

# EXPLICIT CONSTRUCTION OF MODULI SPACE OF BOUNDED COMPLETE REINHARDT DOMAINS IN $\mathbb{C}^n$

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Dedicate to Professor Richard Melrose on the occasion of his 60<sup>th</sup> Birthday.

ABSTRACT. One of the most fundamental problems in complex geometry is to determine when two bounded domains in  $\mathbb{C}^n$  are biholomorphically equivalent. Even for complete Reinhardt domains, this fundamental problem remains unsolved for many years. Using the Bergmann function theory, we construct an infinite family of numerical invariants from the Bergman functions for complete Reinhardt domains in  $\mathbb{C}^n$ . These infinite family of numerical invariants are actually a complete set of invariants if the domains are pseudoconvex with  $C^1$  boundaries. For bounded complete Reinhardt domains with real analytic boundaries, the complete set of numerical invariants can be reduced dramatically although the set is still infinite.

## 1. INTRODUCTION

One of the basic problems in complex geometry is to find a reasonable object which parametrizes all non-isomorphic complex manifolds. This is the well known moduli problem. The concept of moduli goes back to Riemann [Rie57] who shows that the isomorphism class of a Riemann surface of genus  $g \geq 2$  depends on  $3g - 3$  parameters. The deformation of algebraic surfaces was first considered by Noether [Noe88]. It was Frölicher and Nijenhuis [FN57] who first studied deformation of higher dimensional complex manifolds by a differential geometric method. In a series of papers [KS58], Kodaira and Spencer systematically developed the theory of deformation of compact complex manifolds. The local moduli problem was solved by Kuranishi in his famous paper [Kur62] in 1962. However, a precise formulation of global moduli problems in algebraic geometry, the definition of moduli spaces for curves and for certain higher dimensional manifolds have only been given recently ([Mum65]), as well as solutions in some cases ([Gie77],[Vie95]). On the

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other hand, little is known about the moduli problem of open manifolds of complex dimension greater than one. This problem is substantially more difficult because it is infinite dimensional in nature.

Let  $D_1$  and  $D_2$  be two domains in  $\mathbb{C}^n$ . One of the most fundamental problems in complex geometry is to find necessary and sufficient conditions which will imply that  $D_1$  and  $D_2$  are biholomorphically equivalent. For  $n = 1$ , the celebrated Riemann mapping theorem states that any simply connected domains in  $\mathbb{C}$  are biholomorphically equivalent. For  $n \geq 2$ , there are many domains which are topologically equivalent to the ball but not biholomorphically equivalent to the ball. Poincaré [Poi07] studied this problem for the two dimensional complex ball. He found necessary and sufficient conditions on a first order perturbation of the unit ball in  $\mathbb{C}^2$  that the perturbed domain is biholomorphic to the ball. In fact Poincaré studied the invariance properties of the CR manifolds, which are real hypersurfaces in  $\mathbb{C}^n$ , with respect to biholomorphic transformations. The systematic study of such properties for real hypersurface was made by Cartan [Car32] and later by Chern and Moser [CM74]. A main result of the theory is the existence of a complete system of local differential invariants for CR-structures on real hypersurface. In 1974, Fefferman [Fef74] proved that a biholomorphic mapping between two strongly pseudoconvex domains is smooth up to the boundaries and the induced boundary mapping is a CR-equivalence on the boundary. Thus, by the fundamental theorem of Fefferman, one can use Chern–Moser invariants to study the biholomorphically equivalent problem of two strongly pseudoconvex domains. Perhaps the most successful story in this approach is due to Webster [Web78] who gave a complete characterization when two ellipsoids in  $\mathbb{C}^n$  are biholomorphically equivalent. In 1978, Burns, Shnider and Wells [BSW78] showed that the “number of moduli” of a “moduli space” of a strongly pseudoconvex bounded domain has to be infinite. Thus the moduli problem of open manifolds is really a very difficult one. Lempert [Lem88] made significant progress in the subject. He was able to construct the moduli space of bounded strictly convex domains of  $\mathbb{C}^n$  with marking at the origin. Although the theory established by Lempert is beautiful, the computation of his invariants is a hard problem. For a more detail on the global and local equivalence problem for real sub-manifolds in  $\mathbb{C}^n$ , we refer the readers to the survey paper by Huang [Hua04].

Recall that an open subset  $D \subseteq \mathbb{C}^n$  is called a complete Reinhardt domain if, whenever  $(z_1, \dots, z_n) \in D$ , then  $(\zeta_1 z_1, \dots, \zeta_n z_n) \in D$  for all complex numbers  $\zeta_j$  with  $|\zeta_j| \leq 1$ . It seems that the biholomorphically equivalent problem or moduli problem for complete Reinhardt

domains remains open, even though there is a beautiful theorem of Sunada [Sun78] which relates two such domains by a permutation map. In fact, Huang [Hua08] communicated to us a known method to reduce the holomorphic equivalence problem for such domains to a pure point-set identification problem for domains in  $(\mathbb{R}_+)^n$ : Let  $D_i$  be a complete bounded Reinhardt domain in  $\mathbb{C}^n$  for  $n > 1$ . After a scaling transformation of the form  $(z_1, \dots, z_n) \rightarrow (k_1 z_1, \dots, k_n z_n)$ , we can assume that  $P_j = (0, \dots, 0, 1, 0, \dots, 0)$  is the boundary point of  $D_i$  for each  $j$ . Here, in  $P_j$ , 1 is the  $j^{\text{th}}$ -component of  $P_j$ . Write  $T(D_i) = \{(|z_1|, \dots, |z_n|) \in (\mathbb{R}_+)^n : (z_1, \dots, z_n) \in D_i\}$ . Then it follows from a result of Sunada that  $D_1$  and  $D_2$  are holomorphically equivalent if and only if  $T(D_1) = T(D_2)$  as subsets in  $(\mathbb{R}_+)^n$  (up to a permutation in coordinates). However the method can be applied very conveniently to see if  $D_1$  and  $D_2$  are holomorphically equivalent only when  $T(D_1)$  and  $T(D_2)$  are defined by not too-complicated equations in  $(\mathbb{R}_+)^n$ . Thus, one cannot construct any invariant among all complete Reinhardt domains even in  $\mathbb{C}^2$  by this method. In particular, it cannot handle the biholomorphically equivalent problem for those domains with  $C^1$  boundaries. The current method to construct the moduli space  $\mathcal{R}$  of all bounded complete Reinhardt domains is as follows: Let  $T(\mathcal{R})$  be the set of all bounded domains in  $(\mathbb{R}_+)^n$ , that contain 0 and have  $P_j$  as the boundary points for each  $j$ . Moreover, we require that if  $(x_1, \dots, x_n) \in T(\mathcal{R})$ , then  $(k_1 x_1, \dots, k_n x_n) \in T(\mathcal{R})$  when  $1 \geq k_j \geq 0$ . Let  $S_n$  be the automorphisms of  $(\mathbb{R}_+)^n$  consisting of the permutations in coordinates. Then  $S_n$  is precisely the symmetric group of order  $n!$ .  $S_n$  acts on  $T(\mathcal{R})$  in a canonical way: For any  $A \in T(\mathcal{R})$  and  $\sigma \in S_n$ ,  $(\sigma, A) \mapsto \sigma(A) \in T(\mathcal{R})$ . This action defines an equivalence relation in  $T(\mathcal{R})$ :  $A \sim B$  if and only if  $A, B \in T(\mathcal{R})$  and  $A = \sigma(B)$ . Then  $\mathcal{R} = T(\mathcal{R}) / \sim$ , or  $\mathcal{R} = T(\mathcal{R}) / S_n$ . Unfortunately, it is very difficult if not impossible for this method to give any information on the geometry of the moduli space or the embedding of the moduli space.

Recently Du and Yau [DYar] studied the moduli problem of complete Reinhardt domains in  $\mathbb{C}^2$  from a different point of view. The main tool to solve this moduli problem with geometry information is the new biholomorphic invariant Bergman function defined by Yau [Yau04] on pseudoconvex domains in a variety with only isolated singularity. Du and Yau discovered that the moduli problem of complete Reinhardt domains in  $\mathbb{C}^2$  is equivalent to the moduli problem of complete Reinhardt domains in  $A_n$ -variety  $\{(x, y, z) : xy = z^{n+1}\}$ , which is the quotient of  $\mathbb{C}^2$  by a cyclic group of order  $n + 1$ . The great advantage of working

on the moduli problem of domains in  $A_n$ -variety is that the biholomorphic maps between these domains are dramatically smaller. This is because these biholomorphic maps not only have to send the boundary of one domain to the boundary of another domain, but also they have to leave the  $A_n$ -variety invariant. In addition, these biholomorphic maps need to preserve the Bergman functions which are positive functions. Du and Yau used these facts to show that all the biholomorphic maps between complete Reinhardt domains in  $A_n$ -variety must be of the special form: permutation of coordinates modulo scalar multiplication. This result is much stronger than the corresponding result obtained by Sunada [Sun78]. In principle, the method introduced by Du and Yau [DYar] could be used to study the biholomorphic equivalence problem or moduli problem for more general pseudoconvex domains in  $A_n$ -variety.

The purpose of this paper is to show that Yau's Bergman function theory can also solve the biholomorphic equivalence problem or moduli problem for complete Reinhardt pseudoconvex domains in  $\mathbb{C}^n$  for all  $n \geq 2$ . In order to describe the complete biholomorphic invariants of bounded complete Reinhardt domains in  $\mathbb{C}^n$ , we introduce the following notations. Let  $S_n$  be the symmetric group of degree  $n$ . Recall that the group ring  $\mathbb{R}[S_n]$  is a ring of the form  $\mathbb{R}[\tau_1, \tau_2, \dots, \tau_n!]$  with  $\tau_i \in S_n$  for  $1 \leq i \leq n!$ . Let  $\sum_i x_i \tau_i$  and  $\sum_j y_j \tau_j$ , where  $x_i, y_j$  are in  $\mathbb{R}$ , be two elements in  $\mathbb{R}[S_n]$ . Then  $(\sum_i x_i \tau_i)(\sum_j y_j \tau_j) := \sum_{i,j} x_i y_j (\tau_i \cdot \tau_j)$ , where  $\tau_i \cdot \tau_j$  is the product in the group  $S_n$ . We shall consider  $\mathbb{R}[S_n] \times \dots \times \mathbb{R}[S_n]$  the product of the group ring with itself. Such a product has a natural  $S_n$ -module structure in the following manner. Let  $\sigma \in S_n$  and  $(\sum_i x_i \tau_i, \dots, \sum_j y_j \tau_j) \in \mathbb{R}[S_n] \times \dots \times \mathbb{R}[S_n]$ . Then

$$\sigma \left( \sum_i x_i \tau_i, \dots, \sum_j y_j \tau_j \right) := \left( \sum_i x_i (\tau_i \sigma), \dots, \sum_j y_j (\tau_j \sigma) \right). \quad (1.1)$$

**Definition 1.1.** Two elements  $f, g$  in  $\mathbb{R}[S_n] \times \dots \times \mathbb{R}[S_n]$  are said to be equivalent and denoted by  $f \sim g$  if there exists a  $\sigma \in S_n$  such that  $\sigma(f) = g$ .

