Lecture 13

Applications of "fiber dimension"

Example 1 Lines on surfaces in $\mathbb{P}^3$.

Theorem A general surface in $\mathbb{P}^3$ of degree ≥ 4 contains no lines.

Note: To say that a "general" surface" has some property means: look at space of deg d surfaces. This is parametrized by $\mathbb{P}^{\binom{d+3}{3}-1}$. The set of ones that do not have the property is a finite union of proper subvarieties.

$$U = \left\{ [F] \in \mathbb{P}^{\binom{d+3}{3}-1} \,\middle|\, V(F) \text{ has property} \right\} \text{ should be dense Zariski-open.}$$

Proof Look at the incidence variety, $\mathcal{I} = \left\{ (L, X) \in \mathbb{G}(1,3) \times \mathbb{P}^{\binom{d+3}{3}-1},\ L \subseteq X \right\}$.

Let p_1 be the projection to $\mathbb{G}(1,3)$ and p_2 to $\mathbb{P}^{\binom{d+3}{3}-1}$. First, we have to know that I is a projective variety.

$\mathbb{G}(1,3)$ is a union of affine spaces isomorphic to $\mathbb{A}^4$. Notice $\mathbb{G}(2,4) \cong \mathbb{G}(1,3)$. An open affine set in $\mathbb{G}(2,4)$ is given by subspaces of the form: span $\left(\begin{pmatrix} 1 \\ 0 \\ a \\ b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ c \\ d \end{pmatrix} \right)$, where $a, b, c, d \in k$ for choice of basis in k^4. Can also define surface $X \subseteq {}'\mathbb{P}^3$ of deg d with an equation $F(x_0, x_1, x_2, x_3) = 0$. So then any point in L is $\begin{pmatrix} 1 + t \cdot 0 \\ 0 + t \cdot 1 \\ a + t \cdot c \\ b + t \cdot d \end{pmatrix}$, $t \in \mathbb{P}^1$.

$L \subseteq X \Leftrightarrow F(1, t, a + tc, b + td)) \equiv 0$ in t. Expand, collect terms, coeffs at $1, t, t^2, ..., t^5$ give equations. This implies $I \cap \left(\mathbb{A}^4 \times \mathbb{P}^{\binom{d+3}{3}-1} \right)$ is closed subvariety $\Rightarrow$ $I \subseteq \mathbb{G}(1,3) \times \mathbb{P}^{\binom{d+3}{3}-1}$ is a closed subvariety.

What else can we say? Well, p_1 is surjective.

Say $L \in \mathbb{G}(1,3)$ is defined by $L_1 = L_2 = 0$. We can find polynomial F of deg d that contains L. Then $F = G_1 L_1 + G_2 L_2$, where G_1, G_2 are of deg $d - 1$.

$I = \{(L, X) \mid L \subseteq X\} \xrightarrow{p_2} \mathbb{P}^{\binom{d+3}{3}-1}, \quad \xrightarrow{p_1} \mathbb{G}(1,3)$.

- Fiber of p_1: $\mathbb{P}GL(3)$ is transitive on lines, so we can move any line L to the one defined by $x_0 = x_1 = 0$

Polynomials F of deg d s.t. $V(F) \supseteq V(x_0, x_1)$.

$F = Gx_0 + Hx_1$, deg $G = d - 1 =$ deg H, count dimension of such F.

$2\binom{d-1+3}{3} - \binom{d-2+3}{3} \longleftarrow$ not to overcount cases when $G = x_1 G'$, $H = x_0 H'$.

$= 2\frac{(d+2)(d+1)}{6} - \frac{(d+1)d(d-1)}{6} = \frac{d(d+1)(d+5)}{6}$.

- Fiber of $p_1 = \mathbb{P}^{\frac{d(d+1)(d+5)}{6}-1}$.

- Now use theorem of the fibers. Since $\mathbb{G}(1,3)$ is irreducible of dimension 4, by the theorem about fibers, I is irreducible of dim $\frac{d(d+1)(d+5)}{6}+3$.
We want to show that p_2 is not surjective if $d \geq 4$. Then $p_2(I)$ will be a proper closed irreducible subvariety, and we can take $U = \mathbb{P}^{\binom{d+3}{3}-1} - p_2(I)$. To show it, just compare dimensions. We have dim $\mathbb{P}^{\binom{d+3}{3}-1} = \binom{d+3}{3} - 1 = \frac{(d+3)(d+2)(d+1)}{6} - 1$ and dim $I = \frac{d(d+1)(d+5)}{6} + 3$. We see

$$\dim \mathbb{P}^{\binom{d+3}{3}-1} - \dim I = \frac{(d+3)(d+2)(d+1)}{6} - 1 - \frac{d(d+1)(d+5)}{6} - 3 = \frac{(d+1)(d^2+5d-6-d^2-5d)}{6} - 4 = d+1-4 = d-3.$$

Since $d \geq 4$, dim $I <$ dim $\mathbb{P}^{\binom{d+3}{3}-1} \Rightarrow$ Theorem. □
Note: When $d \leq 3$, there *are* lines on any surface.
Case $d=1$: The surface is a plane, so dim $I \geq 2$.
Case $d=2$: We have a quadric surface, so if it's smooth, we can write it as $x_0x_1 = x_2x_3$. Then we could write for example a line $(\alpha t, t, \alpha t, t)$ for α a line in X (infinite family).
Case $d=3$: We have a cubic surface, and in this case, the dimensions are equal. There are exactly 27 lines on any smooth cubic surface.

Example 2 Study the determinental variety. Let M be the space of $m \times n$ matrices up to scale (so this will be a proj space $\cong \mathbb{P}^{mn-1}$). Let M_k be the matrices in M with rank $\leq k$.
Thm Want to show that $M_k \subseteq M$ is an irreducible variety of coimension $(m-r)(n-r)$.
Proof Let $I \subseteq M \times \mathbb{G}(n-r,n)$ so that

$$I = \{(A,\Lambda) \mid A \text{ is a matrix of size } m \times n \text{ and } \Lambda \subseteq \ker A\}.$$

Exercise I is a projective variety, $I \xrightarrow{p_1} M$ and $I \xrightarrow{p_2} \mathbb{G}(n-r,n)$.

-study p_2: Fix subspace Λ of dimension $n-r$. If $\Lambda \subseteq$ ker A, get induced map.

$$k^n/\Lambda \xrightarrow{\overline{A}} k^m.$$

- Dim $n-(n-r)$
- Space of such $\cong k^{rm}$.
$\Longrightarrow$ Fibers of p_2 are $\cong \mathbb{P}^{rm-1}$.
$\Longrightarrow$ I is irreducible and dim $I = (rm-1) + \dim \mathbb{G}(n-r,m) = (rm-1)+(n-r)r$.
- General "fiber" of p_1 is a single pt (dim 0).
(if rk $A = r$, then only $(A, \ker A) \in I$.
$\Rightarrow$ image $p_1(I)$ is irreducible of dim $= (rm-1)+(n-r)r$.

codim$(p_1(I), M) = (mn-1) - (rm-1+rn-r^2) = mn - r - nr + r^2 =$
$(m-r)(n-r)$. □

Define M_r by vanishing of $(r+1) \times (r+1)$ minors.

Lecture 14

Grassmannians

Say we have V a vector space. Want to talk about $\bigwedge^r V$. Say $e_1, ..., e_n$ is a basis of V. Then $\bigwedge^r V$ has as a basis: pick $i_1 < ... < i_s$, then $e_{i_1} \wedge ... \wedge e_{i_s}$. If $\delta \in S_r$, then

$$e_{i_{\delta(1)}} \wedge e_{i_{\delta(2)}} \wedge ... \wedge e_{i_{\delta(r)}} = \text{sign}(\delta)\, e_{i_1} \wedge ... \wedge e_{i_s}.$$

$\sum a_{1i} e_i \wedge \sum a_{2i} e_i \wedge ... \wedge \sum a_{ri} e_i = \det(a_{ij})\, e_{i_1} \wedge ... \wedge e_{i_s}$.

Grassmannians: $G(r,n) = \{r\text{-dim subspaces of } V^n\} = \mathbb{G}(r-1, n-1) =$ space of $\mathbb{P}^{r-1}$ in $\mathbb{P}^{n-1}$.

Plücker embedding

Want to put $G(r,n) \hookrightarrow \mathbb{P}(\bigwedge^r V) \cong \mathbb{P}^{\binom{n}{r}-1}$. If V has dimension n, $\bigwedge^r V$ has dimension $\binom{r}{n}$. If $W^r \subset V$, choose basis for W: $v_1, ..., v_r$. Then send $w \mapsto v_1 \wedge ... \wedge v_r$. Choosing a different basis leads to the same point in $\mathbb{P}(\bigwedge^r V)$ so the map is well defined.

Say we are looking at $e_1 \wedge e_2 + e_3 \wedge e_4$ (in $\bigwedge^2 V$). In general, you can not write it as $v_1 \wedge v_2$.

Remark The Plücker embeding is injective and the image is characterized by those elements in $\bigwedge^r V$ that are completely decomposable.

Take $v \in \bigwedge^r V$ with $w \,|\, v\, (w \in V)$. If $w \wedge v = 0$, then we can write $v = w \wedge v'$, where $v' \in \bigwedge^r V$. Take $u \in V^*$ (the dual). Then we can extend $\bigwedge^r V \overset{\perp}{\rightarrow} \bigwedge^{r-1} V$, with $u(e_{i_1} \wedge ... \wedge e_{i_r}) = \sum (-1)^{j-1} u(e_{ij}) = e_{i_1} \wedge ... \wedge e_{i_{(j-1)}} \wedge e_{i_{(j+1)}} \wedge .... \wedge e_{1_r}$. Then $u_1 \perp (u_2 \perp ...(u_{r-1} \perp x)) \wedge x = 0 \Leftrightarrow x \in \bigwedge^r V$ is completely decomposable. So we get the Plücker relations with basis $e_1, ..., e_n$ and dual basis $e_1^*, ..., e_n^*$, so we can choose $p_{i_1,..,i_r}$ to be the coefficient of $e_{i_1} \wedge ... \wedge e_{i_r}$. So then $\sum (-1)^t$ $p_{i_1,...,i_{r-1}} p_{j_1,...,j_{t-1},j_{t+1},...,j_{r+1}} = 0$. This has to be true for all $i_1, ..., i_{r-1}, j_1, ..., j_{r+1}$. Then if we look at $G(2,4) \equiv \mathbb{G}(1,3)$ (space of lines in $\mathbb{P}^3$). Then $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{24} = 0$ so the Grassmanian of lines in $\mathbb{P}^3$ $(G(2,4))$ is a quadric hypersurface in $\mathbb{P}^5$.

Example: Let $k = \mathbb{C}$ and choose a basis of V. Take $F_i = \text{span}\{e_1, ..., e_n\}$.

Schubert Varieties

Defined in $G(r,n)$. Pick a partition $n - r \geq \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r \geq 0$ and fix a flag $F = 0 \subset F_1 \subset ... \subset F_n = V$. Then

$$\sum_{\lambda_i} F = \; = \{W \in G(r,n) \,|\, \dim(W \cap F_{n-r+i-\lambda_i}) \geq i\}.$$

Example Look at $\mathbb{G}(1,3) = G(2,4)$. Then

$\sum_{1,0} = \{W \in G(2,4) \mid \dim W \cap F_2 \geq 1$ and $\dim W \cap F_4 \geq 2\}$. Then
$F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4$ and we have a subvariety of lines that intersect a fixed line in space.
$\sum_{2,0} = \{W \in G(2,4) \mid \dim W \cap F_1 \geq 1$ and $\dim W \cap F_4 \geq 2\}$ is the set of lines that pass through a fixed point in space.
$\sum_{1,1} = \{W \in G(2,4) \mid \dim W \cap F_2 \geq 1$ and $\dim W \cap F_3 \geq 2\}$ is the set of lines contained in a fixed plane.
$\sum_{2,1} =$?

