

NEW INVARIANTS FOR COMPLEX MANIFOLDS AND ISOLATED SINGULARITIES

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ABSTRACT. In this paper we introduce some new invariants for complex manifolds. These invariants measure in some sense how far away for the complex manifold to have global complex coordinates. For applications, we introduce two new invariants $f^{(1,1)}$ and $g^{(1,1)}$ for isolated surface singularities. We show that $f^{(1,1)} = g^{(1,1)} = 1$ for rational double points and cyclic quotient singularities.

Dedicated to Professor Michael Artin on the occasion of his 75th Birthday.

1. INTRODUCTION

Let M be a complex manifold of dimension n . It is a natural question to ask how far for this complex manifold to have global complex coordinates. In this paper we introduce some new biholomorphic invariants which give some measurements for this purpose.

Definition 1.1. For any $1 \leq p \leq n$, let Ω_M^p be the sheaf of germs of holomorphic p -forms on M . Denote by $\langle \Lambda^p \Gamma(M, \Omega_M^1) \rangle$ the linear span of all p -th wedge products of global holomorphic 1-forms on M . Define $\gamma^{(p)}(M) := \Gamma(M, \Omega_M^p) / \langle \Lambda^p \Gamma(M, \Omega_M^1) \rangle$. Then $\gamma^{(p)}$ is a biholomorphic invariant of M .

If M is a complex submanifold in \mathbb{C}^N , then given any global holomorphic p -form α on M , there exists a holomorphic p -form $\tilde{\alpha}$ on \mathbb{C}^N such that the restriction of $\tilde{\alpha}$ on M is α . Obviously, $\tilde{\alpha}$ is a p -th wedge product of holomorphic 1-forms. We see that $\gamma^{(p)}(M) = 0$. If M is a compact complex torus, then it is easy to see that $\gamma^{(p)}(M) = 0$. It is an interesting question to classify those compact complex manifolds with $\gamma^{(p)} = 0$.

The purpose of this paper is to use the above idea to study singularities. Specifically we introduce two new invariants $f^{(1,1)}$ and $g^{(1,1)}$ for isolated

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surface singularities. Numerical invariants are used for the classification of surface singularities. The fundamental invariants are the geometric genus p_g and the arithmetic genus p_a . In [Ar], Artin introduced a definition for singularity to be rational, i.e. those singularities with $p_g = 0$. Wagreich introduced a definition for singularity to be weakly elliptic, i.e. for those singularities with $p_a = 1$. Later, Laufer [La] studied the so-called minimal elliptic singularities, i.e. for those Gorenstein singularities with $p_g = 1$. In a series of papers ([Ya1], [Ya2], [Ya3], [Ya4], [Ya5]) Yau developed a novel theory of elliptic sequence to study weakly elliptic singularities which may have p_g arbitrarily large. In particular, he classified all weighted dual graphs of hypersurface singularities with $p_g = 2$ ([Ya1]). In 1982, he considered another invariant irregularity q ([Ya6], [Ya7], [Ya8]). Later Wahl ([Wah]), Straten and Steenbrink ([St-St]) studied this invariant further. The invariant $g^{(1,1)}$ in this paper was used by Du and Yau ([Du-Ya]) to solve the regularity problem of the Harvey-Lawson solution to the complex Plateau problem. In this paper, we shall give a detailed study of $f^{(1,1)}$. We also give explicit calculation for $f^{(1,1)}$ and $g^{(1,1)}$ for rational double points and cyclic quotient singularities. The following are our main results.

Theorem A: Let $(V, 0)$ be a 2-dimensional Stein space with 0 as its only singular point with \mathbb{C}^* -action. Then $f^{(1,1)} \geq 1$.

Theorem B: Let $(V, 0)$ be a 2-dimensional Stein space with 0 as its only singular point. If 0 is rational double points or cyclic quotient singularities then $f^{(1,1)} = g^{(1,1)} = 1$.