Before we can describe our main results, we need the following notations. Let  $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$  be an  $n$ -tuple of nonnegative integers. Denote  $\phi_{\vec{\alpha}} = \left( \prod_{i=1}^n z_i^{\alpha_i} \right) dz_1 \wedge dz_2 \wedge \dots \wedge dz_n$ . For a domain  $D$  in  $\mathbb{C}^n$ . We shall use notation  $\|\phi_{\vec{\alpha}}\|_D^2 = \int_D \phi_{\vec{\alpha}} \wedge \Phi_{\vec{\alpha}}$ . In this paper, we show that all the biholomorphic invariants of a bounded complete Reinhardt domain

are contained in  $(\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n]) / \sim$  where there are  $n!$  copies of  $\mathbb{R}[S_n]$  and  $\sim$  is the equivalent relation defined in Definition 1.1.

**Theorem A.** *Let  $D$  be a bounded complete Reinhardt domain in  $\mathbb{C}^n$ . Let  $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$  be a  $n$ -tuple of nonnegative integers. For any  $\tau \in S_n$ , denote*

$$g_D^\tau(\vec{\alpha}) = \frac{\|\phi_{\vec{0}}\|_D^{\sum \alpha_i - 1} \|\phi_{\tau(\vec{\alpha})}\|_D}{\prod_{i=1}^n \|\phi_{\vec{e}_i}\|_D^{\alpha_{\tau(i)}}} \quad (1.2)$$

where  $\tau(\vec{\alpha}) = (\alpha_{\tau(1)}, \dots, \alpha_{\tau(n)})$  and  $\vec{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$  with 1 in the  $i$ th component. Then for all  $n$ -tuples of nonnegative integers  $\vec{\beta}_1, \dots, \vec{\beta}_{n!}$ ,  $\xi_D^{\vec{\beta}_1, \dots, \vec{\beta}_{n!}} = (\sum_{\tau \in S_n} g_D^\tau(\vec{\beta}_1)\tau, \dots, \sum_{\tau \in S_n} g_D^\tau(\vec{\beta}_{n!})\tau)$  as an element in  $(\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n]) / \sim$  is a biholomorphic invariant. In fact, if  $D_1$  and  $D_2$  are two such domains which are biholomorphically equivalent, then there exists a  $\sigma \in S_n$  such that  $g_{D_1}^\tau(\vec{\alpha}) = g_{D_2}^{\tau \circ \sigma}(\vec{\alpha}) \forall \tau \in S_n$  and  $\forall \vec{\alpha}$   $n$ -tuple of nonnegative integers.

The invariants in Theorem A are complete invariants for bounded complete Reinhardt pseudoconvex domains with  $C^1$  boundaries.

**Theorem B.** *Let  $D_i$ ,  $i = 1, 2$ , be two bounded complete Reinhardt pseudoconvex domains in  $\mathbb{C}^n$  with  $C^1$  boundaries. If for all  $\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}$   $n$ -tuples of nonnegative integers,  $\xi_{D_1}^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}} = \xi_{D_2}^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}}$  in  $(\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n]) / \sim$ , where  $\xi_D^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}} = (\sum_{\tau \in S_n} g_D^\tau(\vec{\alpha}_1)\tau, \dots, \sum_{\tau \in S_n} g_D^\tau(\vec{\alpha}_{n!})\tau)$ , then there exists  $\sigma \in S_n$  and a biholomorphic map*

$$\Psi_\sigma(z_1, \dots, z_n) = (a_1 z_{\sigma(1)}, \dots, a_n z_{\sigma(n)}),$$

where  $a_i = \frac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e}_i}\|_{D_2}}{\|\phi_{\vec{e}_{\sigma(i)}}\|_{D_1} \|\phi_{\vec{0}}\|_{D_2}}$ , such that  $\Psi_\sigma$  sends  $D_1$  onto  $D_2$ .

Theorem A and Theorem B above give a complete characterization of two bounded complete Reinhardt domains in  $\mathbb{C}^n$  to be biholomorphically equivalent in terms of the quotient of group ring  $(\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n]) / \sim$ . In case  $n = 2$ , we can actually write down the complete numerical invariants for two bounded complete Reinhardt in  $\mathbb{C}^2$  to be biholomorphically equivalent.

**Theorem C.** *Let  $D_1, D_2$  be two bounded complete Reinhardt pseudoconvex domains in  $\mathbb{C}^2$  with  $C^1$  boundaries. Then  $D_1$  is biholomorphic to  $D_2$  if and only if*

$$g_{D_1}(\alpha_1, \alpha_2) + g_{D_1}(\alpha_2, \alpha_1) = g_{D_2}(\alpha_1, \alpha_2) + g_{D_2}(\alpha_2, \alpha_1) \quad (1.3)$$

$$g_{D_1}(\alpha_1, \alpha_2)g_{D_1}(\alpha_2, \alpha_1) = g_{D_2}(\alpha_1, \alpha_2)g_{D_2}(\alpha_2, \alpha_1) \quad (1.4)$$

$$\begin{aligned} & (g_{D_1}(\alpha_1, \alpha_2) - g_{D_1}(\alpha_2, \alpha_1)) (g_{D_1}(\beta_1, \beta_2) - g_{D_1}(\beta_2, \beta_1)) \\ &= (g_{D_2}(\alpha_1, \alpha_2) - g_{D_2}(\alpha_2, \alpha_1)) (g_{D_2}(\beta_1, \beta_2) - g_{D_2}(\beta_2, \beta_1)) \end{aligned} \quad (1.5)$$

for all nonnegative integers  $\alpha_i, \beta_i$ , where

$$g_{D_i}(\alpha_1, \alpha_2) = \frac{\|\phi_{\bar{0}}\|_{D_i}^{\alpha_1 + \alpha_2 - 1} \|\phi_{(\alpha_1, \alpha_2)}\|_{D_i}}{\prod_{j=1}^2 \|\phi_{\bar{e}_j}\|_{D_i}^{\alpha_j}}.$$

**Corollary D.** *The moduli space of bounded complete Reinhardt pseudoconvex domains with  $C^1$  boundaries in  $\mathbb{C}^2$  can be constructed explicitly as the image of the complete family of numerical invariants:*

$$\begin{aligned} & g_D(\alpha_1, \alpha_2) + g_D(\alpha_2, \alpha_1), \\ & g_D(\alpha_1, \alpha_2)g_D(\alpha_2, \alpha_1), \end{aligned}$$

and

$$(g_D(\alpha_1, \alpha_2) - g_D(\alpha_2, \alpha_1)) \cdot (g_D(\beta_1, \beta_2) - g_D(\beta_2, \beta_1))$$

$\forall \alpha_i, \beta_i$  nonnegative integers.

In order to find the complete numerical biholomorphic invariants of bounded complete Reinhardt domain in  $\mathbb{C}^n$  for  $n \geq 3$ , we need to consider the finite symmetric group  $S_n = \{\sigma_1, \sigma_2, \dots, \sigma_n\}$  of degree  $n$  acting on the affine space  $\mathbb{C}^{n!} = \mathbb{C}^{n!} \times \dots \times \mathbb{C}^{n!}$ , which is the product of  $n!$  copies of  $\mathbb{C}^{n!}$ , in the following manner. Let  $\tau \in S_n$  and  $(x_{\sigma_1}, \dots, x_{\sigma_n}; \dots; y_{\sigma_1}, \dots, y_{\sigma_n}) \in \mathbb{C}^{n!} \times \dots \times \mathbb{C}^{n!} = \mathbb{C}^{n!n!}$ . Then  $\tau(x_{\sigma_1}, \dots, x_{\sigma_n}; \dots; y_{\sigma_1}, \dots, y_{\sigma_n}) = (x_{\sigma_1\tau}, \dots, x_{\sigma_n\tau}; \dots; y_{\sigma_1\tau}, \dots, y_{\sigma_n\tau})$ . Since  $S_n$  is linearly reductive, by Hilbert Theorem, the ring of invariants  $\mathbb{C}[x_{\sigma_1}, \dots, x_{\sigma_n}; \dots; y_{\sigma_1}, \dots, y_{\sigma_n}]^{S_n}$  is finitely generated. Moreover, the generators can be listed explicitly by Göbel's theorem.

**Theorem E.** *Let  $f_1, \dots, f_N \in \mathbb{C}[x_{\sigma_1}, \dots, x_{\sigma_n}; \dots; y_{\sigma_1}, \dots, y_{\sigma_n}]^{S_n}$  be the generators of the ring of invariant polynomials. Let  $D$  be a bounded complete Reinhardt domain in  $\mathbb{C}^n$ . Then, for  $\vec{\alpha}_1, \vec{\alpha}_2, \dots, \vec{\alpha}_{n!}$   $n$ -tuples of nonnegative integers,*

$$f_1(g_D^\sigma(\vec{\alpha}_1), \dots, g_D^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n}, \dots, f_N(g_D^\sigma(\vec{\alpha}_1), \dots, g_D^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n}$$

are biholomorphic invariants, where

$$g_D^\sigma(\vec{\beta}) = \frac{\|\phi_{\bar{0}}\|_D^{\sum \beta_i - 1} \|\phi_{\sigma(\vec{\beta})}\|_D}{\prod_{i=1}^n \|\phi_{\bar{e}_i}\|_D^{\beta_{\sigma(i)}}}, \quad \vec{\beta} = (\beta_1, \beta_2, \dots, \beta_n).$$

The following theorem says that the above invariants are actually complete in case the domain  $D$  is pseudoconvex.