<u>Theorem</u> (from topology) $H^*(G(r,n), \mathbb{Z})$ (cohomology). The Schubert classes given as an additive basis of this cohomology as λ varies over all the partitions
$n - r \geq \lambda_1 \geq ... \geq \lambda_r \geq 0$.

Lecture 17

Take the homogeneous coordinate ring of a closed algebraic set in $\mathbb{P}^n$,

$$S(X) = k[x_0, ..., x_n]/\mathcal{I}(X)$$

and define the Hilbert function $h_X(m) = \dim S(X)_m$ with $m \in \mathbb{N}$, that is, the codimension of the space of homogeneous polynomials of degree m vanishing on X.

Last time, $h_X(m) = d$ if X was d points in $\mathbb{P}^n$ provided $m \geq d - 1$.

<u>Thm</u> Let $X \subset \mathbb{P}^n$ be a closed algebraic set and let h_X be its Hilbert function. Then $\exists p_X$ a polynomial such that $h_X(m) = p_X(m)$ for $m \gg 0$ and deg $p_X = \dim X$.

<u>Bertini's Theorem</u>

For X^k a general linear space, a set $Y = X \cap \Lambda$, and $\mathcal{I}(Y) = \overline{(\mathcal{I}(X), \mathcal{I}(\Lambda))}$ (saturation).

<u>Definition</u> Let $\mathcal{I} \subset k[x_0, ..., x_n]$. The <u>saturation</u> of of

$$\overline{\mathcal{I}} = \{F \in k[x_0, ..., x_n] \mid F(z_0, ..., z_n)^m \subset \mathcal{I}\}.$$

Notice $\overline{\mathcal{I}}/\mathcal{I}$ is Noetherian is equivalent to saying that $\mathcal{I}$ and $\overline{\mathcal{I}}$ agree after a certain degree.

Proof. (of Bertini's) Let $X \cap \Lambda = Y$ where Y is a collection of points,

$$\Lambda = \{L_1 = ... = L_k = 0\}.$$

Then $\mathcal{I}^0 = \mathcal{I}(X) \subset \mathcal{I}^1 = (\mathcal{I}(X), L_1) \subset \mathcal{I}^2 = (\mathcal{I}(X), L_1, L_2) \subset ... \subset I^{(k)}$. But then $h^\alpha(m) = \dim (S(X)/\mathcal{I}^\alpha)_m$ and $h^k(m) =$ constant if $m \gg 0$. We want to calculate $h^0_X(m)$. Consider the exact sequence $S^{\alpha-1}_{(m-1)} \overset{L_\alpha}{\to} S^{\alpha-1}_m \to S^\alpha_m \to 0$.

Then $h^\alpha(m) = h^{\alpha-1}(m) - h^{\alpha-1}(m-1)$. So then

$$h^{\alpha-1}(m+k) = c + \sum_{i=m}^{m+k} h^\alpha(i).$$

Hence, by induction, it follows that $h_X^0(m)$ is a polynomial of degree k.
The leading coefficient of $p_X(m)$ will be very important for us. It will define the degree of the variety. In case X is a curve, $p_X(m) = cm + (1-g)$. Then g is called the genus of the curve c.

<u>Example</u> Let c be a plane curve of deg d. Then it has f of degree d with $\mathcal{I} = (f)$ so then g is a homogeneous polynomial of degree m vanishing in $f \mid g$, and dim $S(X)_m$ is the codimension of the space of deg m polynomials divisible by f. If $m \geq d$, $g = fh$ where h is homogeneous of degree $m-d$. Then the dimension of the space of homogeneous polynomials of degree $m-d$ is

$$\binom{m+2}{2} - \binom{m-d+2}{2} = [(m+2)(m+1) - (m-d+2)(m-d+1)]/2 =$$
$$\frac{(m+1)(m+2)-(m+2)(m+1)+d(m+2)+d(m+1)-d^2}{2} = \frac{d(2m+3)-d^2}{2} = dm + \frac{-d^2+3d}{2}.$$

Then $1 - g = \frac{-d^2+3d}{2}$ so that $\frac{(d-1)(d-2)}{2} = g$ (this is called arithmetic genus). Notice $d = 1, 2$ implies $g = 0$ and $d = 3$ implies $g = 1$, and $d = 4$ means $g = 3$. □

If c is smooth over $\mathbb{C}$, then we can consider c as a complex manifold. Up to homeomorphism, any such complex manifold is a sphere with g handles (like a teacup).

<u>Tangent spaces</u>

Start with $X \subset \mathbb{A}^n$, want to define the tangent space at a point $x \in X$. As a first approximation, let $T_x X$ be the union of all the tangent lines to X at x.

Then take $\mathcal{I}(X) = (F_1, ..., F_m)$, say $x = (0, 0, ..., 0)$. Any line passing through x,

$$L_a = \frac{t(a_1,...,a_n)}{\text{fixed}} \; t \in k.$$

Then $F_1(ta_1, ..., ta_n) = F_2(ta_1, ..., ta_n) = ... = F_m(ta_1, ..., ta_n) = 0$ describes. So each of these polynomials are polynomials of one variable, $F_j(ta) = c_j \prod_n (t - \alpha_j)^{i_n}$.
$f_a(t) = \text{hcf}(F_1(ta), ..., F_m(ta))$.

<u>Definition</u> The <u>multiplicity of intersection</u> of L_a with X is the multiplicity with which $(t-a)$ divides $f_a(t)$. If $f_a(t) \equiv 0$, set this mult to $+\infty$.

L_a is tangent to X at x if the mult. of intersection of L_a with X at x is at least 2.

X is a hypersurface, have $F = 0$. Then express $F = L + G$ where L is linear and order $G \geq 2$. Then $F(ta) = L(ta) + G(ta) = tL(a) + G(ta)$ (where deg t is at least 2 in $G(ta)$). The line ta can be tangent to $f = 0 \;\Leftrightarrow\; L(a) = 0$.

$L = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(0)\, x_i$. The tangent space at a point to a hypersurface $L = 0$.

Example If $F = y^2 - x^3$ at $(0,0)$, then $\frac{\partial F}{\partial x} = 3x^2$ at $(0,0)$ both vanish, and at $\frac{\partial F}{\partial y} = 2y$.

Lecture 18

A line is given by $L_{\vec{a}} = t\vec{a}$. Then $f_{\vec{a}}(t) = \text{hcf}(F(t\vec{a})) \Leftrightarrow \text{hcf}(F_1(t\vec{a}), ..., F_m(t\vec{a}))$ where the F_i generate $\mathcal{I}(X)$, $F \in \mathcal{I}(X)$. To say that $L_{\vec{a}}$ has contact of order ≥ 2 means t^2 divides $f_{\vec{a}}(t)$. For F a hypersurface, the Taylor expansion

$$F = L + F_2 + ...$$

$$F(t\vec{a}) = L(t\vec{a}) + F_2(t\vec{a}) + ... = tL(\vec{a}) + t^2 F_2(\vec{a}) + ...$$

$L_{\vec{a}} = t\vec{a}$ has contact of order ≥ 2 if and only if $L(\vec{a}) = 0$.

$L = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(0)\, x_i$. In general, the tangent space $\vec{x} = (t_1, ..., t_n)$.

$\sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(t_i)\,(x_i - t_i)$.

If X is not a hhypersurface, the tangent space is the intersection fo all the linear spaces to a set of generators $F_1, ..., F_m$ of $\mathcal{I}(X)$. The kernel of the matrix

$$\begin{pmatrix} \frac{\partial F}{\partial x_i} \\ \vdots \\ \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\delta F_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix}$$

The local ring of X at a point x, $\mathcal{O}_{X,x} \subset k(X)$. Then $\mathcal{O}_{X,x} :=$ the subring of the function field $f \in k(X)$ such that f is regular at $x \Leftrightarrow$ localization of $k[x]$ at the maximal ideal of the point x. Recall that this maximal ideal is $m_X = \{$the set of regular functions that vanish at $x\}$. e.g., If $A \supset p$ is a prime ideal, then $A_p := \{(f,g) \,|\, f, g \in A,\ g \notin p\}$ (think of it as $\left(\frac{f}{g}\right)$). But of course $\frac{f}{g} = \frac{f'}{g'}$ if $\exists h \notin p$ s.t. $h(f'g - fg') = 0$. Add and multiply:

$$(f,g) \cdot (f', g') = (ff', gg')$$

$$(f,g) + (f', g') = (fg' + gf', gg')$$

The latter comes from $\frac{f}{g} + \frac{f'}{g'} = \frac{fg' + gf'}{gg'}$.

Differential $d_x F = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(t_i)\,(x_i - t_i)$. Usual properties exist: $d_x(F+G) = d_x F + d_x G$, and $d_x(FG) = G d_x F + F d_x G$. Then $T_x X = \{d_x F_1 = ... = d_x F_m = 0\}$, with $\mathcal{I}(X) = \{F_1, ..., F_m\}$.

Now suppose I have an arbitrary regular function $g \in k[x]$. Say G is a polynomial in $k[x_1, ..., x_n]$ such that $G|_x = g$. Then $d_x g = d_x G$. But this is not well-defined (because G is not uniquely determined, only up to $\mathcal{I}(X)$). Then

$$G + A_1 F_1 + ... + A_m F_m = d_x G + \sum (F_i\, d_x A_i + A_i d_x F_i).$$

Restrict this to the tangent space. Then $d_x g = d_x G|_{T_x X}$ is well-defined.

Note $d_x \alpha = 0$ $(\alpha \in k)$. Hence if we change g by a constant value, then we do not change $d_x g$. Let's assume that $g \in m_x$. Then $d_x : m_{X,x} \to T_x^* X$.

Theorem. The map $d_x : m_{X,x}/m_{X,x}^2 \to T_x^* X$ is an isomorphism. [as in diff. manifolds!]

Proof. Surjectivity is clear, because any linear functional on the tangent space is. Now we just need ot look at the kernel. Any linear form on $T_x X$ is induced by some linear functional:

$$d_x F = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(t_i)\,(x_i - t_i)$$

The kernel $d_x g = 0$ for g induced by some G, $d_x G = \lambda_1 d_x F_1 + ... + \lambda_m d_x F_m$. Then define $G_1 = G - \sum \lambda_i F_i$, so that $G_1|_x = g$. Then Taylor expansion of G_1 has no constant (none in G) or linear terms (cancelled out by each $\lambda_i F_i$) , so $G_1 \in (x_1, ..., x_n)^2$. So $g \in m_{X,x}^2$. Hence this is an isomorphism. ☐

Corollary. $T_x X$ is the space of linear functionals on m_x/m_x^2.

Corollary. Under an isomorphism, the tangent spaces of the corresponding points are isomorphic.

When $X \subset \mathbb{P}^n$ is a quasiprojective variety, $x \subset X$ is a point. Choose affine neighborhood $x \in \mathbb{A}^n$. Do the same count. The closure in $\mathbb{P}^n$ does not depend on choice of affine neighborhood.

Projective tangent space

$$\sum_{i=0}^{n} \frac{\partial F_\alpha}{\partial J_i}(x)\, J_i = 0. \qquad \mathbb{A}^n \times X \supseteq \{(a,x) \,|\, a \in T_x X\,\}.$$

Look at the second projection π_2 to X of $\mathbb{A}^n \times X$. By the theorem on the dimension of fibers, there is a minimal s such that all fibers of π_2 have dimension $\geq s$.

Definition. A point $x \in X$ is non-singular if dim $\pi_2^{-1}(x) = s$. Otherwise it's called singular.

Theorem. The dim $T_x X - \pi_2^{-1}(x) =$ dim X if x is non-singular.