2. INVARIANTS OF SINGULARITIES

Let V be a n -dimensional complex analytic subvariety in \mathbb{C}^N with only isolated singularities. In [Ya7], Yau considered four kinds of sheaves of germs of holomorphic p -forms

- (1) $\bar{\Omega}_V^p := \pi_* \Omega_M^p$, where $\pi : M \rightarrow V$ is a resolution of singularities of V .
- (2) $\bar{\bar{\Omega}}_V^p := \theta_* \Omega_{V \setminus V_{sing}}^p$ where $\theta : V \setminus V_{sing} \rightarrow V$ is the inclusion map and V_{sing} is the singular set of V .
- (3) $\Omega_V^p := \Omega_{\mathbb{C}^N}^p / \mathcal{H}^p$, where $\mathcal{H}^p = \{f\alpha + dg \wedge \beta : \alpha \in \Omega_{\mathbb{C}^N}^p; \beta \in \Omega_{\mathbb{C}^N}^{p-1}; f, g \in \mathcal{I}\}$ and \mathcal{I} is the ideal sheaf of V in \mathbb{C}^N .
- (4) $\tilde{\Omega}_V^p := \Omega_{\mathbb{C}^N}^p / \mathcal{H}^p$, where $\mathcal{H}^p = \{\omega \in \Omega_{\mathbb{C}^N}^p : \omega|_{V \setminus V_{sing}} = 0\}$.

Clearly $\Omega_V^p, \tilde{\Omega}_V^p$ are coherent. $\bar{\Omega}_V^p$ is a coherent sheaf because π is a proper map. $\bar{\bar{\Omega}}_V^p$ is also a coherent sheaf by a theorem of Siu (cf. Theorem A of

[Si]). In case V is a normal variety, the dualizing sheaf ω_V of Grothendieck is actually the sheaf $\bar{\Omega}_V^n$.

Definition 2.1. *The Siu complex is a complex of coherent sheaves J^\bullet supported on the singular points of V which is defined by the following exact sequence*

$$0 \longrightarrow \bar{\Omega}^\bullet \longrightarrow \bar{\Omega}^\bullet \longrightarrow J^\bullet \longrightarrow 0. \quad (2.1)$$

Definition 2.2. *Let V be a n -dimensional Stein space with 0 as its only singular point. Let $\pi : (M, A) \rightarrow (V, 0)$ be a resolution of the singularity with A as exceptional set. The geometric genus p_g , the irregularity q and $g^{(p)}$ invariant of the singularity are defined as follows (cf. [Ya7], [St-St]):*

$$p_g := \dim \Gamma(M \setminus A, \Omega^n) / \Gamma(M, \Omega^n), \quad (2.2)$$

$$q := \dim \Gamma(M \setminus A, \Omega^{n-1}) / \Gamma(M, \Omega^{n-1}), \quad (2.3)$$

$$g^{(p)} := \dim \Gamma(M, \Omega_M^p) / \pi^* \Gamma(V, \Omega_V^p). \quad (2.4)$$

The s -invariant of the singularity is defined in [Lu-Ya] as follows

$$s := \dim \Gamma(M \setminus A, \Omega^n) / [\Gamma(M, \Omega^n) + d\Gamma(M \setminus A, \Omega^{n-1})]. \quad (2.5)$$

In [Lu-Ya], the authors found the relation between some of these invariants.

Lemma 2.3. ([Lu-Ya]) *Let V be a n -dimensional Stein space with 0 as its only singular point. Let $\pi : (M, A) \rightarrow (V, 0)$ be a resolution of the singularity with A as exceptional set. Let J^\bullet be the Siu complex of coherent sheaves supported on 0 . Then*

- (1) $\dim J^n = p_g$,
- (2) $\dim J^{n-1} = q$,
- (3) $\dim J^i = 0$, for $1 \leq i \leq n-2$