**Theorem F.** *Let  $D_i$ ,  $i = 1, 2$ , be two bounded complete Reinhardt pseudoconvex domains in  $\mathbb{C}^n$  with  $C^1$  boundaries. Let  $f_1, \dots, f_N \in \mathbb{C}[x_{\sigma_1}, \dots, x_{\sigma_{n_1}}; \dots; y_{\sigma_i}, \dots, y_{\sigma_{n_i}}]^{S_n}$  be the generators of the ring of invariant polynomials. If for all  $\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}$   $n$ -tuples of nonnegative integers*

$$f_i(g_{D_1}^\sigma(\vec{\alpha}_1), \dots, g_{D_1}^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n} = f_i(g_{D_2}^\sigma(\vec{\alpha}_1), \dots, g_{D_2}^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n},$$

$$i = 1, 2, \dots, N,$$

*then there exists  $\tau \in S_n$  and a biholomorphic map  $\Psi_\tau: \mathbb{C}^n \rightarrow \mathbb{C}^n$ ,  $\Psi_\tau(z_1, \dots, z_n) = (a_1 z_{\tau(1)}, \dots, a_n z_{\tau(n)})$ , where*

$$a_i = \frac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e}_i}\|_{D_2}}{\|\phi_{e_{\tau(i)}}\|_{D_1} \|\phi_{\vec{0}}\|_{D_2}},$$

*such that  $\Psi_\tau$  sends  $D_1$  onto  $D_2$ .*

Remark: The generators of  $\mathbb{C}[x_{\sigma_1}, \dots, x_{\sigma_{n_1}}; \dots; y_{\sigma_i}, \dots, y_{\sigma_{n_i}}]^{S_n}$  can be explicitly listed in view of the Theorem of Göbel (cf. Theorem 4.6 below).

**Corollary G.** *The moduli space of bounded complete Reinhardt pseudoconvex domains with  $C^1$  boundaries in  $\mathbb{C}^n$  can be constructed explicitly as the image of the complete family of numerical invariants:*

$$f_i(g_D^\sigma(\vec{\alpha}_1), \dots, g_D^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n}, \quad 1 \leq i \leq N,$$

*where  $\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}$  are all possible  $n$ -tuples of nonnegative integers.*

For complete Reinhardt pseudoconvex domains with real analytic boundaries, we can use fewer numerical invariants to characterize these domains. More precisely, we have the following theorems.

**Theorem B'.** *Let  $D_i$ ,  $i = 1, 2$ , be two bounded complete Reinhardt pseudoconvex domains in  $\mathbb{C}^n$  with real analytic boundaries. Then  $D_1$  is biholomorphically equivalent to  $D_2$  if and only if for all  $\vec{\alpha}$   $n$ -tuple of nonnegative integers,  $\xi_{D_1}^{\vec{\alpha}} = \xi_{D_2}^{\vec{\alpha}}$  in  $\mathbb{R}[S_n]/\sim$ , where  $\xi_{D_i}^{\vec{\alpha}} = \sum_{\tau \in S_n} g_{D_i}^\tau(\vec{\alpha})\tau$ .*

*In this case, there exists  $\sigma \in S_n$  and a biholomorphic map*

$$\Psi_\sigma(z_1, \dots, z_n) = (a_1 z_{\sigma(1)}, \dots, a_n z_{\sigma(n)}),$$

*where  $a_i = \frac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e}_i}\|_{D_2}}{\|\phi_{\vec{e}_{\sigma(i)}}\|_{D_1} \|\phi_{\vec{0}}\|_{D_2}}$ , such that  $\Psi_\sigma$  sends  $D_1$  onto  $D_2$ .*

**Theorem C'.** *Let  $D_1, D_2$  be two bounded complete Reinhardt pseudoconvex domains in  $\mathbb{C}^2$  with real analytic boundaries. Then  $D_1$  is biholomorphic to  $D_2$  if and only if*

$$g_{D_1}(\alpha_1, \alpha_2) + g_{D_1}(\alpha_2, \alpha_1) = g_{D_2}(\alpha_1, \alpha_2) + g_{D_2}(\alpha_2, \alpha_1)$$

$$g_{D_1}(\alpha_1, \alpha_2)g_{D_1}(\alpha_2, \alpha_1) = g_{D_2}(\alpha_1, \alpha_2)g_{D_2}(\alpha_2, \alpha_1)$$

for all nonnegative integers  $\alpha_1, \alpha_2$ , where

$$g_{D_i}(\alpha_1, \alpha_2) = \frac{\|\phi_{\bar{0}}\|_{D_i}^{\alpha_1 + \alpha_2 - 1} \|\phi_{(\alpha_1, \alpha_2)}\|_{D_i}}{\prod_{j=1}^2 \|\phi_{\bar{e}_j}\|_{D_i}^{\alpha_j}}.$$

**Theorem F'.** *Let  $D_i$ ,  $i = 1, 2$ , be two bounded complete Reinhardt pseudoconvex domains in  $\mathbb{C}^n$  with real analytic boundaries. Let*

$$f_1, \dots, f_N \in \mathbb{C}[x_{\sigma_1}, \dots, x_{\sigma_n}]^{S_n}$$

be the generators of the ring of invariant polynomials. Then  $D_1$  is biholomorphically equivalent to  $D_2$  if and only if for all  $\vec{\alpha}$   $n$ -tuples of nonnegative integers

$$\begin{aligned} f_i(g_{D_1}^{\vec{\alpha}}(\vec{\alpha})_{\sigma \in S_n}) &= f_i(g_{D_2}^{\vec{\alpha}}(\vec{\alpha})_{\sigma \in S_n}), \\ i &= 1, 2, \dots, N. \end{aligned}$$

In this case, there exists  $\tau \in S_n$  and a biholomorphic map  $\Psi_\tau: \mathbb{C}^n \rightarrow \mathbb{C}^n$   $\Psi_\tau(z_1, \dots, z_n) = (a_1 z_{\tau(1)}, \dots, a_n z_{\tau(n)})$ , where  $a_i = \frac{\|\phi_{\bar{0}}\|_{D_1} \|\phi_{\bar{e}_i}\|_{D_2}}{\|\phi_{e_{\sigma(i)}}\|_{D_1} \|\phi_{\bar{0}}\|_{D_2}}$ , such that  $\Psi_\tau$  sends  $D_1$  onto  $D_2$ .

Remark: The generators of  $\mathbb{C}[x_{\sigma_1}, \dots, x_{\sigma_n}]^{S_n}$  can be explicitly listed in view of the Theorem of Göbel (cf. Theorem 4.6 below).

Our paper is organized as follows. In §2 we recall the basic notion of the Bergman function and some well known results which are needed for later discussion. In §3, we construct continuous invariants of bounded complete Reinhardt domains in  $\mathbb{C}^n$  and prove Theorem A and Theorem B (B'). In §4, the Hilbert 14<sup>th</sup> problem and Göbel's theorem are discussed and complete continuous numerical invariants of bounded complete Reinhardt pseudoconvex domains are constructed. Theorem C (C'), Theorem E and Theorem F (F') are proved in this section. In section 5, we give applications to some concrete examples.

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## 2. PRELIMINARIES

In this section, we shall recall some basic definitions and results in our previous papers [Yau04], [DYar] which will facilitate our subsequent discussion. Let  $M$  be a pseudoconvex complex manifold and  $A$  be a compact complex analytic variety in the interior of  $M$ .

**Definition 2.1.** Let  $F_M$  (respectively,  $F_{M,A}$ ) be the space of all  $L^2$ -integrable holomorphic  $n$ -form on  $M$  (respectively, vanishing at the compact analytic subset  $A$  in  $M$ ). Let  $\{\omega_j\}$  (respectively,  $\{\omega_j^A\}$ ) be a complete orthonormal basis of  $F_M$  (respectively,  $F_{M,A}$ ). The Bergman kernel (respectively, Bergman kernel vanishing at  $A$ ) is defined to be  $K_M(z) = \sum_j \omega_j(z) \wedge \overline{\omega_j(z)}$  (respectively,  $K_{M,A}(z) = \sum_j \omega_j^A(z) \wedge \overline{\omega_j^A(z)}$ ).

**Lemma 2.2.** (a) *Bergman kernel  $K_{M,A}(z)$  vanishing at the compact analytic subset  $A$  is independent of the choice of the complete orthonormal basis of  $F_{M,A}$ .*

(b) *Let  $\Phi: (M_1, A_1) \rightarrow (M_2, A_2)$  be a biholomorphic map such that  $\Phi(A_1) = A_2$ . Then  $K_{M_1, A_1}(z) = \Phi^* K_{M_2, A_2}(z)$ .*

**Definition 2.3.** The Bergman function  $B_{M,A}$  on  $M$  is defined to be  $K_{M,A}(z)/K_M(z)$ .

The following Theorem 2.4 can be found in [Yau04].

**Theorem 2.4.** *Let  $A_1$  (respectively  $A_2$ ) be compact analytic variety in complex manifold  $M_1$  (respectively  $M_2$ ). If  $\Phi: (M_1, A_1) \rightarrow (M_2, A_2)$  is a biholomorphic map, then  $B_{M_1, A_1}(z) = B_{M_2, A_2}(\Phi(z))$ .*

In what follows, we shall recall the following beautiful theorem of Sunada.

**Theorem 2.5** (Sunada [Sun78]). *Two  $n$ -dimensional bounded Reinhardt domains  $D_1$  and  $D_2$  are biholomorphically equivalent if and only if there exists a linear transformation  $\Phi: \mathbb{C}^n \rightarrow \mathbb{C}^n$  given by  $z_i \mapsto r_i z_{\sigma(i)}$  ( $r_i > 0$ ,  $i = 1, \dots, n$  and  $\sigma$  being a permutation of the indices  $i$ ) such that  $\Phi(D_1) = D_2$ .*

The following proposition which is a corollary to Satz 1 in [Pfl75], is stated as Proposition 1 in [Ohs80].

**Proposition 2.6** (Pflug [Pfl75]). *The Bergman kernel blows up at every boundary point in a pseudoconvex domain with  $C^1$ -bounding in  $\mathbb{C}^n$ .*

### 3. CONTINUOUS INVARIANTS OF BOUNDED COMPLETE REINHARDT DOMAINS IN $\mathbb{C}^n$

In what follows, we shall use the following notations  $\vec{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$  with 1 in the  $i$ th position.