Lecture 19

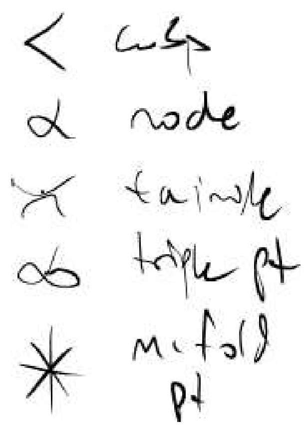
Theorem. If X is a variety, the set of singular points in X is a proper closed subvariety (possibly empty). At a non-singular point $x \in X$, dim $T_x X =$ dim X. (in general, dim $T_x X =$ dim X)

Example. $\mathbb{A}^2 \times \mathbb{P}^1 \supset \{xv = uy\} \xrightarrow{\pi_1} \mathbb{A}^2 \ni 0$. If one of x or $y \neq 0$, this is a birational map (a regular map on a Zariski open set; birational because its inverse is rational).

If X and Y are varieties, $\varphi^* : k(Y) \to k(X)$ is birational if and only if $k(Y) \overset{\varphi}{\xrightarrow{\sim}} k(X)$.

Let X be a variety of dim n. Then tr $\deg(k(X)) = n$. Then $k(x) = k(x_1, ..., x_n, x_{n+1})$, and x_{n+1} can written as a polynomial with coefficients in $k[x_1, ..., x_n]$.

Two isomorphic birational varieties have isomorphic Zariski open subsets.



Cusps have local equation $x^2 = y^3$; nodes have xy; tacnodes have $x^2 = y^4$; triple points have $x^3 - y^3$; m-fold points have $x^m - y^m$. We can associate a finer invariant:

Definition. Given $F = 0$, it is possible to write taylor series expansion $F = F_k + F_{k+1} + ...$ for $k \geq 2$. Then set $F_k = 0$. This is called the <u>tangent cone</u>.

Definition. $u_1, ..., u_n \in \mathcal{O}_x$ are local parameters if $u_i \in m_x$ and $u_1, ..., u_n$ give a basis of m_x/m_x^2.

Notice $du_1 = du_2 = ... = du_n = 0$. The onle solution of this set of equations is 0. $X_i = X \cap (u_i = 0)$, so $T_x X_i = T_x X \cap (du_1 = 0)$

Theorem. If $u_1, ..., u_n$ are local parameters at x, $X_i = X \cap (u_i = 0)$ is non-snigular at x, $\bigcup T_\alpha X_i = 0$.

Definition. $Y_1, ..., Y_r$ non-singular in X are transversal at $x \in \bigcup Y_i$ if

$$\mathrm{codim}_{T_x X} \left(\bigcup_{i=1}^r T_x Y_i\right) = \sum_{i=1}^r \mathrm{codim}_X Y_i.$$

Definition. A formal power series Φ is called a Taylor series for $f \in \mathcal{O}_x$ if $f - S_k \Phi$ (the kth partial sum of $\Phi = F_0 + ... + F_k$) lies in m_x^{k+1}.

Theorem. Every $f \in \mathcal{O}_x$ has a Taylor expansion.

Theorem. If $x \in X$ is non-singular, then a function has a unique Taylor series.

Lecture 20

From last time, we have a local system of parameters, $u_1, ..., u_n \in m_x \subset \mathcal{O}_x$. Then there exists a formal power series expansion in the local parameters.

Theorem. If $x \in X$ is non-singular, then a function has a unique Taylor series.

Proof. It suffices to show $f = 0$ has the zero expansion $u_1, ..., u_n$ with local parameters at x. Then $F_k(u_1, ..., u_n) \in m_x^{k+1} \implies F_k = 0$. Suppose it isn't 0. Then by a linear change, we can assume coefficient reference T_{nb}^k is non-zero. Then

$$\begin{aligned} F_k(T_1, ..., T_n) &= \alpha T_n^k + G_1(T_1, ..., T_{k-1})T_k^{k-1} + ... + G_k(T_1, ..., T_{k-1}) \\ &= \alpha u_n^k + G_1(u_1, ..., u_{n-1})u_n^{k-1} + ... + G_k(u_1, ..., u_{n-1}). \\ F_k(u_1, ..., u_n) &= \mu u_n^k + H_1(u_1, ..., u_{n-1})u_n^{k-1} + ... + H_k(u_1, ..., u_{n-1}). \end{aligned}$$

This says any form in m_x^{k+1} can be written as a polynomial of degree k in $u_1, ..., u_n$ with coefficients in m_x. Then $(u - \alpha)^k u_n^k \in (u_1, ..., u_{n-1})$. We cannot have $\mu - \alpha \notin m_x$ so then $(\mu - \alpha)^{-1} \in \mathcal{O}_x$ so $u_n^k \in (u_1, ..., u_{n-1})$. Then notice $T_x X_n \supset T_x X_1 \cap ... \cap T_x X_{n-1}$ and $X_i = (u_i = 0) \cap X$. But that's a contradiction. Since $u_1, ..., u_n$ are local systems of parameters, $du_1 = ... = du_n = 0$ has only 0 as a solution. Then if X is a variety, x is a n-n singular point of x implies $\mathcal{O}_x \hookrightarrow k[[T_1, ..., T_n]]$ as an inclusion of unique Taylor series expansion. ☐

Corollary. If $x \in X$ is non-singular, then there exists a unique component of X passing through x.

Reason: $k[[T]]$ has no zero-divisors.

In other words, a smooth and connected algebraic set is irreducible. If $X^r \subset \mathbb{A}^n$ and $T_x X$ is a matrix of the form $(\partial f_i / \partial x_j)$, and x is smooth if this matrix has rank $n - r$.

Look at Sard/Bertini's Theorem.

Definition. $f_1, ..., f_n \in \mathcal{O}_x$ are <u>local equations</u> for $x \in Y \subset X$ such a neighorhood of x if there is an affine neighborhood X of x with $f_1, ..., f_m \in k[x']$ and $y' = y \cap x'$ and $I(y') = (f_1, ..., f_m)$ in $k[x']$.

Definition. An irreducible variety $Y \subset X^1$ of codim 1 has a local equation in a neighborhood of a nonsingular point of $x \in X$.

Lecture 21

Let $f_1, ..., f_m \in \mathcal{O}_{x,X}$. Having local equations for $Y \subset X$ means if $\exists$affine neighborhood $X' \subset X$ with $x \in X'$ s.t. $f_1, ..., f_m \in k[X']$ and $I(Y' = Y \cap X') = (f_1, ..., f_m)$ in $k[x']$.

Theorem. If $x \in X$ is nonsingular with $x \in Y \subset X$ an irreducible subvariety of codimension 1, then Y has a local equation at x.

Theorem. If X is nonsingular, and say $\varphi : X \to \mathbb{P}^n$ is a rational map. Then the set of points $\{x \in X \mid \varphi \text{ is not regular at } x\}$ has codimension ≥ 2.

Proof. Let $\varphi : (f_0 : ... : f_n)$. Then this is not well defined when $f_0 = ... = f_n = 0$. If $g|f_i$ for all i then $f_i = gh_i$. Suppose there exists a codimension one component of the locus where $f_i = 0$ for all i. That codimension basis is defined by a local equation around any x. ☐

Corollary. Any rational map of a nonsingular curve to $\mathbb{P}^n$ (projective space) is regular.

Corollary. If two *nonsingular* projective curves are birational, then they are isomorphic.

Remark. $\mathbb{A}^1 \to y^2 = x^3$ is birational but not an isomorphism because the latter is nonsingular ($(0,0)$ is singular on $y^2 = x^3$).

Theorem. Let X be an affine variety and $x \in X$ a nonsingular point. Let $u_1, ..., u_n$ be regular functions on X that form a system of local parameters at x. Then for $m \leq n$, the closed subset defined by $u_1 = ... = u_m = 0$ is nonsingular at x and $I_y = (u_1, ..., u_m)$ in some neighborhood of x. Moreover $u_{m+1}, ..., u_n$ give a system of local parameters at x for Y.

Proof. Induction on m. By previous theorem for $m = 1$, since Y has codimension 1, Y has a local equation. Say $I_y = (f)$ in a neighborhood of x. Write $u_1 = gf$ since u_1 vanishes on Y. Then $du_1 = g(x)\, d_x f$. So$u_1, ..., u_n$ is a system of local parameters at x for X. Note $g(x) \neq 0$. So if x is a nonsingular point on Y, $T_x = T_x X \cap d_x u_1 = 0$. For T_x^*, $du_1, ..., du_n$ give a basis and $du_2, ..., du_n$ give basis for $T_x Y$. ☐

Theorem. If X is a variety $Y^m \subset X^n$ a subvariety, and $x \in Y \subset X$ with x a nonsingular point of Y and X, then there is a locla system of parameters $u_1, ..., u_m$ at x and an affine neighborhood $X \supset U \ni x$ such that $I_{Y \cap U} = (u_1 : ... : u_m)$ in U.

Resolution of singularities

Given X a singular variety, can we find a model of X which is nonsingular. Furthermore, $\exists$? a nonsingular birational moprhism to X, etc. You can ask for more, for instance, φ to be an isomoprhism between $Y - \varphi^{-1}\left(X^{\text{sing}}\right) \to X - X^{\text{sing}}$. You can even reuqire that φ is a simple, easily understood birational morphism.

Theorem. (Hironaka '64) char $k = 0$. Wishes for the conditions in the previous paragraph to be realized.

Normal varieties

R is integrally closed if every element $v \in FF(R)$ (function field) which is integral over R is contained in R. An irreducible affine variety X is normal if $k[X]$ is integrally closed. A quasiprojective variety is normal if every pt $x \in X$ has an affine neighborhood which is normal.

Example. We know $y^2 = x^3$ is not normal. We also know $(y/x)^2 - x = 0$ and y/x is integral $k[x,y]/(y^2 - x^3)$ but not in this ring.

Example. Quadric cone $x^2 + y^2 + z^2 \subset \mathbb{A}^3$ is singular at $(0,0,0)$, but it is normal.

Lecture 22 (Chapter II.5 in Shafarevich)

R is integrally closed if every elements of its fraction field which is integral over R is contained in R. An affine variety X is normal if $k[x]$ is integrally closed.

An affine variety X is normal if $k[x]$ is integrally closed. A quasiprojective variety X is normal if every point has a normal affine neighborhood.

Notice $y^2 = x^3 + x^3 \subset \mathbb{A}^2$. This is <u>not normal</u> so $y/x \notin k[c]$ even though $x \in k[c]$. Notice $(y/x)^2 - (1+x) = 0$.

In $\mathbb{A}^3$, we can look at $x^2 + y^2 = z^2$. This is certainly singular at $(0,0,0)$. We can write every function in $k[\mathbb{Q}]$. Then we can write it as $u + vz$ where $u, v \in k[x,y]$.

More generally, $\varphi \in k[\mathbb{Q}]$ means we can write $\varphi = u + vz$ with $u, v \in k(x,y)$. Suppose $u + vz$ is integral over $k[\mathbb{Q}]$. Furthermore, suppose $u + vz$ is also integral over $k[x,y]$. Then write the minimal polynomial $T^2 - 2uT + u^2 - (x^2+y^2)v^2$. Then $2u \in k[x,y]$. Hence $u \in k[x,y]$. But then $(x^2+y^2)v^2 \in k[x,y]$ since it means the T^0 term is in $k[x,y]$. Then we can write $(x+iy)(x-iy)v^2$. These are irreducible, so $v \in k[x,y]$. Hence $\varphi \in k[\mathbb{Q}]$.

Lemma. If X is normal, then the local ring $\mathcal{O}_Y$ (localization of $k[x]$ along $I(Y)$) at any irreducible variety $Y \subset X$ is integrally closed. In particular, $\mathcal{O}_x$ is integrally closed $\forall x \in X$.