Proposition 2.4. ([Lu-Ya]) *Let V be a n -dimensional Stein space with 0 as its only singular point. Let $\pi : (M, A) \rightarrow (V, 0)$ be a resolution of the singularity with A as exceptional set. Let J^\bullet be the Siu complex of coherent sheaves supported on 0 . Then the s -invariant is given by*

$$s := \dim H^n(J^\bullet) = p_g - q \quad (2.6)$$

and

$$\dim H^{n-1}(J^\bullet) = 0 \quad (2.7)$$

Definition 2.5. *Let $(V, 0)$ be a 2-dimensional Stein analytic space with an isolated singularity at 0 . Let $\pi : (M, A) \rightarrow (V, 0)$ be a resolution of the singularity with A as exceptional set. Define a sheaf of germs $\bar{\Omega}_V^{1,1}$ by the sheaf associated to the presheaf*

$$U \mapsto \langle \Gamma(\pi^{-1}(U), \Omega_M^1) \wedge \Gamma(\pi^{-1}(U), \Omega_M^1) \rangle,$$

where U is an open set of V .

Lemma 2.6. *Let $(V, 0)$ be a 2-dimensional Stein analytic space with an isolated singularity at 0. Let $\pi : (M, A) \rightarrow (V, 0)$ be a resolution of the singularity with A as exceptional set. Then $\bar{\Omega}_V^{1,1}$ is coherent and there is a short exact sequence*

$$0 \longrightarrow \bar{\Omega}_V^{1,1} \longrightarrow \bar{\Omega}_V^2 \longrightarrow \mathcal{F}^{(1,1)} \longrightarrow 0 \quad (2.8)$$

where $\mathcal{F}^{(1,1)}$ is a sheaf supported on the singular point of V . Let

$$F^{(1,1)}(M) := \Gamma(M, \Omega_M^2) / \langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle, \quad (2.9)$$

then $\dim \mathcal{F}_0^{(1,1)} = \dim F^{(1,1)}(M)$.

Proof. Since the sheaf of germ $\bar{\Omega}_V^1$ is coherent by a theorem of Siu (cf Theorem A of [Si]), for any point $w \in V$ there exists an open neighborhood U of w in V such that $\Gamma(\pi^{-1}(U), \Omega_M^1)$ is finitely generated over $\Gamma(\pi^{-1}(U), \mathcal{O}_M)$. So $\Gamma(\pi^{-1}(U), \Omega_M^1) \wedge \Gamma(\pi^{-1}(U), \Omega_M^1)$ is also finitely generated over $\Gamma(\pi^{-1}(U), \mathcal{O}_M)$, which means $\bar{\Omega}_V^{1,1}$ is a sheaf of finite type. It is obvious that $\bar{\Omega}_V^{1,1}$ is a subsheaf of $\bar{\Omega}_V^2$ which is also coherent. So $\bar{\Omega}_V^{1,1}$ is also coherent.

Notice that the stalk of $\bar{\Omega}_V^{1,1}$ and $\bar{\Omega}_V^2$ coincide at each point different from the singular point 0, $\mathcal{F}^{(1,1)}$ is supported at 0. And from Cartan Theorem B

$$\dim \mathcal{F}_0^{(1,1)} = \dim \Gamma(V, \bar{\Omega}_V^2) / \Gamma(V, \bar{\Omega}_V^{1,1}) = \dim F^{(1,1)}(M).$$

Q.E.D.

Observe that $\Gamma(M, \Omega_M^2)$ and $\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle$ are birational invariants. Thus, from Lemma 2.6, we can define a local invariant of a singularity which is independent of resolution.

Definition 2.7. *Let V be a 2-dimensional Stein space with 0 as its only singular point. Let $\pi : (M, A) \rightarrow (V, 0)$ be a resolution of the singularity with A as exceptional set. Let*

$$f^{(1,1)}(0) := \dim \mathcal{F}_0^{(1,1)} = \dim F^{(1,1)}(M). \quad (2.10)$$

We will omit 0 in $f^{(1,1)}(0)$ if there is no confusion from the context.

$f^{(1,1)}$ is independent of the resolution of V .

The following proposition is to show that $f^{(1,1)}$ is bounded above by $g^{(2)}$.