$\mathbb{Z}_+$  = set of nonnegative integers

$$\vec{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n.$$

**Proposition 3.1.** *Let  $D$  be a bounded complete Reinhardt domain in  $\mathbb{C}^n$ . Let  $\phi_{\vec{\alpha}} = \prod_{i=1}^n z_i^{\alpha_i} dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$ ,  $\alpha_i \in \mathbb{Z}_+$ . Then  $\left\{ \frac{\phi_{\vec{\alpha}}}{\|\phi_{\vec{\alpha}}\|_D} \right\}$  is a complete orthonormal base of  $F_D$ , and  $\left\{ \frac{\phi_{\vec{\alpha}}}{\|\phi_{\vec{\alpha}}\|_D} : \vec{\alpha} \neq 0 \right\}$  is a complete orthonormal basis of  $F_{D,0}$ . The Bergman kernel  $K_{D,0}$  vanish at the origin and the Bergman kernel  $K_D$  are given by*

$$K_{D,0} = \Theta_D dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \quad (3.1)$$

and

$$K_D = \left( \frac{1}{\|\phi_{\vec{0}}\|_D^2} + \Theta_D \right) dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \quad (3.2)$$

where

$$\Theta_D = \sum_{\vec{\alpha} \neq 0} \frac{\prod_{i=1}^n |z_i|^{2\alpha_i}}{\|\phi_{\vec{\alpha}}\|_D^2}. \quad (3.3)$$

*Proof.* This is a consequence of the proof of Proposition 3.2 of [Yau04].  
Q.E.D.

**Proposition 3.2.** *Let  $D$  be a bounded complete Reinhardt domain in  $\mathbb{C}^n$ . With the notations in the above proposition,  $\|\phi_{\vec{0}}\|_D^2 \Theta_D$  is invariant under biholomorphic maps which send the origin to the origin.*

*Proof.* Let  $\Psi: D_1 \rightarrow D_2$  be a biholomorphic map between two bounded complete Reinhardt domains such that  $\Psi(0) = 0$ . By Theorem 2.4 and Proposition 3.1, we have

$$\begin{aligned} B_{D_1}(z) &= B_{D_2}(\Psi(z)) \\ \Rightarrow \frac{\Theta_{D_1}(z)}{\frac{1}{\|\phi_{\vec{0}}\|_{D_1}^2} + \Theta_{D_1}(z)} &= \frac{\Theta_{D_2}(\Psi(z))}{\frac{1}{\|\phi_{\vec{0}}\|_{D_2}^2} + \Theta_{D_2}(\Psi(z))} \\ \Rightarrow \|\phi_{\vec{0}}\|_{D_1}^2 \Theta_{D_1}(z) &= \|\phi_{\vec{0}}\|_{D_2}^2 \Theta_{D_2}(\Psi(z)). \end{aligned}$$

Q.E.D.

**Theorem 3.3.** *Let  $D_i$ ,  $i = 1, 2$ , be two bounded complete Reinhardt domains in  $\mathbb{C}^n$ . If  $D_1$  is biholomorphically equivalent to  $D_2$ , then there exists a biholomorphic map  $\Psi_\sigma$  of the following form*

$$\Psi_\sigma(z) = (a_1 z_{\sigma(1)}, \dots, a_n z_{\sigma(n)}) \quad (3.4)$$

where  $\sigma$  is in  $S_n$ , a symmetric group of order  $n$ , and

$$a_i = \frac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e}_i}\|_{D_2}}{\|\phi_{\vec{e}_{\sigma(i)}}\|_{D_1} \|\phi_{\vec{0}}\|_{D_2}}. \quad (3.5)$$

*Proof.* In view of Theorem 2.5, there exists a biholomorphism  $\Psi_\sigma : D_1 \rightarrow D_2$  of the form (3.4). By Proposition 3.2

$$\begin{aligned} \|\phi_{\bar{0}}\|_{D_1}^2 \Theta_{D_1}(z) &= \|\phi_{\bar{0}}\|_{D_2}^2 \Theta_{D_2}(\Psi_\sigma(z)) \\ \Rightarrow \|\phi_{\bar{0}}\|_{D_1}^2 \sum_{\bar{\alpha} \neq 0} \frac{\prod_{i=1}^n |z_n|^{2\alpha_i}}{\|\phi_{\bar{\alpha}}\|_{D_1}^2} &= \|\phi_{\bar{0}}\|_{D_2}^2 \sum_{\bar{\alpha} \neq 0} \frac{\prod_{i=1}^n |a_i z_{\sigma(i)}|^{2\alpha_i}}{\|\phi_{\bar{\alpha}}\|_{D_2}^2}. \end{aligned} \quad (3.6)$$

Comparing the coefficient of  $|z_i|^2$  in (3.6), we can get

$$\begin{aligned} \|\phi_{\bar{0}}\|_{D_1}^2 \frac{1}{\|\phi_{\bar{e}_i}\|_{D_1}^2} &= \frac{\|\phi_{\bar{0}}\|_{D_2}^2 |a_{\sigma^{-1}(i)}|^2}{\|\phi_{\bar{e}_{\sigma^{-1}(i)}}\|_{D_2}^2} \\ \Rightarrow a_{\sigma^{-1}(i)} &= \frac{\|\phi_{\bar{0}}\|_{D_1} \|\phi_{\bar{e}_{\sigma^{-1}(i)}}\|_{D_2}}{\|\phi_{\bar{e}_i}\|_{D_1} \|\phi_{\bar{0}}\|_{D_2}} \\ \text{i.e., } a_i &= \frac{\|\phi_{\bar{0}}\|_{D_1} \|\phi_{\bar{e}_i}\|_{D_2}}{\|\phi_{\bar{e}_{\sigma(i)}}\|_{D_1} \|\phi_{\bar{0}}\|_{D_2}}. \end{aligned}$$

Q.E.D.

Now we are ready to prove Theorem A.

*Proof of Theorem A.* Let  $D_1$  and  $D_2$  be biholomorphically equivalent bounded complete Reinhardt domains. By Theorem 3.3, there exists a biholomorphic map

$$\begin{aligned} \Psi_\sigma : D_1 &\longrightarrow D_2 \\ (z_1, \dots, z_n) &\longrightarrow (a_1 z_{\sigma(1)}, \dots, a_n z_{\sigma(n)}) \end{aligned}$$

where  $a_i = \frac{\|\phi_{\bar{0}}\|_{D_1} \|\phi_{\bar{e}_i}\|_{D_2}}{\|\phi_{\bar{e}_{\sigma(i)}}\|_{D_1} \|\phi_{\bar{0}}\|_{D_2}}$

By Proposition 3.2

$$\begin{aligned} \|\phi_{\bar{0}}\|_{D_1}^2 \sum_{\bar{\alpha} \neq 0} \frac{\prod_{i=1}^n |z_i|^{2\alpha_i}}{\|\phi_{\bar{\alpha}}\|_{D_1}^2} &= \|\phi_{\bar{0}}\|_{D_1}^2 \Theta_{D_1}(z) \\ &= \|\phi_{\bar{0}}\|_{D_2}^2 \Theta_{D_2}(\Psi_\sigma(z)) \\ &= \|\phi_{\bar{0}}\|_{D_2}^2 \sum_{\bar{\alpha} \neq 0} \frac{\prod_{i=1}^n |a_i z_{\sigma(i)}|^{2\alpha_i}}{\|\phi_{\bar{\alpha}}\|_{D_2}^2} \end{aligned} \quad (3.6)$$

Comparing the coefficient of  $|z_1|^{2\alpha_1}|z_2|^{2\alpha_2}\dots|z_n|^{2\alpha_n}$  in (3.6), we get

$$\begin{aligned}
\frac{\|\phi_{\vec{0}}\|_{D_1}^2}{\|\phi_{\vec{\alpha}}\|_{D_1}^2} &= \frac{\|\phi_{\vec{0}}\|_{D_2}^2 \cdot \prod_{i=1}^n |a_{\sigma^{-1}(i)}|^{2\alpha_i}}{\|\phi_{\sigma(\vec{\alpha})}\|_{D_2}^2} \\
&\Rightarrow \frac{\|\phi_{\vec{0}}\|_{D_1}^2}{\|\phi_{\vec{\alpha}}\|_{D_1}^2} \\
&= \frac{\|\phi_{\vec{0}}\|_{D_2}^2}{\|\phi_{\sigma(\vec{\alpha})}\|_{D_2}^2} \prod_{i=1}^n \left( \frac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e}_{\sigma^{-1}(i)}}\|_{D_2}}{\|\phi_{\vec{e}_i}\|_{D_1} \|\phi_{\vec{0}}\|_{D_2}} \right)^{2\alpha_i} \\
&\Rightarrow \frac{\|\phi_{\vec{0}}\|_{D_1}^{\sum \alpha_i - 1} \cdot \|\phi_{\vec{\alpha}}\|_{D_1}}{\prod_{i=1}^n \|\phi_{\vec{e}_i}\|_{D_1}^{\alpha_i}} \\
&= \frac{\|\phi_{\vec{0}}\|_{D_2}^{\sum \alpha_i - 1} \cdot \|\phi_{\sigma(\vec{\alpha})}\|_{D_2}}{\prod_{i=1}^n \|\phi_{\vec{e}_{\sigma^{-1}(i)}}\|_{D_2}^{\alpha(i)}} \\
&= \frac{\|\phi_{\vec{0}}\|_{D_2}^{\sum \alpha_i - 1} \|\phi_{\sigma(\vec{\alpha})}\|_{D_2}}{\prod_{i=1}^n \|\phi_{\vec{e}_i}\|_{D_2}^{\alpha_{\sigma(i)}}} \tag{3.7}
\end{aligned}$$

i.e.,  $g_{D_1}^{\text{Id}}(\vec{\alpha}) = g_{D_2}^{\sigma}(\vec{\alpha})$ .

Similarly, by comparing the coefficient of  $|z_1|^{2\alpha_{\tau(1)}}|z_2|^{2\alpha_{\tau(2)}}\dots|z_n|^{2\alpha_{\tau(n)}}$ , we know

$$g_{D_1}^{\tau}(\vec{\alpha}) = g_{D_2}^{\tau \cdot \sigma}(\vec{\alpha}),$$

which implies

$$\begin{aligned}
\sum_{\tau \in S_n} g_{D_1}^{\tau}(\vec{\alpha}) \tau &= \sum_{\tau \in S_n} g_{D_2}^{\tau \cdot \sigma}(\vec{\alpha}) \cdot \tau \\
&= \sum_{\tau \in S_n} g_{D_2}^{\tau \cdot \sigma}(\vec{\alpha}) \tau \cdot \sigma \cdot \sigma^{-1} \\
&= \sigma^{-1} \left( \sum_{\tau \in S_n} g_{D_2}^{\tau}(\vec{\alpha}) \tau \right). \tag{3.8}
\end{aligned}$$

For the same reason, we have

$$\sum_{\tau \in S_n} g_{D_1}^{\tau}(\vec{\alpha}_i) \tau = \sigma^{-1} \left( \sum_{\tau \in S_n} g_{D_2}^{\tau}(\vec{\alpha}_i) \tau \right), \quad i = 1, \dots, n!. \tag{3.9}$$

From (3.9), we have  $\xi_{D_1}^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}} \sim \xi_{D_2}^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}}$  Q.E.D.