Proof. Let $\alpha \in k(X)$ which is integral over $\mathcal{O}_Y$. Then $\alpha^n + a\alpha^{n-1} + ... + a_n = 0$ where each $a_i \in \mathcal{O}_Y$. But the latter means we can write $a_i = b_i/c_i$ where $b_i, c_i \in k[x]$ but $c_i \notin I(Y)$. Then define $d = c_1c_2...c_n \in k[x]$ but not in $I(Y)$ (because it's a prime ideal--can't have product be in $I(Y)$ without one of the terms being in it). Then $d\alpha^n + d_1\alpha^{n-1} + ... + d_n = 0$ where $d_i = (d/c_i)b_i$. Then multiply by d^{n-1}:

$$(d\alpha)^n + d_1'(d\alpha)^{n-1} + ... + d_n' = 0.$$

So that $d\alpha$ is clearly integral over k$[X]$. Since $k[X]$ is integrally closed, $d\alpha \in k[X]$. Consider the element $d\alpha/d$. Since $d\alpha, d \in k[X]$ but $d \notin I(Y)$, we have $d\alpha/d = \alpha \in \mathcal{O}_Y$ as desired. So $\mathcal{O}_Y$ is integrally closed. ☐

Lemma. If X is an irreducible affine variety and $\forall x \in X$ points, $\mathcal{O}_x$ is integrally closed, then X is normal.

Proof. Let $\alpha \in k(X)$ which is integral over $k[x]$. In particular, α is integral over $\mathcal{O}_x$ for all $x \in X$. So then $\alpha \in \bigcap_{x \in X} \mathcal{O}_x = k[X]$. Hence X is normal. □

Theorem. A non-singular variety is normal.

Proof. If $x \in X$ is non-singular, then $\mathcal{O}_x$ is a UFD. UFD's are integrally closed†. But since $\mathcal{O}_x$ is integrally closed for all $x \in X$, X itself must be normal.

Theorem. If X is normal and $Y \subset X$ is a codimension 1 subvarity, then $\exists$ an affine subset $X' \subset X$ such that $X' \cap Y \neq \emptyset$ and $Y' = X' \cap Y$ and $k[X']$ is principal.

Proof. Can assume X is affine. It's enough to show that $m_Y = (u)$ with $u \in \mathcal{O}_Y$ (this is the maximal ideal of $\mathcal{O}_Y$, the localization of $k[X]$ at $I(Y)$). Suppose $m_Y = (u)$ with $u = \frac{a}{b}$ and $a, b \in k[x]$ but $b \notin I(Y)$. Suppose $I(Y) = (v_1, ..., v_m)$. Then $I(Y) \subset m_Y$. So $v_i = uw_i$ with $w_i \in \mathcal{O}_Y$. Then $w_i = c_i / d_i$ with $c_i, d_i \in k[X]$ and $d_i \notin I(Y)$. Let $X' = X - (V(b) \cup V(d_1) \cup ... \cup V(d_n))$. Take $Y' = Y \cap X$. Then $I(Y') = (u)$.

Now we need to show that ... Take $0 \neq f \in k[X]$ and assume $f \in I(Y) \subset \mathcal{O}_Y$. But of course $f \in I(Y)$ means $Y \subset V(f)$ (the zero locus of f) since it vanishes at $I(Y)$, i.e both are codimension 1). Then $V(f) = Y \cup Y'$ and $\varphi \nsubseteq Y'$ (???), then $X_1 = X - Y'$ and $Y \cap X_1 \neq \emptyset$. By restricting to X, we can assume $Y = V(f)$. Using the Nullstelensatz, $I(Y)^k \subset (f)$ in $k[X]$ and $m_Y^k \subset (f)$ in $\mathcal{O}_Y$. Let k be the minimal such integer. Then there exists $\alpha_1, ..., \alpha_{k-1} \in m_Y$ such that $\alpha_1, ..., \alpha_k \notin (f)$, and $\alpha ... \alpha_{k-1} m_Y \in (f)$. Set $g = \alpha_1 ... \alpha_{k-1}$. Then $u = f/g$. We have $u^{-1} \notin \mathcal{O}_Y$ but $u^{-1} m_Y \subset \mathcal{O}_Y$. Then X normal implies $\mathcal{O}_Y$ is integrally closed so $u^{-1} m_Y \subset m_Y$. Since $\mathcal{O}_Y$ is integrally closed. So $u^{-1} m_Y = \mathcal{O}_Y$ and so m_Y is generated by u. □

Some consequences of this theorem:

Theorem. *The set of singular points of a normal variety has codimension ≥ 2.*

Corollary. Normal curves are non-singular.

Lecture 23

Last time, we did Theorem II.5.2 in Shafarevich. Two corollaries hold:

Corollary. The set of singular points of a normal variety has codimension ≥ 2.

Proof. Suppose X is normal with dimension $n = \dim X$. Let $S \subset X^{\text{sing}}$ be a dimension $n-1$ locus in the singular locus. Then let $y \in S$ be a smooth point of S. Then let $S' = S \cap X'$ with X' as in the theorem. Then we can choose a local system of parameters S' at y with $\mathcal{O}_{S', y}$ the local ring of S' at y and $u_1, ..., u_{n-1}$ a system of parameters. Then

---

†*Then* $\alpha^n + u_1 \alpha^{n-1} + ... + u_n = 0$, *where* $\alpha = \frac{u}{v}$ *where* u, v *have no common factors. Hence* $u_n + u_1 v u^{n-1} + v^n = 0$ *and* $v \mid u^n$. *Since* v *has no common factors it is a unit.*

$I(S') = (u)$, so that $\mathcal{O}_{X',y}/(u) = \mathcal{O}_{S',y}$. Notice $\mathfrak{m}_{X',y}$ is the inverse image of $\mathfrak{m}_{S',y}$ under the map natural map $\mathcal{O}_{X',y} \to \mathcal{O}_{S',y}$. So choose arbitrary images $v_1, ..., v_{n-1}$ of the local parameters. Then dim $\mathfrak{m}_{X',y}/m^2_{X',y} \leq n$ so that y is a non-singular point of X.

Corollary. A normal curve is smooth.

Definition. A *normalization* of an irreducible variety X is an irreducible normal variety X^ν so that $\nu : X^\nu \to X$ is defined such that ν is regular, finite, and birational.

Theorem. An affine irreducible variety X has an affine normalization.

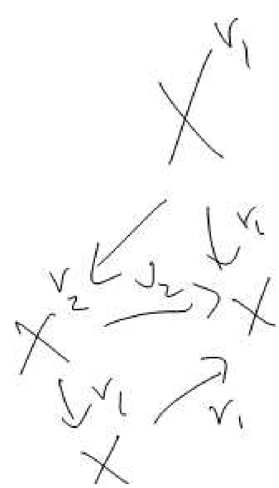
Proof. We know $k[X] \subset k(X)$. Take the integral closure $A = \overline{k[X]}$ in $k(X)$. Then A is a finite module over $k[X]$, i.e., a finitely generated k-algebra with no nilpotents. So let $A = k[Y]$ for Y an affine variety.Then Y is normal and $k[X] \hookrightarrow A$ induces a morhphism $Y \to X$. □

Theorem. (1) Suppose we have a map $g : Y \to X$ that is finite, regular, and birational (for X and Y affine varieties). Then there exists a regular map $h : X^\nu \to Y$ such that the diagram $X^\nu \overset{\nu}{\to} X \overset{g}{\leftarrow} Y \overset{h}{\leftarrow} X^\nu$ is commutative.

(2) If $g : Y \to X$ is regular, $g(Y)$ is dense in X and Y is normal, then there is a regular $h : X^\nu \to Y$ such that the diagram $Y \overset{h}{\to} X^\nu \overset{\nu}{\to} X \overset{g}{\leftarrow} Y$ is commutative.

Corollary. The normalization of an affine variety is unique up to isomorphism.

Proof. Suppose we have two of them X^{ν_1}, X^{ν_2}.. Then we have the diagram



and it is commutative by the theorem so that $X^{\nu_1} \cong X^{\nu_2}$. □

Proof. (of theorem) (1) We have the inclusions $k[X] \subset k[Y] \subset k(X) = k(Y)$ (since they are birational) with $k[Y]$ integral over $k[X]$. Then consider $A = \overline{k[X]}$. Since $k[Y]$ is integral over $k[X]$, $k[Y] \subset A$, so each time you have a ring homomorphism $X^\nu \to Y$. This induces a map between the corresponding affine varieties.

(2) Let $u \in k[X^\nu]$ which is integral over $k[X]$ and contained in $k(X) \subset k(Y)$. But since $k[X] \subset k[Y]$, it must be integral over $k[Y]$. But since Y is normal (so that $k[Y]$ is

integrally closed) $u \in k[Y]$.Thus we have an inclusion $k[X^\nu] \to k[Y]$ which induces a morphism $Y \to X^\nu$. □

Theorem 1. A quasiprojective curve X has a normalization X^ν.

Proof. Let $X = \bigcup U_i$ be a finite, open affine cover of X. By the earlier theorem, let $f_i : U_i^\nu \to U_i$ be the normalization for each U_i. First, notice $\overline{U_i} = X$, and $\overline{U_i^\nu}$ is birational to X. Set $V_j = \overline{U_j^\nu}$. We have a rational map $U_i^\nu \to V_j$ for all i, j. Recall that U_i^ν is normal (in particular it is non-singular), so consider the map $U_i^\nu \to V_j$. Let $W = \prod_j V_j$ and let $\varphi_i = \prod \varphi_{ij} : U_i^\nu \to W$. Then $\varphi_i(u) = (\varphi_{i_1}(u), ...)$. Let $X' = \bigcup \varphi_i(U_i^\nu) \subset W$. We claim that X' is the normalization of X. Consider $U = \bigcap_{i=1}^n U_i$. Then U is a Zariski open dense subset of X. Then $\varphi(U^\nu) \subset \varphi_i(U_i^\nu) \subset \overline{\varphi(U^\nu)}$. Notice that $\overline{\varphi(U^\nu)} - X'$ consists of finitely many points. So then the map $X' \to X$ is finite and birational. But we need that X' is normal. First, notice $\varphi_i : U_i^\nu \to \varphi_i(U_i^\nu)$. Then $(u_1, ..., u_n) \mapsto \varphi_{ii}^{-1}(u_i)$ has an inverse to φ_i. Since U_i^ν is normal, X' is normal. □

Theorem 2. The normalization of a projective curve is projective.

Corollary. Any projective curve is birational to a smooth projective curve.

Lecture 24 - Shafarevich §II.5-6

Proposition. [II.5.4.L] A finite map $f : X \to Y \subset \mathbb{P}^n$ is an isomorphic embedding if and only if f is bijective.

Proof. This follows from Nakayama's lemma. First, note it suffices to assume X and Y are affine. Then we have $f^* : A[Y] \to A[X]$. By Nullstelensatz, since f is a bijection between points, f^* is a bijection between maximal ideals. Since $T_x X = (n/n^2)$, $d_x f$ is injective, so then $m/m^2 \to n/n^2$ is surjective.

Corollary. A bijection between $f : X \to Y$ with injective differential everywhere is an isomorphism.

Theorem. [II.5.4.T1] Let X be a smooth, projective variety of dimension k. Then X admits an embedding to $\mathbb{P}^{2k+1}$.

Corollary. [II.5.4.C1] Let $X \subset \mathbb{P}^n$ be a variety with $p \in \mathbb{P}^n \setminus X$. Suppose every line passing through p either does not intersect X or intersects X at one point transversely. Then $\pi_p : X \to Y \subset \mathbb{P}^{n-1}$ is an isomorphism.

Bertini Theorems [II.6]

Theorem. If X is a quasiprojective variety over k with char $k = 0$, then $f : X \to \mathbb{P}^n$ is a regular map. Let H be a general hyperplane in $\mathbb{P}^n$. Set $Y = f^{-1}(H)$. Then $Y_{\text{sing}} = X_{\text{sing}} \cap f^{-1}(H)$.

