V =$ deg Gauss : $\partial M \to S^{n-1}$.

Notice then Ind $V =$ Ind V' for any two of [redacted]tor fields.

Lecture 27

We had a compact manifold M, with a vector field V on M with isolated zeroes so that

$$\text{Ind}\, V = \sum_{V(Z)=0} \text{Ind}_Z V = \sum_{V(Z)=0} \deg f_Z$$

with $f_Z : S_\varepsilon \to S^n$.

Definition. If U is an open neighorhood of z, an isolated zero of a vector field V, then z is non-degenerate if dV_z is non-singular.

Example. For $f(x, y)$, if $V = \nabla f = \langle \partial f/\partial x,\, \partial f/\partial y\rangle$, then

$$V(x, y) = \frac{\partial f}{\partial x}(x, y)\vec{i} + \frac{\partial f}{\partial y}(x, y)\vec{j}$$

so that the matrix dV_z with respect to $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ is the Hessian matrix

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}.$$

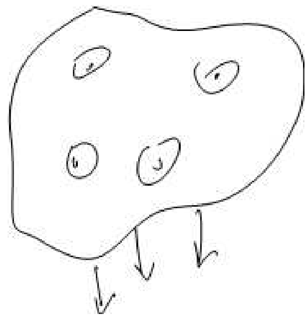
Lemma 2. Suppose V has an isolated zero at z. Then an arbitrarily small perturbation of V will have $\deg_z(V)$ non-degenerate zeroes in a small neighborhood of z.

Lemma 1. If z is a non-degenerate zero of V, then $\deg_z V = \pm 1$.

Proof. V (thought of as a diffeomorphism) is smoothly isotopic to the identity or to a reflection (degree is 1 or -1, respectively).

Lecture 28

Theorem. If M is a smooth n-manifold and V is a vector field on M with isolated

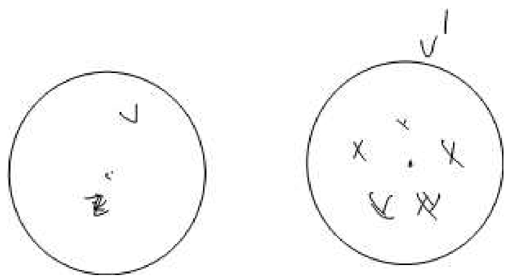


(removed points circled)

If z is a nondegenerate zero of V, then if dV_z is non-singular, in this case

$$\mathrm{Ind}_z\, V = \text{œ} \begin{matrix} 1 & \text{if det } dV_x > 0 \\ -1 & \text{if det } dV_x < 0 \end{matrix}.$$

If V is arbitrary with isolated zeroes, then we can perturb iV to a vector field with non-generated zeroes of the same index.



with □ $\mathrm{ind}_{Z_i} V' = \mathrm{ind}_z V$.

Embed M in $\mathbb{R}^n$ for some N. Define $N_\varepsilon \subset \mathbb{R}^n$ by $N_\varepsilon =$ e$x \in \mathbb{R}^n$: dist□x, N□ $\leq \varepsilon$f.

Assume:

(1) for small enough ε, N_ε is a smooth manifold.

(2) for even smaller Ae

if08.591( )y0TJ /R12 8p479431(a)3.31(c)-2.36417(o)-0.956417(u)-0.9eb i Tls57 9(t)-2.53658(h)-0.9

$$T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n.$$

Then $TM \subset M \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n$ at each point of M, T_xM is a subspace of $T_x\mathbb{R}^n$ with $\sum a_i\, \partial/\partial x_i$ tangent vectors to $\mathbb{R}^n$.

Assume $M \subset \mathbb{R}^n$. Then if $NM \subset M \times \mathbb{R}^n$,$NM = \{(x, v) \mid v \perp T_xM$ (subspace of $\mathbb{R}^n$)$\}$

Then NM is a submanifold of $M \times \mathbb{R}^n \subset T\mathbb{R}^n \equiv \mathbb{R}^n \times \mathbb{R}^n$. Using Gramm-Schmidt process we can find smothing that is smooth ($NM \to \mathbb{R}^n$ by $(x, v) \to x + v$, $x \in N$, $v \perp T_xM$).

Assume: (1) for small enough ε, N_ε is a smooth manifold, and (2) for even smaller ε there is a well-defined map $r : N_\varepsilon \to M$ such that r is smooth and $r(x)$ is the closest point on M.