In order to prove Theorem B, we need to establish the following theorem.

**Theorem 3.4.** *Let  $D_i$ ,  $i = 1, 2$ , be two bounded complete Reinhardt domains in  $\mathbb{C}^n$ . Suppose for all  $\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}$   $n$ -tuples of nonnegative integers,  $\xi_{D_1}^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}} = \xi_{D_2}^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}}$  in  $(\mathbb{R}[S_n] \times \dots \times \mathbb{R}[S_n]) / \sim$ , i.e.,  $\exists \sigma^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}} \in S_n$  such that*

$$\xi_{D_1}^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}} = \sigma^{\alpha_1, \dots, \alpha_{n!}} \left( \xi_{D_2}^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}} \right) \quad (3.10)$$

Then, there exists  $\sigma \in S_n$  and a biholomorphic map

$$\begin{aligned} \Psi_\sigma: \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ (z_1, \dots, z_n) &\longrightarrow (a_1 z_{\sigma(i)}, \dots, a_n z_{\sigma(n)}), \end{aligned}$$

where  $a_i = \frac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e}_i}\|_{D_2}}{\|\phi_{e_{\sigma(i)}}\|_{D_2} \|\phi_{\vec{0}}\|_{D_2}}$ , such that  $B_{D_1}(z) = B_{D_2}(\Psi_\sigma(z))$ .

*Proof.* We claim that  $\exists \sigma \in S_n$  such that  $g_{D_1}^{\vec{\tau}}(\vec{\beta}) = g_{D_2}^{\vec{\tau} \circ \sigma}(\vec{\beta}) \forall \vec{\tau} \in S_n$  and  $\forall \vec{\beta}$   $n$ -tuple of nonnegative integers. For any  $\vec{\alpha}$   $n$ -tuple of nonnegative integers, let

$$I_{\vec{\alpha}} = \{\sigma \in S_n : g_{D_1}^{\vec{\tau}}(\vec{\alpha}) = g_{D_2}^{\vec{\tau} \circ \sigma}(\vec{\alpha}), \quad \forall \vec{\tau} \in S_n\}.$$

If our claim is not true, then  $\forall \sigma_i \in S_n = \{\sigma_1, \dots, \sigma_{n!}\}$ ,  $\exists \vec{\alpha}_i$ ,  $n$ -tuple of nonnegative integers such that  $\sigma_i \notin I_{\vec{\alpha}_i}$ . It follows that

$$\bigcup_{i=1}^{n!} (S_n \setminus I_{\vec{\alpha}_i}) = S_n,$$

which implies  $\bigcap_{i=1}^{n!} I_{\vec{\alpha}_i} = \emptyset$ . On the other hand for these  $n$ -tuples of integers  $\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}$ , we have  $\xi_{D_1}^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}} = \xi_{D_2}^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}}$ , i.e.,  $\exists \sigma^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}} \in S_n$  such that (3.10) hold. Let  $\tau_{(\vec{\alpha}_1, \dots, \vec{\alpha}_{n!})} = (\sigma^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}})^{-1}$ . In view of (3.10), one easily sees that  $\tau_{(\vec{\alpha}_1, \dots, \vec{\alpha}_{n!})} \in \bigcap_{i=1}^{n!} I_{\vec{\alpha}_i}$ . This leads to a contradiction and our claim is proven.

We now take  $\sigma$  in the claim and let

$$\begin{aligned} \Psi_\sigma: \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ (z_1, \dots, z_n) &\longrightarrow (a_1 z_{\sigma(i)}, \dots, a_n z_{\sigma(n)}), \end{aligned}$$

where  $a_i = \frac{\|\phi_{\bar{0}}\|_{D_1} \|\phi_{\bar{e}_i}\|_{D_2}}{\|\phi_{e_{\sigma(i)}}\|_{D_2} \|\phi_{\bar{0}}\|_{D_2}}$ . After computation as in the proof of Theorem A, we get

$$\|\phi_{\bar{0}}\|_{D_1}^2 \sum \frac{|z_1|^{2\alpha_1} \cdots |z_n|^{2\alpha_n}}{\|\phi_{\bar{\alpha}}\|_{D_1}} = \|\phi_{\bar{0}}\|_{D_2}^2 \sum \frac{\prod_{i=1}^n |a_i z_{\sigma(i)}|^{2\alpha_i}}{\|\phi_{\sigma(\bar{\alpha})}\|_{D_2}^2}.$$

It follows that  $B_{D_1}(z) = B_{D_2}(\Psi_{\sigma}(z))$ . Q.E.D.

Now we are ready to prove Theorem B.

*Proof of Theorem B.* It is easy to see that the Bergman function  $B_{D_i}$  is zero at the origin and  $0 < B_{D_i} < 1$  on  $D_i \setminus \{(0, 0, \dots, 0)\}$ . In view of Proposition 2.6,  $B_{D_i}$  is identically equal to 1 on  $\partial D_i$ . By Theorem 3.4, there exists a biholomorphic map  $\Psi_{\sigma}$  from  $\mathbb{C}^n$  to  $\mathbb{C}^n$  such that  $B_{D_1}(z) = B_{D_2}(\Psi_{\sigma}(z))$ . In particular  $\Psi_{\sigma}$  preserves the level sets of the Bergman functions. It follows that  $\Psi_{\sigma}$  sends  $\partial D_1$  to  $\partial D_2$ . Q.E.D.

#### 4. COMPLETE CONTINUOUS NUMERICAL INVARIANTS

We shall now construct the complete continuous numerical invariants for complete Reinhardt domains with  $C^1$  boundaries. For  $n = 2$ , we have Theorem C stated in §1.

*Proof of Theorem C.* Since (1.3), (1.4) and (1.5) hold for all non-negative integers  $\alpha_i, \beta_i$ , we have either

$$(1) \quad \begin{aligned} g_{D_1}(\alpha_1, \alpha_2) &= g_{D_2}(\alpha_1, \alpha_2), & g_{D_1}(\alpha_2, \alpha_1) &= g_{D_2}(\alpha_2, \alpha_1), \\ g_{D_1}(\beta_1, \beta_2) &= g_{D_2}(\beta_1, \beta_2), & g_{D_1}(\beta_2, \beta_1) &= g_{D_2}(\beta_2, \beta_1), \end{aligned}$$

or

$$(2) \quad \begin{aligned} g_{D_1}(\alpha_1, \alpha_2) &= g_{D_2}(\alpha_2, \alpha_1), & g_{D_1}(\alpha_2, \alpha_1) &= g_{D_2}(\alpha_1, \alpha_2), \\ g_{D_1}(\beta_1, \beta_2) &= g_{D_2}(\beta_2, \beta_1), & g_{D_1}(\beta_2, \beta_1) &= g_{D_2}(\beta_1, \beta_2), \end{aligned}$$

for all nonnegative integers  $\alpha_i, \beta_i$ . It is easy to see that in both cases, we get

$$\begin{aligned} \xi_{D_1}^{\bar{\alpha}, \bar{\beta}} &:= \left( g_{D_1}(\alpha_1, \alpha_2) \cdot \text{Id} + g_{D_1}(\alpha_2, \alpha_1) \cdot \sigma, \right. \\ &\quad \left. g_{D_1}(\beta_1, \beta_2) \cdot \text{Id} + g_{D_1}(\beta_2, \beta_1) \cdot \sigma \right) \\ &\sim \xi_{D_2}^{\bar{\alpha}, \bar{\beta}} := \left( g_{D_2}(\alpha_1, \alpha_2) \cdot \text{Id} + g_{D_2}(\alpha_2, \alpha_1) \cdot \sigma, \right. \\ &\quad \left. g_{D_2}(\beta_1, \beta_2) \cdot \text{Id} + g_{D_2}(\beta_2, \beta_1) \cdot \sigma \right) \end{aligned} \quad (4.1)$$

where  $S_2 = \{\text{Id}, \sigma\}$ . In view of Theorem B, we know that  $D_1$  is biholomorphic to  $D_2$ .

Conversely if  $D_1$  is biholomorphic to  $D_2$ , then by Theorem A, (4.1) holds. Then it is easy to check that (1.3), (1.4) and (1.5) hold. Q.E.D.

Theorem C says that

$$\begin{aligned} &g_D(\alpha_1, \alpha_2) + g_D(\alpha_2, \alpha_1), \\ &g_D(\alpha_1, \alpha_2) \cdot g_D(\alpha_2, \alpha_1) \end{aligned}$$

and

$$(g_D(\alpha_1, \alpha_2) - g_D(\alpha_2, \alpha_1))(g_D(\beta_1, \beta_2) - g_D(\beta_2, \beta_1)),$$

$\forall \alpha_i, \beta_i \geq 0$ ,  $\alpha_i, \beta_i \in \mathbb{Z}$ , are complete numerical biholomorphic invariants of bounded complete Reinhardt pseudoconvex domains in  $\mathbb{C}^n$  with  $C^1$  boundaries. If we want to find the complete numerical biholomorphic invariants for bounded complete Reinhardt pseudoconvex domains in  $\mathbb{C}^n$ ,  $n \geq 3$ , we need to consider the following problem.

Let  $S_n$  be the symmetric group of order  $n$ .  $S_n$  acts on  $\mathbb{C}[x_\sigma, \dots, y_\sigma]$  in the following manner.

$$\tau \in S_n, \quad \tau(x_\sigma) = x_{\sigma \cdot \tau}, \dots, \tau(y_\sigma) = y_{\sigma \cdot \tau}$$

Let  $R$  be subring of invariants in  $\mathbb{C}[x_\sigma, \dots, y_\sigma]$ . In other words,

$$R := \mathbb{C}[x_\sigma, \dots, y_\sigma]^{S_n} = \{f \in \mathbb{C}[x_\sigma, \dots, y_\sigma] : \forall \tau \in S_n, \tau(f) = f\}$$

We want to know whether  $R$  is finitely generated and what are the generators of  $R$ . Actually the first problem is a special case of Hilbert's 14<sup>th</sup> problem. At the International Congress of Mathematicians at Paris in 1900, David Hilbert asked the following question.