Example. For (x, y, z, w, u), note $xy + zw - u^2 \subset \mathbb{P}^4$ with char $k = 2$.

Proof. (of theorem) Consider the universal hyperplane section

$$\Gamma = \{(p, H) \mid f(p) \in H\} \subset X \times (\mathbb{P}^n)^*$$

This is irreducible of dimension dim $X+n-1$. Let $p \in X - X_{\text{sing}}$ (a smooth point of X). Choose a coordinate such that $p=(0,0,...,0,1)$ and $H=(Z_0=0)$. We can write f locally, $[f_0(x), f_1(x), ..., f_{n-1}(x), 1]$. We can write a hyperplane close to H as $Z_0+\alpha_1 Z_1+...+\alpha_n Z_n=0$. This is the equation of Γ. Then $f_0+\alpha_1 f_1+...+\alpha_n=0$. Since $\partial F/\partial \alpha_n \neq 0$, Γ is smooth at a point whose projection is a smooth point of x. But $(\mathbb{P}^n)^* \overset{\pi_2}{\leftarrow} \Gamma_{\text{smooth}} \subset \Gamma$. By Sard's Theorem, the general fiber of π_2 is smooth. This concludes the proof.

Corollary. Let $F_1,...,F_k$ be general polynomials of degree $d_1,...,d_k$ in $n+1$ variables. The corresponding hypersurfaces $F_1=...=F_k=0$ intersect transversely. The variety defined by $F_1=...=F_k=0$ is nonsingular of dimension $n-k$.

Furthermore, $I(X)=(F_1,...,F_k)$, and X is called a complete intersection.

Corollary. Let X be a smooth projective variety of dimension k. Let $L_1,...,L_k$ be general linear forms. Then $Y=X\cap\{L_1=...=L_k=0\}$ is smooth and the ideal of Y is generated by $(I(X), L_1, ..., L_k)$.

Remark. This still holds in characteristic p.

Lecture 25

Degree [Shafarevich pg 143−]

Unlike dimension, smoothness, etc. degree is extrinsic not intrinsic.

Suppose you have a finite map $f: X^n \to X^y$ with $k(Y) \hookrightarrow k(X)$ a finite field extension. Then you can define deg $f=[k(X):k(Y)]$. A notion over $\mathbb{C}$ of degree of a map $X^n \xrightarrow{f} Y^n$ count # of inverse images $f^{-1}(y)$.

Theorem. If $f: X\to Y$ is a finite map between irreducible varieties, and Y is normal, then the number of points # $f^{-1}(y) \leq$ deg f.

Proof. If X, Y are affine, then $k[X]$ is an integral extension of $k[Y]$, and Y is normal so that $k[Y]$ is integrally closed. Let $f^{-1}=\{x_1,...,x_m\}$. Take $a\in k[X]$ to be such that $a(x_i)=0\,\forall i=1,...,m$. Then write the minimal polynomial of a over $k[Y]$. If $F=F^N-\alpha_1 T^{N-1}+...+\alpha_N$, then $\#m \leq N$.

Ramification f is unramified over y is $\#f^{-1}(y)=$ deg f. Otherwise, f is ramified at y.

Theorem. The set of ramification points of a map f is open and non-empty if $f^*(k(Y)) \hookrightarrow k(X)$ is separable.

Proof. Take a generating element and look at its minimal polynomial F. Let deg $f=n$. Then $T^n+\alpha_{n-1}T^{n-1}+..+\alpha_0$ has the property that at each point y you get a polynomial. So then $p=T^n+\alpha_{n-1}T^{n-1}+..+\alpha_0$. To say f is unramified means p evaluated at y has no double roots. $D(p)=0 \Leftrightarrow$ ramification points. ☐

Remark: Since $X\subset\mathbb{P}^n$ is a hypersurface, X is defined by a single polynomial, so we can think of deg $X=$ deg F.

Degree: Let $X \subset \mathbb{P}^n$ be an irreducible (possibly quasiprojective) variety of dimension k. Then the degree of X is defined by any of the following ways:

(1) The projection from a general linear space of dimension $n-k-1$ gives a finite surjective map $\pi : X \longrightarrow \mathbb{P}^k$ with $\deg(X) = \deg \pi = \deg[k(X) : k(\mathbb{P}^n)]$.

(2) The general projection from $X \rightarrow \mathbb{P}^{k+1}$ gives a birational map from X to the image in $\mathbb{P}^{k+1}$, $\pi : X \rightarrow Y \subset \mathbb{P}^{k+1}$ with $\deg X = \deg Y =$ deg of the polynomial defining Y.

(3) A general linear space of dimension $n-k$ will intersect X in finitely many points by the Bertini Theorem, so we can define deg $X =$ # pts in $X \cap \Lambda$ where Λ is a general linear space of dim $n-k$.

(4) Consider the Hilbert polynomial $p_X(m)$ of X. Then $\deg X = k!$, the leading coefficient of $p_X(m)$.

[...rest of lecture not understandable, didn't bother taking notes...]

Lecture 26

For a projective variety $X^k \subset \mathbb{P}^k$

(1) $X \xrightarrow{\pi} \mathbb{P}^k$ of deg π (2) $x \xrightarrow{\pi} \mathbb{P}^{k+1}$ of deg hypersurface

(3) General $n-k-1$ plane, # of int points $X \cap \Lambda$

(4) Hilbert polynomial of deg $k!$, the leading coeff

Examples. (1) <u>Veronese varieties</u> Take $v_d(\mathbb{P}^n) \hookrightarrow \mathbb{P}^{\binom{n+d}{d}-1}$. Then deg $v_d(\mathbb{P}^n) =$?

(2) <u>Hilbert polynomial</u> Consider a polynomial of deg m in $\binom{n+d}{d}$ variables. If we restrict $v_d(\mathbb{P}^n)$ to a polynomial in $n+1$ variables of deg md, we have Hilbert polynomial $\binom{md+n}{n} = (md+n)...(md+1)\,/\,n! = \frac{d^n m^d}{n!} +$ l.o.t. in (m). Then deg$ree = d^n$.

Remarks: In particular, rational normal curve of degree d has really degree d. The Veronese surface $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ has degree 4.

Then the $\left(\binom{n+d}{d} - 1 - n\right)$-plane $L_1 = ... = L_{\binom{n+d}{d}-1-n} = 0$ for $L_i \cap v_d(\mathbb{P}^n)$ gives a hypersurface of degree d in $\mathbb{P}^n$. In how many points do n general hypersurfaces of deg d intersect? d^n. For each of the hypersurfaces of deg d you can take $L_{i1} = ... = L_{id} = 0$ (product of linear forms).

Examples. (1) <u>Segre varieties</u> Have $\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{(n+1)(m+1)-1}$, what is the degree? Have a Hilbert polynomial. A polynomial of deg m in $(n+1)(m+1)$ variables induces a homogeneous polynomial of bidegree (k,k) in $(n+1)$ and $(m+1)$ variables. Then the Hilbert polynomial is given by $p_x(k) = \binom{k+n}{n}\binom{k+m}{m} = \frac{(k+n)...(k+1)}{n!}\,\frac{(k+m)...(k+1)}{m!} = \frac{k^{n+m}}{n!m!} +$ l.o.t.(k). Then degree $\frac{(n+m)!}{n!m!} = \binom{n+m}{n}$. If $n = m = 1$, then we have indeed a quadric surface in $\mathbb{P}^3$.

Bezout's Theorem. Let X, Y be closed sets of $\mathbb{P}^n$ of pure dimension k and l (with $k+l \geq n$). Then X and Y intersect naturally: deg $X \cap Y =$ deg $X \cdot$ deg Y. In particular, $k+l=n$ means X and Y intersect at deg X $\cdot$ deg Y points. ☐

Suppose X and Y intersect properly (dim $X \cap Y = k+l-n$). Given an irreducible component $Z \subset X \cap Y$, one can associate an intersection multiplicity $m_Z(X,Y)$ of X and Y along Z.

Bezout's Theorem (general). If X and Y are closed subsets of pure dimension intersecting properly, then deg$(X) \cdot$ deg$(Y) = \sum_{Z \subset X, Y \text{ irred}} m_Z(X,Y) \cdot$ deg(Z).

Properties of $m_Z(X,Y)$: (1) $m_Z(X,Y) = m_Z(Y,X)$, (2) $\mathbb{Z} \ni m_Z(X,Y) \geq 1 \Leftrightarrow Z \subset X \cap Y$, (3) $m_Z(X,Y) = 1$ if X and Y intersect transversally at general points of $\mathbb{Z}$. (4) $m_Z(X \cup X', Y) = m_Z(X,Y) + m_Z(X',Y)$ if X and X' have no common components, and $X \cup X'$ include Y properly.

Corollary. If X and Y are closed subsets of $\mathbb{P}^n$ intersecting properly of pure dimension intersecting properly, then the deg $X \cap Y \leq$ deg X $\cdot$ deg Y.

Corollary. Suppose $X, Y \subset \mathbb{P}^n$ are subvarieties intersecting properly and deg $X \cap Y =$ deg $X \cdot$ deg Y. Then X and Y are smooth at general points of $X \cap Y$.

Corollary. Suppose $X^k \subset \mathbb{P}^n$ is a variety of degree 1. Then X is a linear space of dimension k.

Proof. (sketch) We can do this by induction on k. If $k=1$, pick two points $p_1, p_2 \in X$ and look at all the hyperplanes containing p_1, p_2 then the int cannot be proper, so every $H \ni p_1, p_2$ has to contain X. But the hyperplanes containing p_1, p_2 generate the ideal of the line containing $p_1$and p_2. X is the line spanned by p_1 and p_2. Keep going for $k=2$. Pick three points on X that are not collinear. Consider hyperplanes containing p_1, p_2, p_3. By the case $k=1$, the int $H \cap X$ cannot be proper to $X \subset H$. Etc. ☐

The Picard Group

Let X be an irreducible variety. A prime divisor on X is an irreducible codimension 1 subvariety of X. Then the divisor of X, Div(X), is the free abelian group generated by prime divisors $D \in$ Div(X). Then $D = \sum_{i=1}^k c_i D_i$ where c_i and D_i are prime divisors on X. Let $f \in k(X)$. Take D to be a prime divisor. Each prime divisor D determines a valuation on $k(X)$ provided X is nonsingular in codimension 1. *Assumption:* X is nonsingular in codimension 1.

The valuation is the order of the zero or pole of f along D. Pick open set $U \subset X$ such that $X - X^{\text{sing}}$ and $D \cap U \neq \emptyset$. Since U consists of nonsingular points, D is defined by a local equation around each point $x \in U$. Let π be the local equation of D. Then $f \in k[X]$. So $\exists k$ such that $f \in (\pi^k)$, but $f \notin (\pi^{k+1})$ so $v_D(f) = k$.

Lecture 27

X irreducible variety nonsingular in codimension 1

A prime divisor D is an irreducible codimension 1 subvariety of X.

Div X - free abelian group generated on prime divisor

$D = \sum_{i=1}^{N} k_i\, D_i$ for $k_i \in \mathbb{Z}$.

Let $f \in k(X)$ ($f \neq 0$) and let D be a prime divisor. Then we can define (a valuation) $v_D(f)$ ("the order of zero or pole of f along D"). Take U open intersecting D and consisting only of nonsingular points of X. Possibly after shrinking U, we can say D has a local equation in U with $\pi = 0$. First assume $f \in k[X]$. Then there exists say m such that $f \in (\pi^m)$ (π divides f) but $f \notin (\pi^{m+1})$. Then define $v_D(f) = m$.