If we choose a $d\psi_{(x,0)}$ nonsingular, then the inverse function theorem shows ψ is a homeomorphism of a neighborhood of $x \in M$.

Hence, $NM_\varepsilon = \{(x, v) \in NM \mid \|v\| < \varepsilon\} \to \mathbb{R}^n$. We claim $\exists \varepsilon > 0$ such that $\psi : NM_\varepsilon \to \mathbb{R}^n$ is injective. Otherwise, we find a sequence $(x_n, v_n), (x'_n, v'_n)$ so that $\psi(x, v_n) = \psi(x'_n, v'_n)$ and $\|v_n\| \to 0$. We find a subsequence of (x'_n) converging to x. For larger (x_n, v_n) and (x'_n, v'_n) of a trivial inside of the embedded neighborhood of x. So then r is a projection $\circ\, \psi^{-1}$. □

Lecture 29

Finite dimensional real vector space

Definition. A function $f : V_1 \times ... \times V_k \to W$ is *multilinear* if

$$f(v_1, ..., v_{i-1}, \cdot, v_{i+1}, ..., v_k)$$

is linear for each i.

Example. A basic example is if $f_1 : V_1 \to W$, ..., $f_k : V_k \to W$, then $f_1 \cdot ... \cdot f_k$ is multilinear. An algebra A is a vector space with a product that satisfies $(\alpha \cdot v)w = v \cdot (\alpha w) = \alpha(v \cdot w)$ and A is a ring.

There exists a vector space $T = T(v_1, ..., v_k)$ which is "universal" for multilinear functions in the sense: (1) there is a multilinear function $\varphi : V_1 \times ... \times V_k \to T$, and (2) if $f : V_1 \times ... \times V_k \to W$ is multilinear, then it factors as

$$V_1 \times ... \times V_k \xrightarrow{\varphi} T$$

$$\searrow$$

$$(v, \alpha v_2 + \beta v_2') - \alpha(v_1, v_2) - \beta(v_1, v_2'),$$
$$(\alpha v_1 + \beta v_1', v_2) - \alpha(v_1, v_2) - \beta(v_1', v_2),$$
$$(\alpha v, w) - \alpha(v, w), \text{ and}$$
$$(v, \beta w) - \beta(v, w).$$

Notation: $\varphi(v_1, \ldots, v_k)$ is written $v_1 \otimes \ldots \otimes v_k$.

Suppose $e_1, \ldots, e_n$ is a basis for v and $f_1, \ldots, f_m$ is a basis for w. We can construct a basis for $v \otimes w$, say by $\{e_i \otimes f_j\}$. We can show these span $v \otimes w$. Suppose $f : V \times W \to Z$ is multilinear. Let $v = \sum a_i e_i \in V$ and $w = \sum b_j f_j \in W$. Then

$$f(v, w) = f\left(\sum a_i e_i, \sum b_j f_j\right) = \sum_{i=1}^{n} a_i b_j f(e_i, e_j).$$

This means f is uniquely determined by specifying $f(e_i, e_j)$. If

$$\delta_{(i,j)}\left(\sum a_i e_i, \sum b_j f_j\right) = a_i b_i$$

then multilinear functions form a vector space with basis $\delta_{(i,j)}$ (dim $= mn$). So this corresponds to $e_i \otimes e_j$.

Definition. $f : V_1 \times \ldots \times V_k \to W$ is an *alternating mutilinear function* if

(1) f is multilinear.

(2) $f(v_1, \ldots, v_i, v_{i+1}, \ldots, v_k) = -f(v_1, \ldots, v_{i+1}, v_i, \ldots, v_k)$.

Example. If dim $V = d$, consider det : $V^k \to \mathbb{R}$. From linear algebra, we know this is an example of an alternating multilinear function.

Example. $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ by $(v, w) \to v \times w$ (cross product).

$$f\left(\sum a_{1j} e_j, \sum a_{2j} e_j, \ldots, \sum a_{nj} e_j\right) = \sum f\left(e_{\sigma(1)}, \ldots, e_{\sigma(n)}\right) = k \cdot f(e_1, \ldots, e_n).$$

Lecture 30

We were taking vector spaces V, W so that $f : V \times W \to Z$ is multilinear.