Proposition 2.8. *Let V be a 2-dimensional Stein space with 0 as its only singular point. Then $f^{(1,1)} \leq g^{(2)}$.*

Proof. Since

$$\begin{aligned} f^{(1,1)} &= \dim \Gamma(M, \Omega_M^2) / \langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle, \\ g^{(2)} &:= \dim \Gamma(M, \Omega^2) / \pi^* \Gamma(V, \Omega_V^2). \end{aligned}$$

and

$$\pi^*\Gamma(V, \Omega_V^2) = \langle \pi^*\Gamma(V, \Omega_V^1) \wedge \pi^*\Gamma(V, \Omega_V^1) \rangle \subseteq \langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle,$$

the result follows. Q.E.D.

Definition 2.9. Let $(V, 0)$ be a Stein germ of a 2-dimensional analytic space with an isolated singularity at 0. Let $\pi : (M, A) \rightarrow (V, 0)$ be a resolution of the singularity with A as exceptional set and $\theta : V \setminus \{0\} \hookrightarrow V$ be the embedding map. Define a sheaf of germs $\Omega_{V \setminus 0}^{1,1}$ by the sheaf associated to the presheaf

$$U \mapsto \langle \Gamma(\pi^{-1}(U), \Omega_M^1) \wedge \Gamma(\pi^{-1}(U), \Omega_M^1) \rangle,$$

where U is an open set of $V \setminus 0$. Let $\bar{\bar{\Omega}}_V^{1,1} := \theta_*(\Omega_{V \setminus 0}^{1,1})$.

Lemma 2.10. Let V be a 2-dimensional Stein space with 0 as its only singular point in \mathbb{C}^N . Let $\pi : (M, A) \rightarrow (V, 0)$ be a resolution of the singularity with A as exceptional set. Then $\bar{\bar{\Omega}}_V^{1,1}$ is coherent and there is a short exact sequence

$$0 \longrightarrow \bar{\bar{\Omega}}_V^{1,1} \longrightarrow \bar{\bar{\Omega}}_V^2 \longrightarrow \mathcal{G}^{(1,1)} \longrightarrow 0 \quad (2.11)$$

where $\mathcal{G}^{(1,1)}$ is a sheaf supported on the singular point of V . Let

$$G^{(1,1)}(M \setminus A) := \Gamma(M \setminus A, \Omega_M^2) / \langle \Gamma(M \setminus A, \Omega_M^1) \wedge \Gamma(M \setminus A, \Omega_M^1) \rangle, \quad (2.12)$$

then $\dim \mathcal{G}_0^{(1,1)} = \dim G^{(1,1)}(M \setminus A)$.

Proof. Since the sheaf of germ $\bar{\bar{\Omega}}_V^1$ is coherent by a theorem of Siu (cf Theorem A of [Si]), for any point $w \in V$ there exists an open neighborhood U of w in V such that $\Gamma(\theta^{-1}(U), \Omega_{V \setminus 0}^1)$ is finitely generated over $\Gamma(\theta^{-1}(U), \mathcal{O}_{V \setminus 0})$. So $\Gamma((\theta \circ \pi)^{-1}(U), \Omega_{M \setminus A}^1)$ is finitely generated over $\Gamma((\theta \circ \pi)^{-1}(U), \mathcal{O}_{M \setminus A})$. Then $\Gamma((\theta \circ \pi)^{-1}(U), \Omega_{M \setminus A}^1) \wedge \Gamma((\theta \circ \pi)^{-1}(U), \Omega_{M \setminus A}^1)$ is finitely generated over $\Gamma((\theta \circ \pi)^{-1}(U), \mathcal{O}_{M \setminus A})$, which means $\bar{\bar{\Omega}}_V^{1,1}$ is a sheaf of finite type. It is obvious that $\bar{\bar{\Omega}}_V^{1,1}$ is a subsheaf of $\bar{\bar{\Omega}}_V^2$ which is also coherent. So $\bar{\bar{\Omega}}_V^{1,1}$ is coherent.