Observe that $v_D(f_1 f_2) = v_D(f_1) + v_D(f_2)$ with $v_D(f_1 + f_2) \geq \min\{v_D(f_1), v_D(f_2)\}$, assuming of course $f_1 + f_2 \neq 0$. So now suppose that $f \in k(X)$. Then write $f = g/h$ where $g, h \in k[X]$. Then we can define $v_D(f) = v_D(g) - v_D(h)$. Then

(1) H does not depend on the representation of f,

(2) It does not depend on the choice of U: if $V \subset U$ is open then π is a local equation of D also in V. Take $W \cap V$ and again that it's well-defined.

Notice it does not make sense to talk about $v_D(f)$ at a point, only at a divisor.

Terminology If $v_D(f) = k > 0$, we say that f has a zero of order k along D. Similarly, if $v_D(f) = -k < 0$, then we say f has a pole of order k along D.

It's important to note these only make sense for codimension 1 subvarieties.

Given $f \in k(X)$, there are finitely many prime divisors D such that $v_D(f) \neq 0$. If X is affine and $f \in k[X]$, then if D is not a component of $V(f)$, then $v_D(f) = 0$. But there are only finitely many components of V. If $f \in k(X)$, express $f = g/h$ with $g, h \in k[X]$. Then $v_D(f) = 0$ unless D is a component of $V(g)$ or $V(h)$.

If X is a quasiprojective cover X by finitely many affines, then since in each piece there exist finitely many D with $v_D(f) \neq 0$, it follows $\exists$ finitely many D such that $v_D(f) \neq 0$. So given a rational function $f \neq 0 \in k(X)$, we can associate a divisor to it,

$$\text{div } f = \sum_D v_D(f) D$$

Definition. The divisor of $f \neq 0 \in k(X)$ is called a principal divisor.

div $f = \sum k_i Di$. The divisor of zeroes of f, $\text{div}_0\, f = \sum_{k_i > 0} k_i D_i$. The divisor of poles of f, $\text{div}_\infty(f) = \sum_{k<0} k_i D_i$.

(1) div $(f_1 \cdot f_2) =$ div $(f_1) +$ div(f_2). If $f \in k$, div$(f) = 0$. If $f \in k[X]$, div$(f) \geq 0$ (the divisor is effective).

Definition. A divisor $\sum k_i D_i$ is called *effective* if $k_i \geq 0\ \forall i$. We write $D \geq 0$ to mean that D is effective.

Proposition. Suppose X is irreducible and nonsingular. If $f \neq 0 \in k(X)$ and if $\mathrm{div}(f) \geq 0$, then $f \in k[X]$. In particular, if in addition X is projective and div $f \geq 0$, then $f \in k$.

Proof. Suppose f is not regular at a point $x \in X$. Express $f = g/h$ where $g, h \in \mathcal{O}_x$. Since X is nonsingular, $\mathcal{O}_x$ is a UFD. We can assume that g, h have no common factor. Suppose π is irreducible, $\pi \mid h$ but $\pi \nmid g$. In some neighborhood, $V(\pi)$ is irreducible and of codimension 1, say D, so $v_D(f) < 0$. Hence div (f) is not effective.☐

Corollary. In a nonsingular projective variety, a rational function f is determined up to a constant by its divisor.

If div $f =$ div g, then div $f/g = 0$, so by proposition $f/g = c \in k$.

Principal divisors form a subgroup of Div(X). The quotient is the class group Cl$(X) =$ Div$(X)/P(X)$ (divisors modded out by principal divisors). This is an important invariant of a variety.

Two divisors are called linearly equivalent if Div$(D_1) -$ Div$(D_2) =$ div(f) (is prinicipal).

Example 1. Start with $\mathbb{A}^n$. What is the class group of $\mathbb{A}^n$, Cl$(\mathbb{A}^n)$? It is 0 because on $\mathbb{A}^n$ every codimension 1 subvariety is defined by a single equation and so is a principal divisor:

For $\sum_{i=1}^{m} k_i D_i$, say $D_i = (F_i = 0)$, $D =$ div $(F_1^k ... F_m^k)$.

Example 2. Cl$(\mathbb{P}^n) = \mathbb{Z}$. Given a prime divisor D, we can define D as the zero locus of a single homogeneous equation: $f = F/G$ with F, G homogenous of the same degree. Define a homomorphism deg : (Div$(\mathbb{P}^n)) \to \mathbb{Z}$ where $\sum k_i D_i \mapsto \sum k_i$ deg D_i. This is certainly onto. $kH \mapsto k$ (for H a hyperplane), so the kernel is precisely the principal divisors. The kernel is precisely the prinicipal divisors $\sum k_i$ deg $D_i = 0$ with $D = \sum k_i D_i$. Split it into 2 pieces, so

$$D_0 = \sum_{k_i > 0} k_i D_i \text{ and } D_\infty = \sum_{k_i < 0} k_i D_i.$$

Each D_i is defined by homogenous polynomials of degree D_i, so we have

$$\prod_{i \in D_0} F_i^{k_i} \Big/ \prod_{i \in D_\infty} F_i^{k_i}$$

where the numerator and denominator have the same degree, and are in $k(\mathbb{P}^n)$.

Example 3. Cl$(\mathbb{P}^{n_1} \times ... \times \mathbb{P}^{n_r}) \cong \mathbb{Z}^r$ by a similar argument.

Lecture 28

Locally principal divisors

If X is a nonsingular variety, then every prime divisor $D \subset X$ around any point $x \in D$ can be defined by a local equation.

If $U \ni x$, D is generated by one function. Suppose you have U_i and U_j, and we define U_i by f_i and U_j by f_j. Then we have $\mathrm{div}(f_i) = \mathrm{div}(f_j)$. What this means is if I look at f_i/f_j, then is regular on $U_i \cap U_j$ and it is everywhere non-zero.

Definition. Let $\{U_i\}$ be an open cover of X, and let $\{f_i\}$ be a *compatible system* of functions corresponding to the open covering $\{U_i\}$. Then f_i/f_j is a regular function on $U_i \cap U_j$ which is nowhere zero.

Any compatible system of funcitons defines a divisor $\sum k_i D_i$. Take an open set U_i such that $U_i \cap D_i \neq \emptyset$. Then $k_i = v_{D_i}(f_i)$. This is well defined if $U_j \cap D_i \neq \emptyset$.

Two systems of compatible functions $\{f_i, U_i\}$ and $\{g_j, V_j\}$ define the same divisor if and only if f_i/g_j is regular and nowhere zero.

Now let $\varphi : X \to Y$ be a regular map of nonsingular varieties Let $D \subset Y$ be a prime divisoysteIri4 12 T3536(a)320.5177(w)0.621304(e)3.15789( )-20.(r)22[(o)-0.957028(n)-0.957028(l)-2.53567

$$\text{Pic}(X) = \frac{\text{Cartier divisors}}{\text{Principal divisors.}}$$

Remark: Suppose X is nonsingular. Then $\text{Pic}(X) \cong \text{Cl}(X)$ with $v_D(fg) = v_D(f) + v_D(g)$. Also, $\text{Pic}(\mathbb{P}^n) = \mathbb{Z}$ and $\text{Pic}(\mathbb{P}^{n_1} \times ... \times \mathbb{P}^{n_r}) \cong \mathbb{Z}^r$. How do you think about $\text{Pic}(\mathbb{P}^n)$? If L_1 and L_2 are both linear forms, $f = L_1/L_2$.

Suppose X is a project variety. Then if $\mathbb{P}^1 \to X \subset \mathbb{P}^3 \supset L$ with

$$(u,t) \mapsto (t^4, t^3u, u^3t, u^4)$$

then $L \cap X$ is a divisor.

Two divisors are linearly equivalent if they differ by a principal divisor.

Definition. The *Riemann-Roch space* of a divisor D is $\{f \in k(X)\}$ such that

$$D + \text{dim}(f) \geq 0.$$

This is a vector space.

Lecture 29

Riemann-Roch Spaces

For $\mathbb{P}^1$, how does one characteristic polynomial of degree d, for $f \in k(\mathbb{P}^1)$ such that

$$\text{dim } f + dx_\infty \geq 0.$$

If X is a nonsingular variety, fix a divisor D so that $f \in k(X)$ with div $f + D \geq 0$.

Definition. The *Riemann-Roch space* of D is the space of functions

$$\mathcal{L}(D) = H^0(X, \mathcal{O}_x(D))$$

is the sub-vector space of $k(X)$ such that div $f + D \geq 0$.

This is an important concept in algebraic geometry, and a fundamental problem since the 19th century is:

Problem. Given a divisor D, determine $\mathcal{L}(D)$ (determine the dimension of $\mathcal{L}(D)$).

Remark: If D_1 and D_2 are linearly equivalent then $\ell(D_1) = \ell(D_2)$.

$$D_1 \sim D_2 \Longrightarrow D_1 - D_2 = \text{div } g.$$

If $f \in \mathcal{L}(D_1)$, div $(fg) + D_2 = \text{div}(f) + \text{div}(g) + D_2 \geq 0$, so $g\mathcal{L}(D_1) \subset \mathcal{L}(D_2)$. So multiplication by g gives an isomorphism between the two. You can associate a dimension $\ell(D)$ for any $D \in \text{Cl}(X)$.

Suppose φ is a rational map $\varphi : X \to \mathbb{P}^n$ (assume image of X, $\overline{\varphi(x)}$ is nondegenerate). Consider $(f_0, ..., f_n)$ with $f_i \in k(X)$. Let $D_1, ..., D_m$ be finitely many divisors such that

$$D_i = \sum h_j \, F_{ij} \text{ with } F_{ij} \text{ prime divisors.}$$

Then the highest common divisor $\text{hcd}(D_1, ..., D_m) = \sum_{i,j} \ell_j \, F_{ij}$ where $\ell_j = \min_i\{k_{ij}\}$.

Set $D = \text{hcd}(\text{div}(f_0), ..., \text{div}(f_n))$ with $D'_i = \text{div}(f_i) - D$.

A rational map φ fails to be regular precisely at the points $\bigcap_i \text{supp}(D'_i)$ (the base locus).

Consider the vector space generated by D'_i. Say $X \subset \mathbb{P}^n$ is non-degenerate with $X \hookrightarrow \mathbb{P}^n$. Take the hyperplane H with $X \cap H \subset X$ a divisor. Consider the effective divisors on X, linearly equivalent to $X \cap H = D$. Then there is always a maximal linear algebra called the *complete linear system* $|D|$. All effective divisors are linearly equivalent to $X \cap H$. If $M \subset |D|$, then $\varphi : X \to \mathbb{P}(|D|)$ and $\varphi_m : X \to \mathbb{P}M$. Choose a basis for M, say $f_{D_1}, ..., f_M$. Complete to a basis of $|D|$. Every rational map $X \to \mathbb{P}^n$ is given by the map given by the complete linear followed by a projection.

Example. Consider $\mathbb{P}^n \xrightarrow{\varphi} \mathbb{P}^m$ with $\text{Cl}(\mathbb{P}^n) \cong \mathbb{Z}$. The linear systems on $\mathbb{P}^n$ are determined by specfiying the degree of the polynomials. So the complete linear system of deg d. We then get the Veronese map $\varphi_{|\mathcal{O}_{\mathbb{P}^n}(d)|} : \mathbb{P}^n \to \mathbb{P}\big(_{|\mathcal{O}_{\mathbb{P}^n}(d)|}\big)$. Hence every rational map (non-degenerate) is obtain by a projection of a Veronese variety.

Consider $\mathbb{P}^1 \to X \subset \mathbb{P}^3$ with $(u, t) \mapsto (t^4, u^3 t, t^3 u, u^4)$. We get the map $\mathbb{P}^1 \to \mathbb{P}^4$ that is a rational normal curve of deg 4 and projection $(0, 0, 1, 0, 0)$. $\square$

A divisor is *very ample* if it is the hyperplane section of X under an embedding of $X \to \mathbb{P}^n$ for some n. A divisor is called *ample* if some positive multiple is very ample.

Let X be a compact complex manifold. When is X a projectivery variety? As an example, $\wp'^2 = c\wp^2 = a\wp + b$ where $\wp$ is the Weierstrass $\wp$-function.