Theorem. There exists a unique vector space $V \otimes W$ such that for any multilinear $f : V \times W \to Z$ there exists a unique linear $T : V \otimes W \to Z$ so that

$$V \times W \to V \otimes W$$
$$\searrow \qquad \downarrow T$$
$$Z.$$

Proof. Construct a vector space with bases $\{(v, w) \in V \times W\}$, R with R a subspace generated by the relations

$$(v, \alpha v_2 + \beta v_2') - \alpha(v_1, v_2) - \beta(v_1, v_2'),$$
$$(\alpha v_1 + \beta v_1', v_2) - \alpha(v_1, v_2) - \beta(v_1', v_2),$$

$$\langle\alpha v, w\rangle - \alpha\langle v, w\rangle, \text{ and}$$
$$\langle v, \beta w\rangle - \beta\langle v, w\rangle.$$

Then we have $V \times W \to H \to H/R$ with $V \times W \to Z$ so that there is a unique mapping $H \to Z$ and $H/R \to Z$ that makes the diagram commute. Then we call $H/R = V \otimes W$.

$$\text{Multi}(V, W; Z) = \{\text{multilinear functions } V \times W \to Z\}.$$

It is clear this is finite dimensional. Choose bases $e_1, ..., e_n$ for - $f_1, ..., f_{n+1}$ for W. If $F : V \times W \to Z$ is multilinear, then $F(\sum a_i e_i, \sum b_j f_j) = \sum a_i b_j F(e_i, f_j)$. Then $E_{ij} \in \text{Multi}(V, W, Z)$. Then $E_{ij}(e_k, f_l) = 1$ if $i = k, j = 1$ and 0 otherwise. So then $F = \sum F(e_i, e_j) \cdot E_{ij}$. So this is saying that $\text{Multi}(V, W; \mathbb{R}) \approx \text{Linear}(V \otimes W, \mathbb{R}) = W^*$ (the dual). Therefore $(V \otimes W)^*$ is finite dimensional with dimension $\dim V \cdot \dim W$. Then V is unnaturally isomorphic to V^*, meaning we can construct an isomorphism by choosing a basis $e_1, ..., e_n$ of V and considering $e_1^*, ..., e_n^* \in V^*$. This is defined by $e_i^*(e_j) = \delta_{ij}$. Then $f \in V^*$ means we can uniquely write $f = \sum f(e_i) e_i^*$. If V is a Hilbert space, this is canonical. If e_i are an orthonormal basis, then $e_i \to e_i^*$. Then $(V^*)^* \approx V$ cononically, so v$(f) = f(v)$.

Then the E_{ij} form a basis for Multi$(V, W; \mathbb{R})$ so that $E_{ij} \in (V \otimes W)^*$ is dual to $e_i \otimes e_j \equiv$ image of (e_i, e_j). We can then construct the *tensor algebra*

$$T(V) = \bigoplus_{n=0}^{\infty} T_n(V),$$

with $T_0(V) = \mathbb{R}$, $T_1(V) = V$, and $T_n(V) = T_{n-1}(V) \otimes V$. Then $T(V)$ is an algebra with $T_n(V) \otimes T_m(V) \to T_n(V) \otimes T_m(V) \equiv T_{m+n}(V)$. Then $T(V)$ is a graded algebra.

A similar proof can be given for $f : V \times ... \times V \to Z$. □

Lecture 31

Continued lecture from Warner chapter 2.

Lecture 32 [Warner 62-66]

We let $\Lambda(M)$ be so that the fiber over p is $\Lambda(T_p M)$. Then $\mathcal{T}_{r,s}(M)$, the fiber over p, is

$$\underbrace{T_p M \otimes \cdots \otimes T_p M}_{r \text{ times}} \otimes \underbrace{T_p M^* \otimes \cdots \otimes T_p M^*}_{s \text{ times}}.$$

Denote $\mathfrak{X}(M) = \{\text{vector fields on } M\}$ as a module over C^∞-smooth functions on M ([Warner; 64]). Notice k-forms are also a module over $C^\infty(M)$. If a k-form ω is an alternating k-linear (multilinear) funciton from the $C^\infty(M)$-module to $\mathfrak{X} \times ... \times \mathfrak{X} \to C^\infty(M)$. Then $\omega(X_1, ..., X_k)/(p) = \omega_p(X_1(p), ..., X_k(p))$ is a differential k-form if and only if alternating k-linear functions from the $C^\infty(M)$-module $\mathfrak{X} \times ... \times \mathfrak{X}$ to $C^\infty(M)$.