*Hilbert 14<sup>th</sup> Problem.* If an algebraic group acts linearly on a polynomial ring in finitely many variables, is the ring of invariants always finitely generated?

Recall that an algebraic group  $G$  is said to be linearly reductive if, for every epimorphism  $\phi: V \rightarrow W$  of  $G$  representations, the induced map on  $G$ -invariants  $\phi^G: V^G \rightarrow W^G$  is surjective. The answer to the Hilbert 14<sup>th</sup> problem is positive if the group  $G$  is linearly reductive [Hil90]. In fact, the answer is also positive if  $G$  is additive (non-reductive) group  $\mathbb{C}$  (or, more generally, the field  $k$ . See [Wei32] and [Ses62]). In view of the following lemma, we know that finding the complete numerical biholomorphic invariants is equivalent to finding the generators of  $R$ .

**Lemma 4.1.** *Let the symmetric group  $S_n = \{\sigma_1, \dots, \sigma_n\}$  act on  $\mathbb{C}^{n!n!}$  via the following formula*

$$\tau \in S_n, \quad p = (x_{\sigma_1}, \dots, x_{\sigma_n!}, \dots, y_{\sigma_1}, \dots, y_{\sigma_n!}) \in \mathbb{C}^{n!n!}$$

$$\tau(p) = (x_{\sigma_1\tau}, \dots, x_{\sigma_{n!}\tau}, \dots, y_{\sigma_1\tau}, \dots, y_{\sigma_{n!}\tau}).$$

Let  $\pi: \mathbb{C}^{n!n!} \rightarrow \mathbb{C}^{n!n!}/S_n$  be the quotient map. For any  $p$  and  $p'$  in  $\mathbb{C}^{n!n!}$ , the following two statements are equivalent

- (1)  $\pi(p) = \pi(p')$
- (2) For any  $f \in R = \mathbb{C}[x_{\sigma_1}, \dots, x_{\sigma_{n!}}, y_{\sigma_1}, \dots, y_{\sigma_{n!}}]^{S_n}$ ,  $f(p) = f(p')$ .

*Proof.* See p. 167 Theorem 5.16 of [Muk03].

In view of Lemma 4.1, we want to show that  $R = \mathbb{C}[x_1, \dots, x_{n!}, \dots, y_1, \dots, y_{n!}]^{S_n}$  is finitely generated.

**Proposition 4.2.** *Every finite group is linearly reductive.*

**Theorem 4.3** (Hilbert [Hil90]). *For linearly reductive group, the ring of invariant polynomials is finitely generated.*

From Proposition 4.2 and Theorem 4.3, we know that

$$R = \mathbb{C}[x_{\sigma_1}, \dots, x_{\sigma_{n!}}, \dots, y_{\sigma_1}, \dots, y_{\sigma_{n!}}]^{S_n}$$

is finitely generated. Now we assume that the generators of  $R$  are  $f_1(x_\sigma, \dots, y_\sigma), \dots, f_N(x_\sigma, \dots, y_\sigma)$  where  $\sigma$  runs over all the elements in  $S_n$ . We want to show that

$$f_1(g_D^\sigma(\vec{\alpha}_1), \dots, g_D^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n}, \dots, f_N(g_D^\sigma(\vec{\alpha}_1), \dots, g_D^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n},$$

for all  $n$ -tuples of nonnegative integers  $\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}$ , are complete numerical invariants of bounded complete Reinhardt pseudoconvex domains in  $\mathbb{C}^n$ .

*Proof of Theorem E.* Let  $D_1, D_2$  be two bounded complete Reinhardt domains in  $\mathbb{C}^n$ . If  $D_1$  is biholomorphic to  $D_2$ , then there exists  $\tau \in S_n$  such that  $g_{D_1}^\sigma(\vec{\alpha}) = g_{D_2}^{\sigma\tau}(\vec{\alpha})$ ,  $\forall \sigma \in S_n$ ,  $\forall \vec{\alpha}$   $n$ -tuple of nonnegative integers. We take

$$P_i = (g_{D_i}^{\sigma_1}(\vec{\alpha}_1), \dots, g_{D_i}^{\sigma_{n!}}(\vec{\alpha}_1), \dots, g_{D_i}^{\sigma_1}(\vec{\alpha}_{n!}), \dots, g_{D_i}^{\sigma_{n!}}(\vec{\alpha}_{n!})) \in \mathbb{C}^{n!n!}$$

where  $i = 1, 2$  and  $S_n = \{\sigma_1, \sigma_2, \dots, \sigma_{n!}\}$ . Then  $\pi(P_1) = \pi(P_2)$  where  $\pi: \mathbb{C}^{n!n!} \rightarrow \mathbb{C}^{n!n!}/S_n$  is the quotient map in Lemma 4.1. It follows that

$$f_i(g_{D_1}^\sigma(\vec{\alpha}_1), \dots, g_{D_1}^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n} = f_i(g_{D_2}^\sigma(\vec{\alpha}_1), \dots, g_{D_2}^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n},$$

$$1 \leq i \leq N.$$

Q.E.D.

*Proof of Theorem F.* Since

$$f_i(g_{D_1}^\sigma(\vec{\alpha}_1), \dots, g_{D_1}^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n} = f_i(g_{D_2}^\sigma(\vec{\alpha}_1), \dots, g_{D_2}^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n},$$

$$i = 1, 2, \dots, N$$

and  $\{f_1, \dots, f_N\}$  generates  $\mathbb{C}[x_{\sigma_1}, \dots, x_{\sigma_{n!}}, \dots, y_{\sigma_1}, \dots, y_{\sigma_{n!}}]^{S_n}$ , we know by Lemma 4.1 that

$$P_1 = (g_{D_1}^{\sigma_1}(\vec{\alpha}_1), \dots, g_{D_1}^{\sigma_{n!}}(\vec{\alpha}_1), \dots, g_{D_1}^{\sigma_1}(\vec{\alpha}_{n!}), \dots, g_{D_1}^{\sigma_{n!}}(\vec{\alpha}_{n!}))$$

and  $P_2 = (g_{D_2}^{\sigma_1}(\vec{\alpha}_1), \dots, g_{D_2}^{\sigma_{n!}}(\vec{\alpha}_{n!}), \dots, g_{D_2}^{\sigma_1}(\vec{\alpha}_{n!}), \dots, g_{D_2}^{\sigma_{n!}}(\vec{\alpha}_{n!}))$  are in the same  $S_n$  orbit. Hence there exists  $\sigma^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}} \in S_n$  such that

$$g_{D_1}^{\sigma}(\vec{\alpha}_1) = g_{D_2}^{\sigma \cdot \sigma^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}}}(\vec{\alpha}_1), \dots, g_{D_1}^{\sigma}(\vec{\alpha}_{n!}) = g_{D_2}^{\sigma \cdot \sigma^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}}}(\vec{\alpha}_{n!}) \quad (4.2)$$

(4.2) implies that  $\xi_{D_1}^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}} = \xi_{D_2}^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}}$  in  $(\mathbb{R}[S_n] \times \dots \times \mathbb{R}[S_n]) / \sim$ . By Theorem 3.4, there exists  $\sigma \in S_n$  and a biholomorphic map  $\Psi_\sigma: \mathbb{C}^n \rightarrow \mathbb{C}^n$  such that  $B_{D_1}(z) = B_{D_2}(\Psi_\sigma(z))$ . The proof of Theorem B shows that  $\Psi_\sigma$  sends  $D_1$  onto  $D_2$ . Q.E.D.

For complete Reinhardt pseudoconvex domains with real analytic boundaries, we can use fewer numerical invariants to characterize these domains.

*Proof of Theorem B'.*

“ $\Rightarrow$ ” This follows from the proof of Theorem A.

“ $\Leftarrow$ ” Suppose that for all  $\vec{\alpha}$   $n$ -tuple nonnegative integers,

$$\xi_{D_1}^{\vec{\alpha}} = \xi_{D_2}^{\vec{\alpha}} \text{ in } R[S_n] / \sim.$$

Then  $\exists \sigma_{\vec{\alpha}} \in S_n$  such that

$$\begin{aligned} \sum_{\tau \in S_n} g_{D_1}^{\tau}(\vec{\alpha})\tau &= \sigma_{\vec{\alpha}} \left( \sum_{\tau \in S_n} g_{D_2}^{\tau}(\vec{\alpha})\tau \right) \\ &= \sum_{\tau \in S_n} g_{D_2}^{\tau}(\vec{\alpha})\tau \cdot \sigma_{\vec{\alpha}}, \\ &= \sum_{\tau \in S_n} g_{D_2}^{\tau \cdot \sigma_{\vec{\alpha}}^{-1}}(\vec{\alpha})\tau \\ &\Rightarrow g_{D_1}^{\tau}(\vec{\alpha}) = g_{D_2}^{\tau \cdot \sigma_{\vec{\alpha}}^{-1}} \end{aligned} \quad (4.3)$$

Let  $S_n = \{\tau_1, \dots, \tau_{n!}\}$ . Denote  $I_j = \{\vec{\alpha} \in (\mathbb{Z}_+)^n: \sigma_{\vec{\alpha}}^{-1} = \tau_j\}$ ,  $j = 1, 2, \dots, n!$ . We shall introduce the concept of partial Bergman function in the following manner. Let

$$\Theta_D^{(j)}(z) = \sum_{\vec{\alpha} \in I_j} \frac{\prod_{k=1}^n |z_k|^{2\alpha_k}}{\|\phi_{\vec{\alpha}}\|_D^2}, \quad B_D^{(j)}(\vec{z}) = \frac{\Theta^{(j)}(z)}{\frac{1}{\|\phi_{\vec{\alpha}}\|_D^2} + \Theta^{(j)}(z)}.$$

Clearly we have

$$\Theta_D(z) = \sum_{\vec{\alpha} \neq 0} \frac{\prod_{k=1}^n |z_k|^{2\alpha_k}}{\|\phi_{\vec{\alpha}}\|_D^2} = \sum_{j=1}^{n!} \Theta_D^{(z)}(z) \quad (4.4)$$