Notice that the stalk of $\bar{\bar{\Omega}}_V^{1,1}$ and $\bar{\bar{\Omega}}_V^2$ coincide at each point different from the singular point 0, $\mathcal{G}^{(1,1)}$ is supported at 0. And from Cartan Theorem B

$$\dim \mathcal{G}_0^{(1,1)} = \dim \Gamma(V, \bar{\bar{\Omega}}_V^2) / \Gamma(V, \bar{\bar{\Omega}}_V^{1,1}) = \dim G^{(1,1)}(M \setminus A).$$

Q.E.D.

Definition 2.11. Let V be a 2-dimensional Stein space with 0 as its only singular point. Let $\pi : (M, A) \rightarrow (V, 0)$ be a resolution of the singularity with A as exceptional set. Let

$$g^{(1,1)}(0) := \dim \mathcal{G}_0^{(1,1)} = \dim G^{(1,1)}(M \setminus A). \quad (2.13)$$

We will omit 0 in $g^{(1,1)}(0)$ if there is no confusion from the context.

The following proposition is to show that $g^{(1,1)}$ is bounded above.

Proposition 2.12. *Let V be a 2-dimensional Stein space with 0 as its only singular point. Then $g^{(1,1)} \leq p_g + g^{(2)}$.*

Proof. Since

$$g^{(1,1)} = \dim \Gamma(M \setminus A, \Omega_M^2) / \langle \Gamma(M \setminus A, \Omega_M^1) \wedge \Gamma(M \setminus A, \Omega_M^1) \rangle,$$

$$p_g = \dim \Gamma(M \setminus A, \Omega_M^2) / \Gamma(M, \Omega_M^2),$$

$$g^{(2)} := \dim \Gamma(M, \Omega^2) / \pi^* \Gamma(V, \Omega_V^2).$$

and

$$\begin{aligned} \pi^* \Gamma(V, \Omega_V^2) &= \langle \pi^* \Gamma(V, \Omega_V^1) \wedge \pi^* \Gamma(V, \Omega_V^1) \rangle \\ &\subseteq \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \\ &\subseteq \Gamma(M \setminus A, \Omega_M^1) \wedge \Gamma(M \setminus A, \Omega_M^1), \end{aligned} \quad (2.14)$$

the result follows. Q.E.D.

Theorem 2.13. *Let V be a 2-dimensional Stein space with 0 as its only normal singular point with \mathbb{C}^* -action. Let $\pi : (M, A) \rightarrow (V, 0)$ be a minimal good resolution of the singularity with A as exceptional set, then $f^{(1,1)} \geq 1$.*

Proof. It suffices to prove there is a holomorphic 2-form ω on M which is not contained in $\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle$. Embed $(V, 0)$ locally into \mathbb{C}^m and let z_1, \dots, z_m be coordinate function of \mathbb{C}^m .

If $p_g > 0$, there exists a holomorphic 2-form ω_0 on $M \setminus A$ but not on M , i.e. $\omega_0 \in \Gamma(M \setminus A, \Omega_M^2) \setminus \Gamma(M, \Omega_M^2)$. So ω_0 must have pole along some irreducible component A_1 of A . Suppose ω_1 has the lowest order of pole along A_1 such that $\omega_1 \in \Gamma(M \setminus A, \Omega_M^2) \setminus \Gamma(M, \Omega_M^2)$. Then $\pi^*(z_j)\omega_1$ is holomorphic along A_1 for all j , $1 \leq j \leq m$. If $\pi^*(z_j)\omega_1 \notin \Gamma(M, \Omega_M^2)$, it must have pole along another irreducible component A_2 of A . Suppose $\omega_2 \in \Gamma(M \setminus A, \Omega_M^2)$ has the lowest order of poles along A_2 and holomorphic along A_1 . Then $\pi^*(z_j)\omega_2$ is holomorphic along A_1 and A_2 , for all j , $1 \leq j \leq m$. Since the number of irreducible components of A is finite, by induction, there exists a holomorphic 2-form $\omega_k \in \Gamma(M \setminus A, \Omega_M^2)$ which has the lowest order of pole along some irreducible component A_k and holomorphic along A_1, \dots, A_{k-1} , and $\pi^*(z_j)\omega_k \in \Gamma(M, \Omega_M^2)$ for all j , $1 \leq j \leq m$. Choose a point b in A_k which is a smooth point of A . Let (x_1, x_2) be a coordinate system centered at b such that A_k is given locally by $x_1 = 0$ at b . Take the power series expansion of $\pi^*(z_j)$ around b :