Lemma. Suppose $\omega : \mathfrak{X} \times .... \times \mathfrak{X} \to C^\infty(M)$ is alternating multilinear. Also let $(X_1, ..., X_k), (Y_1, ..., Y_k) \in \mathfrak{X} \times ... \times \mathfrak{X}$ are such that $X_i(p) = Y_i(p)$. Then

$$\omega|_p(X_1, ..., X_k) = \omega|_p(Y_1, ..., Y_k).$$

Proof. It suffices to assume $X_1(p), ..., X_k(p) = 0$. We choose a chart neighborhood U around p. In U, $X_i = \sum a_i \frac{\partial}{\partial X_i}$ with $a_i \in C^\infty(U)$ and $a_i(p) = 0$. We then choose a bump function $\varphi \equiv 1$ on $W \subset U$ and $\varphi \equiv 0$ on $M - U$. Then $\overline{X_i} = \sum (a_i\varphi)(\varphi \frac{\partial}{\partial x_i})$. Then $X_i = \sum (a_i\varphi)(\varphi \frac{\partial}{\partial x_i}) + (1 - \varphi^2)X_i = \overline{X_i} + (1 - \varphi^2)X_i$ so $\omega(\overline{X_1}, \overline{X_n}) = 0$ at p because $(X_1, ..., X_n)$ is 0 at p. □

Graded modules, e.g., $E(M)$, for homomorphisms f from $E(M) \to E(M)$:

- f has degree i if $f : E^n(M) \to E^{n+1}(M)$.
- f is a derivation if $f(\omega \wedge \eta) = f(\omega) \wedge \eta + \omega \wedge f(\eta)$.
- f is an antiderivation if $f(\omega \wedge \eta) = f(\omega) \wedge \eta + (-1)^p \omega \wedge f(\eta)$ where $\omega \in E^p(M)$ and $\eta \in E(M)$.

Theorem. There is a unique antiderivation $d : E(M) \to E(M)$ of degree -1 satisfying $d^2 = 0$ and $d(f) = df$ for $f \in E^0(M) \equiv C^\infty(M)$.

In $\mathbb{R}^n$ with $f(x_0, ..., x_n)$ smooth, $df = \sum \frac{\partial f}{\partial x_i} dx_i$, where dx_i is a linear functional on tangent vectors, $dx_i(X) \in C^\infty$, $dx_i|_p(v) \in \mathbb{R}$ with $v \in T_p\mathbb{R}^n$. Then dx_i is the dual to the standard basis vector field.

Lecture 34

Homework. Warner, chapter 2: 9, 10, 12, 13.

If $E(M)$ are differential k forms with a module over $C^\infty(M)$ bundle with fibers $\Lambda(T_p)^*$ thath as product $\wedge$. Also, there is a unique degree 1 antiderivation $d : E(M) \to E(M)$ such that $d^2 = 0$ ($d(d(a)) = 0$) with $d(f) = df$. Furthermore,

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \wedge d\eta \text{ when } \omega \in E^p(M).$$

Also, $E^0(M) \equiv C^\infty(M)$. Suppose $f, g \in E^0(M)$ (C^∞ functions), with $\omega \in E^p(M)$ and $f \wedge \omega = \omega \wedge f$. Further, $\omega \wedge \eta = (-1)^{pq}\ \eta \wedge \omega$ when $\omega \in E^p$, $\eta \in E^q$. The convention is that $f \wedge \omega \equiv f\omega = \omega f$ and $\omega \wedge f \equiv f\omega = \omega f$. If we have $f, g \in C^\infty(M)$, then $\mathrm{d}(fg) = \mathrm{d}f \cdot g + g \cdot \mathrm{d}f = g \cdot \mathrm{d}f + f \cdot \mathrm{d}g$. Then $\mathbb{R}^n = M$ and $E^0 \equiv C^\infty$ with $E_1, ..., E_n$ constant vector fields, $E_i(p) = e_i$, $E_i \equiv \frac{\partial}{\partial x_i}$. We know E_1 has basis $\mathrm{d}x_1, ..., \mathrm{d}x_n$. Then

$$\mathrm{d}x_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij},$$

which is a constant function arising from the differential form applied to the partial. Notice $\mathrm{d}x_1 \wedge \mathrm{d}x_2(X, Y) \in C^\infty$, with $X = a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + ...$ and $Y = b_1\frac{\partial}{\partial x_1} + ...$, with

$$\mathrm{d}x_1 \wedge \mathrm{d}x_2(X < Y) = \mathrm{d}x_1(X)\mathrm{d}x_2(Y) - \mathrm{d}x_1(Y)\mathrm{d}x_2(X) = a_1b_2 - a_2b_1.$$