In view of (4.3), by routine computation we get

$$\|\Phi_{\vec{0}}\|_{D_1}^2 \Theta_{D_1}^{(j)}(z) = \|\phi_{\vec{0}}\|_{D_2}^2 \Theta_{D_2}^{(j)}(\Psi_{\tau_j}(z)) \quad (4.5)$$

and

$$B_{D_1}^{(j)}(z) = B_{D_2}^{(j)}(\Psi_{\tau_j}(z)) \quad (4.6)$$

where

$$\begin{aligned} \Psi_{\tau_j} : \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ (z_1, \dots, z_n) &\longrightarrow (a_1 z_{\tau_j(1)}, \dots, a_n z_{\tau_j(n)}) \\ a_i &= \frac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e}_i}\|_{D_2}}{\|\phi_{e_{\tau_j(i)}}\|_{D_1} \|\phi_{\vec{0}}\|_{D_2}} \end{aligned}$$

By Proposition 2.6, the Bergman kernel and hence  $\Theta_{D_1}$  blows up at the boundary points. It follows from (4.4) that there exists some  $m$  such that  $\Theta_{D_1}^{(m)}$  is infinite on a nonempty open set  $\mathcal{U}_1$  of  $\partial D_1$ . It follows that  $B_{D_1}^{(m)}(z)$  is equal to 1 in the nonempty open set  $\mathcal{U}_1$  of  $\partial D_1$ . Recall that by the definition,  $B_{D_i}^{(m)}$  is zero at the origin and is  $0 < B_{D_i}^{(m)} < 1$  on  $D_i \setminus \{(0, \dots, 0)\}$ . Since  $B_{D_1}^{(m)}(z) = B_{D_2}^{(m)}(\Psi_{\tau_m}(z))$ , we see immediately that  $\Psi_{\tau_m}$  sends  $\mathcal{U}_1 \subseteq \partial D_1$  to an open subset of  $\partial D_2$ . In view of the fact that  $\partial D_i$ ,  $i = 1, 2$ , are real analytic and compact, it follows easily that  $\Psi_{\tau_m}$  sends  $\partial D_1$  to  $\partial D_2$ . Q.E.D.

*Proof of Theorem C'.* This follows immediately from Theorem B'. Q.E.D.

*Proof of Theorem F'.*

“ $\Rightarrow$ ” This follows from the proof of Theorem E.

“ $\Leftarrow$ ” Suppose that for all  $\vec{\alpha}$   $n$ -tuple nonnegative integers, we have

$$f_i(g_{D_1}^{\vec{\alpha}})_{\sigma \in S_n} = f_i(g_{D_2}^{\vec{\alpha}})_{\sigma \in S_n}, \quad i = 1, \dots, N.$$

Then by the analogy of Lemma 4.1, we have

$$\pi(g_{D_1}^{\tau_1}(\vec{\alpha}), \dots, g_{D_1}^{\tau_{n!}}(\vec{\alpha})) = \pi(g_{D_2}^{\tau_1}(\vec{\alpha}), \dots, g_{D_2}^{\tau_{n!}}(\vec{\alpha}))$$

when  $\pi : \mathbb{C}^{n!} \longrightarrow \mathbb{C}^{n!}/S_n$ . Therefore there exist  $\sigma_{\vec{\alpha}}$  such that  $g_{D_1}^{\vec{\alpha}} = g_{D_2}^{\sigma_{\vec{\alpha}}}(\vec{\alpha})$  for all  $\sigma \in S_n$ . By the proof similar to the proof of Theorem B', we get a biholomorphic map  $\Psi_{\tau}$  which sends  $D_1$  onto  $D_2$ . Q.E.D.

Actually, the generators of  $R$  can be listed explicitly by a theorem discovered by M. Göbel ([Göb95]). Before we give the statement of Göbel's theorem, we need to introduce some definitions.

**Definition 4.4.** *Suppose that a finite group  $G$  acts as permutations on a finite set  $X$ . We then refer to  $X$  together with the  $G$ -action as a finite  $G$ -set. A subset  $B \subset X$  is called an orbit if  $G$  permutes the elements of  $B$  among themselves and the induced permutation action of  $G$  on  $B$  is transitive. The formal sum  $\sum_{x \in B} x$  is called orbit sum of any  $x \in B$ .*

**Definition 4.5.** *If  $K = (k_1, \dots, k_n)$  is an  $n$ -tuple of nonnegative integers, then  $K$  is called an exponent sequence. The associated partition of  $K$  is the ordered set consisting of the  $n$  numbers  $k_1, \dots, k_n$  rearranged in weakly decreasing order. We denote by  $\lambda(K)$  the partition associated to  $K$ , so*

$$\lambda(K) = (\lambda_1(K) \geq \lambda_2(K) \geq \dots \geq \lambda_n(K))$$

*and the  $n$ -tuple  $(\lambda_1(K), \lambda_2(K), \dots, \lambda_n(K))$  is a permutation of  $k_1, \dots, k_n$ . The monomial  $x^K$  is called special if the associated partition  $\lambda(K)$  of the exponent sequence  $K$  satisfies*

- (1)  $\lambda_i(K) - \lambda_{i+1}(K) \leq 1$  for all  $i = 1, \dots, n-1$ , and
- (2)  $\lambda_n(K) = 0$ .

Notice that if two exponent sequences  $A$  and  $B$  are permutations of each other, then  $\lambda(A) = \lambda(B)$ .

**Theorem 4.6** (M. Göbel). *Let  $G$  be a finite group,  $X$  a finite  $G$ -set, and  $R$  a commutative ring. Then the ring of invariants  $R[X]^G$  is generated as an algebra by  $e_{|X|} = \prod_{x \in X} x$ , the top degree elementary symmetric polynomial in the elements of  $X$ , and the orbit sums of special monomials.*

So, using Göbel's theorem to our main theorems we completely give numerical invariants for our domains.

## 5. APPLICATION IN CONCRETE EXAMPLES

In [Hua08], Huang communicated to us the following state of art of determining when two complete Reinhardt domains is biholomorphically equivalent. Assume that  $D_1, D_2$  are two bounded Reinhardt domains.  $D_i$  is defined by real analytic function  $f_i(|z_1|, \dots, |z_n|) < 0$ . Let  $T(D_i) = \{(|z_1|, \dots, |z_n|) : (z_1, \dots, z_n) \in D_i\}$ . By Sunada's Theorem,  $D_1$  is biholomorphic to  $D_2$  if and only if there is a permutation  $\sigma \in S_n$  such that  $\sigma(T(D_1)) = T(D_2)$ . Thus the biholomorphic problem for

Reinhardt domains is converted to the identification problem up to permutation for domains in  $(\mathbb{R}^+)^n$  with points  $P_j = (0, \dots, 0, 1, 0, \dots, 0)$  (in the  $j$ th component) being the boundary points. This method is effective when  $f_i$ ,  $i = 1, 2$  are irreducible polynomial. For example, let

$$D_1 = \{(z_1, z_2) \in \mathbb{C}^2 : f(|z_1|, |z_2|) = |z_1|^N + |z_2|^N + \sum_{j=1}^{N-1} e_j |z_1|^j |z_2|^{N-j} - 1 < 0\}$$

$$D_2 = \{(z_1, z_2) \in \mathbb{C}^2 : f(|z_1|, |z_2|) = |z_1|^N + |z_2|^N + \sum_{j=1}^{N-1} e_j |z_1|^j |z_2|^{N-j} - 1 < 0\}$$

under the assumption that  $f_1, f_2$  are irreducible, we get that  $D_1$  is biholomorphic to  $D_2$  if and only if either  $(e_1, \dots, e_{N-1}) = (c_1, \dots, c_{N-1})$  or  $(e_1, \dots, e_{N-1}) = (c_{N-1}, c_{N-2}, \dots, c_1)$ . But this method cannot handle the case when  $f_i$  is not irreducible.

**Example 1.** Let

$$D_1 = \{z \in \mathbb{C}^2 : f_1(|z_1|, |z_2|) = (|z_1||z_2| + |z_1| + |z_2| - 1) \left( \frac{1}{2} |z_1||z_2| + |z_1| + |z_2| - 1 \right) < 0\}$$

$$D_2 = \{z \in \mathbb{C}^2 : f_2(|z_1|, |z_2|) = (|z_1||z_2| + |z_1| + |z_2| - 1) \left( \frac{1}{3} |z_1||z_2| + |z_1| + |z_2| - 1 \right) < 0\}.$$

Notice that  $P_1 = (1, 0)$  and  $P_2 = (0, 1)$  are both the outermost boundary points for  $D_1$  and  $D_2$ . Also,  $T(D_j)$  are invariant under the action of  $S_2$ . Hence,  $D_1$  and  $D_2$  are holomorphically equivalent if and only if  $T(D_1) = T(D_2)$  in  $(\mathbb{R}^+)^n$ . But this is impossible, for  $(1/3, 4/7)$  is a boundary point of  $T(D_1)$  while it is an interior point of  $T(D_2)$ . From this example, we can see that there is no easy way for this method to handle the biholomorphic equivalent problem for general Reinhardt domains.