Remark. In higher dimensions, tyou cannot always embed X into $\mathbb{P}^n$.

Example. (*Hopf surface*) Look at $\mathbb{C}^2 \backslash \{0\}$ and place the equivalence relation $(x_1, y_1) \sim (x_2, y_2)$ if $\exists n \in \mathbb{Z}$ such that $(x_2, y_2) = (x_1^n, y_1^n)$.

Divisors on curves

Let X be a nonsingular projective curve. Then $D = \sum k_i\, p_i$ where p_i are points on X with deg$(D) = \sum k_i$.

Theorem. Let $f : X - \gg Y$ be a map between non-singular projective curves. Then

$$\deg(f) = [k(X) : k(Y)] \text{ and } \deg(f) = \deg(f^*(y))$$

for any point $y \in Y$.

Corollary. The degree of a principal divisor on any non-singular projective curve is C. Then $f \in k(X)$ defines a map $f : X \to \mathbb{P}^1$ with $\text{div}(f) = \text{div}_0(f) - \text{div}_\infty(f)$, and $\deg(\text{div}(f)) = \deg(k^*(f(D))) - \deg f^*(\infty) = 0$.

Lecture 30 [Shafarevich pg. 168 - 171]

Divisors on curves

Let X be a nonsingular curve. Then $D = \sum k_i\, p_i$ with deg $D = \sum k_i$.

Theorem. If $f : X \to Y$ is a regular surjective morphism of nonsingular projective curves, then deg $f = [k(X) : k(Y)] = \deg(f^*(y))$ with $k(Y) \hookrightarrow k(X)$, for any point $y \in Y$.

Corollary. The degree of a principal divisor on a nonsingular projective curve is 0.

Proof. If $f \in k(X)$, then f gives a regular non-constant map, with $f : X \to \mathbb{P}^1$. Then deg (div $(f)) = \deg(\text{div}_0(f)) - \deg(\text{div}_\infty(f)) = \deg f - \deg f = 0$. □

Under the hypothesis, $f^* : k(Y) \to k(X)$, identify $k(Y)$ with a subfield of $k(X)$. Given finitely many points, $x_1, ..., x_r \in X$, let $\widetilde{\mathcal{O}}_{x_1,...,x_r} = \bigcap_{i=1}^r \mathcal{O}_{x_i}$. If $y \in Y$ and $f^{-1}(y) = \{x_1, ..., x_r\}$, let $\widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}_{x_1,...,x_r}$. Note we can identify $\mathcal{O}_y$ as a subring of $\widehat{\mathcal{O}}$.

Theorem A. $\widetilde{\mathcal{O}}$ is a principal ideal domain with finitely many prime ideals. There exists elements $t_i \in \widetilde{\mathcal{O}}$ such that $v_{x_j}(t_i) = \delta_{ij}$. Moreover, if $u \in \widetilde{\mathcal{O}}$, then $u = t_1^{k_1}...t_r^{k_r}$ such that $v_{x_i}(u) = k_i$ and v is invertible in $\widetilde{\mathcal{O}}$.

Theorem B. If $\{x_1, ..., x_r\} = f^{-1}(y)$, then $\widetilde{\mathcal{O}}$ is a free $\mathcal{O}_y$-module of rank $= \deg f = n$

Proof. (Theorem A + B $\implies$ main Theorem) Let t be a local parameter at $y \in Y$. Then $t = t_1^{k_1}...t_r^{k_r}$, v where $v_{x_i}(t) = k_i$ and invertible. Then $\deg(f^*(y)) = \sum k_i$ since $f^*(y) = \sum k_i x_i$. Then $t_1, ..., t_r$ are relatively prime so that $\widetilde{\mathcal{O}}/(t) \cong \bigoplus_{i=1}^r \widetilde{\mathcal{O}}/\left(t_i^{k_i}\right)$. Compare the dimensions as $\mathcal{O}_y/(t)$-modules. Then $n = \deg\ f = \sum k_i$. So $\deg(f^*(y)) = \deg f$. Observe that if D is a divisor on a nonsingular variety X and $x \in X$, then $\exists D' \sim D$ such that $x \notin$ Supp D'. (Exercise) □

Proof. (*of Theorem A*) Choose local parameters u_i at x_i. Then $\text{div}(u) = x_i + D$. If we change by linear equivalence, we can assume that supp $D \not\ni \{x_1, ..., x_r\}$. Once we choose our u_i as such, $v_{x_i}(u_1) = 1$, $v_{x_j}(u_u) = 0$. Set $t_i = u_i$ chosen as such. Let $u \in \widetilde{\mathcal{O}}$. Let $u \in \widetilde{\mathcal{O}}$, $v_{x_i}(u) = k_i$ and $w = t_1^{-k_1}...t_r^{-k_r}u$. Then $v_{x_i}(w) = 0\,\forall x_i$ by choice of the k_i. Both v and v^{-1} are regular at x_i, with $w, w^{-1} \in \widetilde{\mathcal{O}}$. Then $u = t_1^{k_1}...t_r^{k_r}w$. Finally, to check $\widetilde{\mathcal{O}}$ is a PID, let $a \in \widetilde{\mathcal{O}}$ be an ideal. Set $k_i = \inf_{u \in a} v_{x_i}(u)$. Let $\alpha = t_1^{k_1}...t_r^{k_r}$.We want to say $a = \langle\alpha\rangle$. Then $u\alpha^{-1} \in \widetilde{\mathcal{O}}$ for any $u \in a$, with $a \subset \langle\alpha\rangle$. Let α' be the set of functions $u\alpha^{-1}$ for $u \in a$. Then $\min_{u \in a'} v_{x_i}(u) = 0$ with $\beta = \sum u_j\, t_i^{k_1}...t_{j-1}^{k_{j-1}} t_{j+1}^{k_{j+1}}...t_r^{k_r}$. Then $v_{x_i}(\beta) = 0\ \forall i$. So $\beta\alpha^{-1} \in \mathcal{O}$. So $\alpha \in a$. □

Proof. (*of Theorem B*) If $f : X \to Y$ is a finite map of curves and X is nonsingular, then X is nonsingular is given by $f^{-1}(y) = \{x_1, ..., x_r\}$, with $\widetilde{\mathcal{O}} = \bigcap \mathcal{O}_{x_i}$, where $\widetilde{\mathcal{O}}$ is a finite $\mathcal{O}_j$-module. We can assume X and Y are affine. If $A = k[X]$ and $B = k[Y]$, then since this is a finite map and A is integral over B, A is a finite B-module. We want to prove the generators of A over B give you generators of $\widetilde{\mathcal{O}}$ over $\mathcal{O}$. Here, $\widetilde{\mathcal{O}} = k[X]\mathcal{O}_y$. So let a function $\varphi \in \widetilde{\mathcal{O}}$. Take z_i to be the poles of φ. Then $f(z_i) = y_i \neq y$. Then $\exists h \in k[Y]$ such that $h(y) \neq 0$ and $h(y_i) = 0$, and $\varphi h \in \mathcal{O}_{z_i}$. Hence, $\varphi h \in k[X]$. By construction, $h^{-1} \in \mathcal{O}_y$. In other words, $\varphi \in k[X]\mathcal{O}_y$. Hence, generators of A over B generate $\widetilde{\mathcal{O}}$ over $\mathcal{O}_y$. So then $\widetilde{\mathcal{O}}$ is a finitely generated module, so it is a direct sum of a free module $\mathcal{O}_j$ and a torsion module (by the structure theorem for finitely generated modules over a PID). The torsion module has to be zero, so $\widetilde{\mathcal{O}}$ is a free-module, say $\widetilde{\mathcal{O}} \cong (\mathcal{O}_y)^m$. Then $[k(X) : k(Y)] = n =$ deg f with $m \leq n$. Pick n elements that give a basis of $k(X)$ over $k(Y)$, say $\alpha_1, ..., \alpha_n$. We can multiply by appropriate powers of t_i's to make these regular. But since they are independent over $k(Y)$, the degree has to be the dgree of the field extension. □

Theorem. A nonsingular projective curve is rational $\Leftrightarrow$ $\mathrm{Cl}^0(X) = 0$.

Proof. If then Cl em y jectiv(r)2.3678( )-22662 7(o)-0.

Theorem. Let X be a smooth projective curve with D a divisor on X. Then

$$\dim \mathcal{L}(D) = \ell(D) \leq \mathrm{g}(D) - \mathrm{g}(X) + 1.$$

Theorem. $\mathcal{L}(D)$ is a finite dimensional vector space for any effective divisor D on the nonsingular projective curve X.

Prooj. If $D = D_1 - D_2$ where both D_1 and D_2 are effective, then $\mathcal{L}(D) \subseteq \mathcal{L}(D_1)$, with $f \in \mathcal{L}(D)$ implying $\mathrm{div}(f) + D \geq 0$ so that $\mathrm{div}(f) + D_1 \geq 0$. Take $D \geq 0$. Then if x is a point with multiplicity r, $\widetilde{D} = (r-1)x + (D - rx)$. Notice that deg $\widetilde{D}$ = deg $D - 1$ and $\widetilde{D} \geq 0$. Let t be a local parameter at x. Then for any $f \in \mathcal{L}(D)$, $\lambda(f) = t^r f(x)$ is a linear function on $\mathcal{L}(D)$. What is the kernel of λ? Well, it must be precisely $\mathcal{L}\left(\widetilde{D}\right)$, fhose functions for which the order of $t^r f$ at $x \geq 1$. In particular, $\ell(D) \leq \ell\left(\widetilde{D}\right) + 1$. We can keep going so that $\ell(D) \leq \ell(0) + \deg D$, with $\ell(0) = 1$ and $f \in k(X)$ such that $\mathrm{div}(f) \geq 0$. Thus f is regular, but all regular for a projective ariety means constant, so that $\mathcal{L}(0) \cong k$. Then $\dim(\mathcal{L}(D)) = \ell(D) \leq \deg D + 1$. If $X \ncong P^n$, then in fact $\ell(D) < \deg D + 1$. Suppose there exists D of degree 1, then $\ell(D) = \deg D + 1 = 2$. In other words, $\exists$ a non-constant map $f \in k(X)$ with $\mathrm{div}(f) + 0 \geq 0$ and $f : X \to \mathbb{P}^1$ with $\deg(f) = 1$ so that both are nonsingular, proj, so $X \cong \mathbb{P}^1$.

Theorem. Let $\alpha_0 \in X$ be a point on a non-singular cubic curve $X \subset \mathbb{P}^2$.

Exercises. (1) For $X : zy^2 = x^3 + axz^2 + bz^3$, X is non-singular if and only if the discriminant $4a^3 - 27b^2 \neq 0$.

(2) Given any non-singular cubic, you can make a change of variables so that it has this form (dehomogeonize): $f = f_1(xy) + f_2(x,y) + f_3(x,y)$. Make substitution $y = tx$ so that $f = x[f_1(1,t) + xf_2(1,t) + x^2 f_3(1mt)]$. Then complete the square, $s^2 = p(t)$. Then send one of the roots of $p(t)$ to ∞, with $y^2 = x^3 + ax^2 + bx + c$. Then $\alpha \mapsto [\alpha - \alpha_0] \in \mathrm{Cl}^0(X)$ defines a 1-1 correspondence between Y and $\mathrm{Cl}^0(X)$. In particular, any nonsingular cubic in the plane inherits a group structure via this correlation.

Lecture 32

Nonsingular plane cubics

Theorem. Let X be a nonsingular plane cubic ($y^2 = x^3 + ax + b$ with $4a^2 - 27b^3 \neq 0$ and $\mathrm{char}(k) \neq 2, 3$). Then we can get a map $X \xrightarrow{\varphi} \mathrm{Cl}^0(X)$. Fix a point α_0 (e.g., $(0,1,0) = \alpha_0$). Then $\alpha \mapsto [\alpha - \alpha_0]$ defines a 1-1 correspondence between X and $\mathrm{Cl}^0(X)$. In particular, X inherits a group structure via this map.