Now, if we take $f \in C^\infty$, then

$$\mathrm{d}f = \frac{\partial f}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f}{\partial x_2}\mathrm{d}x_2 + \ldots$$

with $\mathrm{d}x_i \in E'$, $\mathrm{d}(\mathrm{d}x_1) = 0$,

$$\mathrm{d}(f\mathrm{d}x_1) = \mathrm{d}f \wedge \mathrm{d}x_1 + f\mathrm{d}(\mathrm{d}x_1) = \left[\frac{\partial f}{\partial x_1}\mathrm{d}x_1 + \frac{\partial f}{\partial x_2}\mathrm{d}x_2 + \frac{\partial f}{\partial x_3}\mathrm{d}x_3\right] \wedge \mathrm{d}x_1$$

$$= -\frac{\partial f}{\partial x_2}\mathrm{d}x_1 \wedge \mathrm{d}x_2 - \frac{\partial f}{\partial x_3}\mathrm{d}x_1 \wedge \mathrm{d}x_3,$$

so that $\sum a_i \mathrm{d}x_i \in E'$ and so

$$\mathrm{d}\left(\mathrm{d}\left(\sum a_i\, \mathrm{d}x_i\right)\right) = \mathrm{d}\left[\sum_{i,j}\left[\frac{\partial a_i}{\partial x_j}\mathrm{d}x_j\right] \wedge \mathrm{d}x_i\right] = \sum_{i,j,k}$$ [BOARD WAS ERASED].

In $\mathbb{R}^3$, $\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z$ with

$$\mathrm{d}(\mathrm{d}f) = \frac{\partial^2 f}{\partial x^2}\mathrm{d}x \wedge \mathrm{d}x + \frac{\partial^2 f}{\partial y \partial x}\mathrm{d}y \wedge \mathrm{d}x + \frac{\partial f}{\partial z \partial x}\mathrm{d}z \wedge \mathrm{d}x + \frac{\partial^2 f}{\partial x \partial y}\mathrm{d}x \wedge \mathrm{d}y + \frac{\partial^2 f}{\partial y^2}\mathrm{d}y \wedge \mathrm{d}y + \ldots$$

If I, J, K are constant vector fields, then the Riemannian metric given by

$$I \cdot I = J \cdot J = K \cdot K = 1 \text{ and } I \cdot J = J \cdot K = K \cdot I = 0$$

give the pairing $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ which induces an isomorphism T to its dual space T^*. Now let

$$\mathrm{d}x \leftrightarrow I, \mathrm{d}y \leftrightarrow J, \mathrm{d}z \leftrightarrow K \text{ for } E^1 \leftrightarrow \mathfrak{X}$$

with $p\,dx + q\,\mathrm{d}y + r\,\mathrm{d}z \to pI + qJ + rK$. Then E^3 is a one dimensional C^∞-module so the basis $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ is given by $E^3 \leftrightarrow C^\infty$. Then there is a natural pairing (called the Hodge star) $E^2 \times E^1 \to E^3 \equiv C^\infty$ with $\omega \times \eta \to \omega \wedge \eta$ and $E^2 \equiv (E^1)^* \equiv \mathfrak{X}$. Then we need to show $E^0 \leftrightarrow C^\infty$, $E^1 \leftrightarrow \mathfrak{X}$, $E^2 \leftrightarrow \mathfrak{X}$, and $E^3 \leftrightarrow C^\infty$. Then we'll get

$$p\,dx + q\,\mathrm{d}y + r\,\mathrm{d}z \leftrightarrow pI + qJ + rK$$

$$p\,dx \wedge \mathrm{d}y + q\,\mathrm{d}y \wedge \mathrm{d}z + r\,\mathrm{d}z \wedge \mathrm{d}x \leftrightarrow pK + qI + rK$$

for $f\ \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \to f$, with

$$\mathrm{d} : E^0 \to E^1 \leftrightarrow f \leftrightarrow \nabla f$$

$$\mathrm{d} : E^1 \to E^2 \quad X \to \nabla \times X \quad \text{(curl)}$$

$$\mathrm{d} : E^2 \to E^3 \quad X \to \nabla \cdot X \quad \text{(div)}$$

[I really don't know where this is going...]