However we can easily use our theory to show that  $D_1$  is not biholomorphic to  $D_2$  as follows. Let

$$\begin{aligned} \|\phi_{\bar{\alpha}}\|_D^2 &= \int_D \phi_{\bar{\alpha}} \wedge \bar{\phi}_{\bar{\alpha}} = \int_D |z_1|^{2\alpha_1} |z_2|^{2\alpha_2} dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \\ &= (4\pi)^2 \iint_{\{(r_1, r_2) : f(r_1, r_2) < 0\}} r_1^{2\alpha_1+1} r_2^{2\alpha_2+1} dr_1 dr_2. \end{aligned}$$

By computation, we get

$$\begin{aligned}\|\phi_{\vec{0}}\|_{D_1}^2 &= \left(-\frac{57}{4} + 42 \ln 3 - 46 \ln 2\right) \cdot (4\pi)^2 \\ \|\phi_{(1,0)}\|_{D_1}^2 &= \|\phi_{(0,1)}\|_{D_1}^2 = \left(-\frac{2903}{24} + 312 \ln 3 - 320 \ln 2\right) \cdot (4\pi)^2 \\ \|\phi_{(2,0)}\|_{D_1}^2 &= \|\phi_{(0,2)}\|_{D_1}^2 = \left(1824 \ln 3 + 836 \ln 2 - \frac{2925}{4}\right) \cdot (4\pi)^2 \\ \|\phi_{\vec{0}}\|_{D_2}^2 &= (-49 + 356 \ln 2 - 180 \ln 3) \cdot (4\pi)^2 \\ \|\phi_{(1,0)}\|_{D_2}^2 &= \|\phi_{(0,1)}\|_{D_2}^2 = \left(5824 \ln 2 - \frac{2500}{3} - 2916 \ln 3\right) \cdot (4\pi)^2 \\ \|\phi_{(2,0)}\|_{D_2}^2 &= \|\phi_{(0,2)}\|_{D_2}^2 = \left(-\frac{163457}{45} + 75804 \ln 2 - 37908 \ln 3\right) \cdot (4\pi)^2.\end{aligned}$$

Let  $S_2 = \{\tau_1 = \text{Id}, \tau_2 = (1, 2)\}$  be the symmetric group of degree 2. We shall take  $\vec{\beta} = (2, 0)$ . Then

$$\begin{aligned}g_{D_i}(2, 0) &= \frac{\|\phi_{\vec{0}}\|_{D_i} \|\phi_{(2,0)}\|_{D_i}}{\|\phi_{(1,0)}\|_{D_i}^2}, \quad i = 1, 2 \\ g_{D_i}(0, 2) &= \frac{\|\phi_{\vec{0}}\|_{D_i} \|\phi_{(0,2)}\|_{D_i}}{\|\phi_{(0,1)}\|_{D_i}^2} = g_{D_i}(2, 0), \quad i = 1, 2\end{aligned}$$

Since  $g_{D_1}(2, 0) + g_{D_1}(0, 2)$ , which is approximately equal to 25.277 is not equal to  $g_{D_2}(2, 0) + g_{D_2}(0, 2)$ , which is approximately equal to 2.637, we conclude that  $D_1$  is not biholomorphic to  $D_2$  by Theorem C.

In 1907 Poincaré discovered the following theorem. [Ran86, p. 24]

**Theorem 5.1** (Poincaré). *Let  $B_n = \{z \in \mathbb{C}^n \mid \sum_{i=1}^n |z_i|^2 < 1\}$  be the unit ball and  $\Delta_n = \{z \in \mathbb{C}^n \mid |z_i| < 1, i = 1, \dots, n\}$  be the unit polydisc in  $\mathbb{C}^n$ , then there exists no biholomorphic map between  $\Delta_n$  and  $B_n$ .*

Using our method, we can easily get this result.

Notation:  $\vec{0} = (0, \dots, 0)$ ,  $\vec{e}_i = (0, \dots, 1, \dots, 0)$ ,  $\vec{\alpha} = (2, 0, \dots, 0)$ .

*Proof of Theorem 5.1.*

For any  $\vec{\beta} \in \mathbb{Z}^n$ ,  $D = \{z \in \mathbb{C}^n \mid f(z_1, \dots, z_n) < 0\}$  is the bounded Reinhardt domain.

$$\begin{aligned}\|\phi_{\vec{\beta}}\|_D &= \int_D \phi_{\vec{\beta}} \wedge \bar{\phi}_{\vec{\beta}} \\ &= \int_D |z_1|^{2\beta_1} \dots |z_n|^{2\beta_n} dz_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_1 \wedge \dots \wedge d\bar{z}_n\end{aligned}$$

$$= (4\pi)^n \int \cdots \int_{\{\bar{r}: f(r_1, \dots, r_n) < 0\}} \prod_{j=1}^n r_j^{2\beta_j+1} dr_1 \wedge \cdots \wedge dr_n$$

After computation, we get

$$\begin{aligned} \|\phi_{\bar{0}}\|_{\Delta_n}^2 &= (4\pi^n) \int_{\Delta_n} \cdots \int r_1 r_2 \cdots r_n dr_1 \cdots dr_n = (2\pi)^n \\ \|\phi_{\bar{e}_1}\|_{\Delta_n}^2 &= (4\pi^n) \int_{\Delta_n} \cdots \int r_1^3 r_2 \cdots r_n dr_1 \cdots dr_n = \frac{1}{2} \cdot (2\pi)^n \\ \|\phi_{\bar{\alpha}}\|_{\Delta_n}^2 &= (4\pi^n) \int_{\Delta_n} \cdots \int r_1^5 r_2 \cdots r_n dr_1 \cdots dr_n = \frac{1}{3} \cdot (2\pi)^n \\ \|\phi_{\bar{0}}\|_{B_n}^2 &= (4\pi^n) \int_{B_n} \cdots \int r_1 r_2 \cdots r_n dr_1 \cdots dr_n = \frac{(4\pi)^n}{\prod_{i=1}^n 2i} \\ \|\phi_{\bar{e}_1}\|_{B_n}^2 &= (4\pi^n) \int_{B_n} \cdots \int r_1^3 r_2 \cdots r_n dr_1 \cdots dr_n = \frac{(4\pi)^n}{\prod_{i=1}^{n+1} 2i} \\ \|\phi_{\bar{\alpha}}\|_{B_n}^2 &= (4\pi^n) \int_{B_n} \cdots \int r_1^5 r_2 \cdots r_n dr_1 \cdots dr_n = \frac{(4\pi)^n}{\prod_{i=3}^{n+2} 2i} \end{aligned}$$

For  $\Delta_n$  and  $B_n$  are symmetric domains

$$\begin{aligned} \|\phi_{\bar{e}_i}\|_{\Delta_n} &= \|\phi_{\bar{e}_j}\|_{\Delta_n}, \quad \|\phi_{\bar{e}_i}\|_{B_n} = \|\phi_{\bar{e}_j}\|_{B_n}, \quad \forall i, j \in \{1, \dots, n\} \\ \|\phi_{\bar{\alpha}}\|_{\Delta_n} &= \|\phi_{\tau(\bar{\alpha})}\|_{\Delta_n}, \quad \|\phi_{\bar{\alpha}}\|_{B_n} = \|\phi_{\tau(\bar{\alpha})}\|_{B_n}, \quad \forall \tau \in S_n \end{aligned}$$

$$\begin{aligned} g_{\Delta_n}^{\tau}(\bar{\alpha}) &= g_{\Delta_n}^{\text{Id}}(\bar{\alpha}) = \frac{\|\phi_{\bar{0}}\|_{\Delta_n}^{\sum \alpha_i - 1} \|\phi_{\bar{\alpha}}\|_{\Delta_n}}{\prod_{i=1}^n \|\phi_{\bar{e}_i}\|_{\Delta_n}^{\alpha_i}} = \frac{2\sqrt{3}}{3} \\ g_{B_n}^{\tau}(\bar{\alpha}) &= g_{B_n}^{\text{Id}}(\bar{\alpha}) = \frac{\|\phi_{\bar{0}}\|_{B_n}^{\sum \alpha_i - 1} \|\phi_{\bar{\alpha}}\|_{B_n}}{\prod_{i=1}^n \|\phi_{\bar{e}_i}\|_{B_n}^{\alpha_i}} = \frac{\sqrt{2(n+1)(n+2)}}{n+2} \end{aligned} \quad (5.1)$$

So  $\sum_{\tau \in S_n} g_{\Delta_n}^{\tau}(\bar{\alpha}) \cdot \tau \neq \sum_{\tau \in S_n} g_{B_n}^{\tau}(\bar{\alpha}) \cdot \tau$  in  $\mathbb{R}[S_n]/\sim$ .

Then  $\Delta_n$  is not biholomorphic to  $B_n$  by Theorem A.

Q.E.D.

**Example 2.** In addition, we can show  $A_n = \{z \in \mathbb{C}^n \mid \sum_{i=1}^n |z_i| < 1\}$  is not biholomorphic to  $B_n$

$$\begin{aligned} \|\phi_{\vec{0}}\|_{A_n} &= (4\pi^n) \int_{A_n} \cdots \int r_1 r_2 \cdots r_n dr_1 \cdots dr_n = \frac{(4\pi)^n}{\prod_{i=1}^{2n} i} \\ \|\phi_{\vec{e}_1}\|_{A_n} &= (4\pi^n) \int_{A_n} \cdots \int r_1^3 r_2 \cdots r_n dr_1 \cdots dr_n = \frac{(4\pi)^n}{2^{(n+1)} \prod_{i=4}^{2n} i} \\ \|\phi_{\vec{\alpha}}\|_{A_n} &= (4\pi^n) \int_{A_n} \cdots \int r_1^5 r_2 \cdots r_n dr_1 \cdots dr_n = \frac{(4\pi)^n}{2^{(n+2)} \prod_{i=6}^{2n} i} \end{aligned}$$

Using the same method, we get

$$g_{A_n}^{\vec{\tau}}(\vec{\alpha}) = g_{A_n}^{\text{Id}}(\vec{\alpha}) = \frac{\|\phi_{\vec{0}}\|_{A_n}^{\sum \alpha_i - 1} \|\phi_{\vec{\alpha}}\|_{A_n}}{\prod_{i=1}^n \|\phi_{\vec{e}_i}\|_{A_n}^{\alpha_i}} = \frac{\sqrt{5(2n+1)(2n+2)}}{\sqrt{3(2n+3)(2n+4)}}$$

So  $\sum_{\tau \in S_n} g_{B_n}^{\vec{\tau}}(\vec{\alpha}) \cdot \tau \neq \sum_{\tau \in S_n} g_{A_n}^{\vec{\tau}}(\vec{\alpha}) \cdot \tau$  in  $\mathbb{R}[S_n]/\sim$ .

Then  $B_n$  is not biholomorphic to  $A_n$  by Theorem A.

**Example 3.**  $\Delta_n$  is not biholomorphic to  $A_n$ .

According to the computation above,

$$\sum_{\tau \in S_n} g_{\Delta_n}^{\vec{\tau}}(\vec{\alpha}) \cdot \tau \neq \sum_{\tau \in S_n} g_{A_n}^{\vec{\tau}}(\vec{\alpha}) \cdot \tau \text{ in } \mathbb{R}[S_n]/\sim$$

So  $\Delta_n$  is not biholomorphic to  $A_n$ .

From the examples above, it is reasonable to suspect the problem lies with the fact that  $B_n$  has a  $C^\infty$  smooth boundary, whereas  $\Delta_n$  and  $A_n$  have ‘‘corners’’ in its boundaries. However  $\Delta_n$  is not biholomorphic to  $A_n$  though their boundaries both have ‘‘corners’’. So the biholomorphic equivalence problem of domains in  $\mathbb{C}^n$  is very complicated even in the Reinhardt domain case.

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