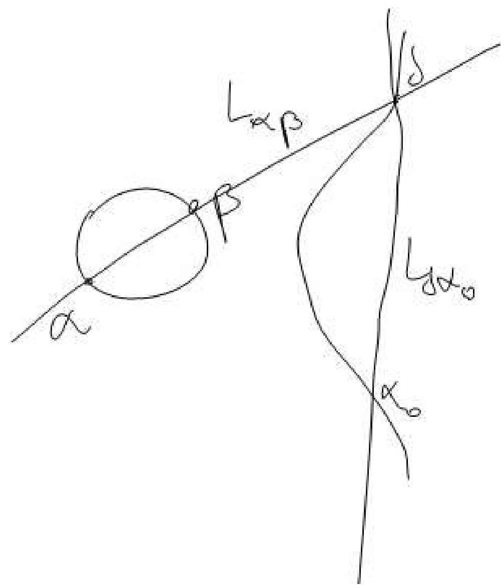
Proof. Observe that $X : zy^2 = x^3 + axz^2 + bz^3$ is not rational as follows. Then X has an automorphism (termed the elliptic/hyperelliptic involution). Then the map

$$(x, y, z) \overset{\sigma}{\mapsto} (x, -y, z)$$

is an obvious automorphism (since $y^2 = (-y)^2$). What are the fixed points of σ? Well, either $y = 0$, or the point $(0,1,0)$. If p is a fixed point of σ, then either $p = (0,1,0)$ or

$y = 0$, $z = 0$, and x is a root of $f(x) = x^3 + ax + b$. The polynomial f has 3 distinct roots, so σ has 4 fixed points. If $X \cong \mathbb{P}^1$, then the automorphisms are given by $\mathbb{P}\mathrm{GL}(2)$. So then any automorphism of $\mathbb{P}^1$ that has more than two fixed points is the identity (since a matrix can only have two eigenvalues). So σ has four points. Then $\alpha - \alpha_0 \sim \beta - \alpha_0$ means $\alpha \sim \beta$ so that $\alpha - \beta$ is principal but this is only true if and only if $\alpha = \beta$. We know the curve is not rational, because otherwise $\mathrm{div}(f) = \alpha - \beta$ with $f : X \to \mathbb{P}^1$ non-continuous of degree 1. But since X is not rational this is not possible. So $\varphi : X \to \mathrm{Cl}^0(X)$is injective.

Now we show surjectivity. Suppose D is an effective divisor on X, then $D \sim \alpha + k\alpha_0$ where $\alpha \in X$ is a point. If deg $D = 1$, then $k = 0$ works. So we can assume deg $D > 1$. Using induction, assume we can do it up to $\deg D - 1$. Then $D = D' + \beta$ gives $D \sim \alpha + \beta + k\alpha_0$, and it's enough to show that $\alpha + \beta \sim \gamma + \alpha_0$.



Then if $\delta \in L_{\alpha\beta} \cap X$, $f = L_{\alpha\beta}/L_{\delta\alpha_0}$ is a rational function on X. Also, $\alpha + \beta + \delta \sim \alpha_0 + \gamma + \delta$ where $\gamma \in L_{\delta\alpha_0} \cap X$. If $\alpha = \beta$, let $L_{\alpha\beta}$ be the tangent line to X at α. Let $D \in \mathrm{Cl}^0(X)$. Then $D = D_1 - D_2$ where D_1, D_2 are efficient, with $D \sim \alpha - \beta$. By what we proved, $\alpha + \alpha_0 \sim \gamma + \beta$ (is the same thing as). Then use the result that for any effective divisor $\alpha + \alpha_0$ and any point, there exists a point γ such that $\alpha - \beta \sim \gamma - \alpha_0$. $\square$

Theorem. If D is an effective divisor on X nonsingular ($y^2 = x^3 + ax + b$, $4a^3 - 27b^2 \neq 0$) , then $\ell(D) = \deg(D)$. Conversely, let X be a nonsingular curve such that for any effective divisor D, $\ell(D) = \deg(D)$. Then X can be realized as a smooth cubic in $\mathbb{P}^2$.

Proof. For two linear equivalent divisors $D \sim D'$, $\ell(D) = \ell(D')$, we can assume $D = \alpha + k\alpha_0$. Since we know X is not rational, $\ell(D) \leq \deg(D)$. If $k = 0$, $\ell(D)$ consists only of constants. If $k = 1$, then $\ell(D)$ has a non-constant for $f(D) = 2 = \deg\ D$. Let $k > 1$. Then it suffices to find a function $f_k : \mathcal{L}(k\alpha_0)$ such that $\mathrm{div}_\infty f_k = k\alpha_0$. Furthermore, $\mathcal{L}(k\alpha_0) \subseteq \mathcal{L}(\alpha + k\alpha_0)$, with $f_k \notin \mathcal{L}(\alpha - (k-1)\alpha_0)$. In other words, the vector space $\mathcal{L}(\alpha + k\alpha_0)$ has dimension $\ell(\alpha + (k-1)\alpha_0) + 1$. =Pick pX. Then $\ell(p) = 1$ constants, and $\ell(2p) = 2$ so $\exists$ nonconstant funcitons f_x, and $\ell(3p) = 3$, so $\exists$ another function with a pole of order exactly $3p$, saya y. Then $\ell(4p) = 4$ has x^2 as a pole. Then $\ell(5p) = 5$ has xy and $\ell(6p) = 6$ has x^3, y^2 as poles. So there has to be a linear relation among these functions, since we found seven functions in a seven dimensional vector space, say $\alpha y^2 + \beta xy + \delta y = ax^3 + bx^2 + cx + d$. You can complete the square for y to get $y^2 = x + ax^2 + bx + c$. $\square$

Lecture 33

We can put a group law on $y^2 = x^3 + ax + b$ with disc $\neq 0$. Fix a point α_0 with

$$x \mapsto \text{Cl}^0(X) \quad \text{and} \quad \alpha \mapsto [\alpha - \alpha_0].$$

To write down formulas, one lets α_0 be the point at ∞. This means if we projectivize the curve $(zy^2 = x^3 + z^2ax + bz^3)$ we can write down the formule where one lets α_0 be the point at ∞, $(0,1,0) = \alpha_0$. This is an inflection point of X.

Now we notice $[\alpha - \alpha_0] + [\beta - \alpha_0] \sim [\gamma - \alpha_0]$. Look at the line drawn between α and β on the circular component, and then it hits some δ, so draw the line between α_0 (at infinity) and δ (this will be a vertical intersection of δ), so that we cross the curve at another point δ. But then $\alpha + \beta \sim \gamma + \alpha_0$ with $f = L_{\alpha\beta}/L_{\gamma\delta_0}$. Then

$$\alpha \in (x_1, y_1) \in X \text{ and } \beta \in (x_2, y_2) \in X,$$

so the line $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ or $y = y_1 + m(x - x_1)$ combined with the equation of the curve $y^2 = x^3 + ax + b$ give

$$(y_1 + m(x - x_1))^2 = x^2 + ax + b$$

$$y_1^2 + m^2(x - x_1)^2 + 2y_1m(x - x_1) = x^3 + ax + b.$$

The coefficient of x^2 is m^2, so if we plug in $x = x_1$ and $x = x_2$ we see these are roots of the equations. Then $x_3 = m^2 - x_1 - x_2 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) - x_1 - x_2$. And y_3 is determined by the euquation of the line. Since the origin $0 = (0,1,0) = \alpha_0$, if $\alpha = \beta$, then $L_{\alpha\alpha}$ is the tangent line to X at α. Then the slope is given by $3x^2 + a$ (since $y^2 = x^3 + ax + b$) so the tangent line $y - y_1 = \left(\frac{3x_1 + a}{2y_1}\right)(x - x_1)$. Then $x_{2p} = -\frac{(3x_1^2 + a)^2}{4(x_1^3 + ax_1 + b)} - 2x_1$. Then $p = (x, y) \mapsto (x, -y) = -p$. Notice then that addition and inversion are regular maps on X. X is called a group variety.

Definition. If X is a variety together with maps $X \overset{(-1)}{\to} X$ inverse and $X \times X \overset{\cdot}{\to} X$ multiplicative which are regular maps; these maps should satisfy the axioms of a group: $\exists$ point $e \in X$ s.t. $e \times X \to X$ is id, $X \times e \to X$ is id, and associativity, and $X \times X \overset{\cdot}{\to} X$ means $(x, x^{-1}) \mapsto e$. If we look at the matrix groups $GL(n)$, $SL(n)$, $SO(n)$, etc. Then the group just defined, E, is compact, and we can see it is an abelian group. Then if X is projective and the group structure is abelian, we call X an *abelian variety*.

If we call the elliptic curve $E : y^2 = x^3 + ax + b$. Let k be a number field. Suppose E is defined over k. Then we can look at the $E(k)$ points whose coordinates $(x, y) \in E$ are in k. So $E(k)$ is a subgroup of $E(\mathbb{C})$.

If $x^2 + y^2 = z^2$ there are infinitely many rational solutions to this equation. So now consider this for $E : y^2 = x^3 + ax + b$. We ask the question: *can you find finitely many points on E such that you can generate all points on $E(\mathbb{Q})$ by the secant and tangent method?* Since E has a group structure, we can ask the same in "modern day" language: is $E(k)$ a finitely generated abelian group?

Theorem. (*Mordell*) $E(k)$ is finitely generated.

$E(k) \cong \mathbb{Z}^r \oplus \text{Torsion}$ where r is the rank of the elliptic curve over K. For $k = \mathbb{Q}$, the Torsion part is fairly well understood.

Differential forms and vector bundles

Suppose f is a regular function on a variety X. Then we can form the different $d_x f$ at any point. We saw how to do this. What kind of object is this thing? Well, if we let $d_x f$ be the diff form at every $x \in X$, then $d_x f \in T_x X$. Now we introduce vector bundles to make discussion of these gadgets simpler. Let M be a differentiable manifold. Then for a C^∞ complex vector bundle, at each point there will be an associated vector space, and these should vary differentially. So a C^∞-complex vector bundle is a collection of complex vector spaces for every point in M, i.e., $\{E_x\}_{x \in M}$, together with a C^∞ manifold structure on $E = \bigcup_{x \in M} E_x$. Then we have a natural projection map $\pi : E \to M$ given by $E_x \mapsto x$. Then (1) π is a C^∞-map. (2) For every $x \in M$ there is a neighborhood, $U \ni x$, and a diffeomorphism $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{C}^r$ such that the map is linear on the fibers. If this is the case, then we call E a vector bundle of rank r (each of the component vector spaces has dimension r). Then φ_U is called a *trivialization* of E along U. E_x is called the fiber of E over x.

Then on $U \cap V$, $\varphi_U : \pi^{-1}(U \cap V) \xrightarrow{\sim} (U \cap V) \times \mathbb{C}^r$, and $\varphi_V : \pi^{-1}(U \cap V) \xrightarrow{\sim} (U \cap V) \times \mathbb{C}^r$. Then we have what's called a *transition function* $g_{UV} = \varphi_V = \varphi_U^{-1}$ is a map from $U \cap V \to GL(r)$. Then $g_{UV} g_{VU} = I$ on $U \cap V$, and $g_{UV} g_{VW} g_{WU} = I$ on $U \cap V \cap W$.

Variations

Suppose M is a complex manifold. Then we can define holomorphic vector bundles on M by requiring E to be a complex manifold, π to be holomorphic, and φ_{UV} to be holomorphic. Similarly, you can ask M to be a variety, E to be a variety, π to be a regular map, the cover to be by Zariski opens, and then we get an algebraic vector bundle of g_{UV} regular maps, etc.