Lecture 35

Pull-backs

If $E : M \to N$ is smooth and $\omega \in E^k(N)$, then f determines a pull-back of ω,

$$\delta f(\omega) \in E^k(M) \quad \text{with} \quad \delta f(\omega)(v_1, \ldots, v_k) = \omega(df(v_1), \ldots, df(v_k))$$

with $v_i \in T_pM$. One way of describing a differential form is something you can integrate over an M-manifold. A k-form is something you can integrate over a singular k-chain. We want to find a singular k-chain σ and its boundary $\partial\sigma$, where we define $\int_\sigma \omega$ with $\omega \in E^k(M)$. Then we want to prove Stokes' Theorem [Warner pg 144]: $\int_{\partial\sigma}\omega = \int_\sigma d\omega$. Here, σ is a $(k+1)$-chain and $d\omega$ is a $(k+1)$-form, and $\partial\sigma$ is a k-chain and ω is a k-form. Then consider smooth singular k-simplexes. Define a smooth singular k-chain to be a formalism $\sum_{i=1}^{n} a_i\sigma_i$ with $a_i \in \mathbb{R}$ where σ_i is a smooth singular k-simplex.

For simplexes, Warner's notation is $\Delta^0 = \{0\}$, $\Delta^1 = [0,1]$, and

$$\Delta^n = \left\{(x_1, ..., x_n) \,\middle|\, \sum_{0 \le x_i \le 1} x_i \le 1\right\}.$$

Suppose σ is a singular k-simplex, i.e., $\sigma : \Delta^k \to M$. We can define (ω a k-form)

$$\int_\sigma \omega = \int_{\Delta^k} \partial\sigma(\omega) = \int_{\Delta^k} f$$

where the boundary $\partial\sigma(\omega) = f\,dx_1 \wedge dx_2 \wedge ... \wedge dx_k$, with $\partial\sigma(\omega) \in E^k(\Delta^k)$.

Before we continue, let's examine what we have done so far in light of what we know from calculus. For line integrals, let $\sigma : [0,1] \to \mathbb{R}^2$ with $\sigma(t) = (x(t), y(t))$ be a path with a vector field $\vec{F} = P\vec{i} + Q\vec{j}$. Then

$$\oint_\sigma \vec{F} \cdot d\vec{s} = \int_0^1 P(x(t), y(t)) \cdot x'(t) + Q(x(t), y(t)) \cdot y'(t)\,dt.$$

Then we know that this is independent of the parameterization with the same endpoints. Then $\vec{i} \to dx$, $\vec{j} \to dy$, and $\vec{F} \to \omega = P\,dx + Q\,dy$. Then

$$\int_\sigma \omega = \int_0^1 \partial\sigma(\omega) = \int_0^1 f(t)\,dt,$$

where $\partial\sigma(\omega) = f \wedge dt$. So then

$$\partial\sigma(Pdx + Qdy) = (P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t))\,dt$$

with $\sigma(t) = (x(t), y(t))$ so that $\partial\sigma(dx) = x'(t)\,dt$ and $\partial\sigma(dy) = y'(t)\,dt$, and of course $\partial\sigma(P)(t) = P(x(t), y(t))$, and $\partial\sigma(Q)(t) = Q(x(t), y(t))$.

Now let's go back to the general case. Take a k-chain σ so that

$$\sum a_i\sigma_i \quad \text{where } \sigma_i \text{ are simplexes,}$$

then

$$\int_\sigma = \sum a_i \int_{\sigma_i} \omega.$$

Lecture 36

We want to define the boundary of a smooth singular k-simplex $\sigma : \Delta^k \to M$ (denoted by $\partial\sigma$). The boundary of Δ^0 (a point) is 0. For Δ^1, it is $\sigma(1) - \sigma(0)$ ("distance" from 0 to 1). We want to create a map $K_0^1(0) = 0$, $K_1^1(0) = 1$, $K_0^1 : \Delta^0 \to \Delta^1$ and $K_1^1 : \Delta^0 \to \Delta^1$. For a 2-simplex (triangle), the boundary is given by $\sigma^0 - \sigma^1 + \sigma^2 = \sigma \circ K_0^2 - \sigma \circ K_1^2 +$

$\sigma \circ K_2^2$ with $K_i : \Delta^1 \to \Delta^2$. We can map the 1-simplex (unit interval) onto the three edges of the 2-simplex (triangle). For a 3-simplex (tetrahedron), $\partial\sigma = \sigma_0 - \sigma_1 + \sigma_2 - \sigma_3$.

Some basic homology and application to proving Stokes' Theorem (see Warner)