$$\pi^*(z_j) \doteq x_1^{r_j} f_j, \quad 1 \leq j \leq m, \quad (2.15)$$

where f_j is holomorphic function such that $f_j(0, x_2) \neq 0$ and “ \doteq ” means local equality around b . Without loss of generality, we can suppose $r_1 =$

$\min\{r_1, \dots, r_m\}$. By local calculation, we know that for any element ψ in $\pi^*\Gamma(V, \Omega_V^2)$, the vanishing order of ψ along A_k , which is denoted by $Ord_{A_k}\psi$, is at least $2r_1 - 1$. However $Ord_{A_k}\pi^*(z_1)\omega_k$ is at most $r_1 - 1$. So $\pi^*(z_1)\omega_k \in \Gamma(M, \Omega_M^2) \setminus \pi^*\Gamma(V, \Omega_V^2)$. We pick such kind $\omega \in \Gamma(M, \Omega_M^2) \setminus \pi^*\Gamma(V, \Omega_V^2)$ which has the lowest order of zeros, r , along A_k , i.e. $Ord_{A_k}\omega = r$. So $r < r_1$.

Let $\xi_V \in \Gamma(V, \Theta_V)$, where $\Theta_V := \mathcal{H}om_{\mathcal{O}_V}(\Omega_V^1, \mathcal{O}_V)$, denote the generating vector field of the \mathbb{C}^* -action and let i_{ξ_V} denote the contraction map. For some $\alpha \in \Gamma(V, \bar{\Omega}_V^1)$, write α as a sum $\sum \alpha^j$ of quasi-homogeneous elements where α^j is a quasi-homogeneous element of degree $l_j > 0$. Let $L_{\xi_V} = i_{\xi_V}d + di_{\xi_V}$ be the Lie derivation. Then

$$l_j\alpha^j = L_{\xi_V}\alpha^j = i_{\xi_V}d(\alpha^j) + di_{\xi_V}(\alpha^j).$$

So

$$\Gamma(V, \bar{\Omega}_V^1) = d(\Gamma(V, \mathcal{O}_V)) + i_{\xi_V}(\Gamma(V, \bar{\Omega}_V^2)). \quad (2.16)$$

Since for minimal good resolution, we have $\pi_*\Theta_M = \Theta_V$ (cf [Bu-Wa]), where Θ_M is the vector field on M . Thus, there exists ξ_M which is a lift of ξ_V , i.e. $\pi_*\xi_M = \xi_V$. We know that ξ_M is tangential to the exceptional set, so

$$\xi_M \doteq x_1^{a_1}p\frac{\partial}{\partial x_1} + x_1^{a_2}q\frac{\partial}{\partial x_2}, \quad a_1 \geq 1, a_2 \geq 0$$

where p and q are holomorphic functions.

Let $i_{\xi_M} : \Gamma(M, \Omega_M^2) \rightarrow \Gamma(M, \Omega_M^1)$ be the contraction map corresponding to i_{ξ_V} . If $\zeta \in \Gamma(M, \Omega_M^2)$ and $\zeta \doteq x_1^u g dx_1 \wedge dx_2$, then

$$i_{\xi_M}(\zeta) \doteq i_{\xi_M}(x_1^u g dx_1 \wedge dx_2) = -x_1^{u+a_2} q g dx_1 + x_1^{u+a_1} p g dx_2.$$

From (2.16),

$$\Gamma(M, \Omega_M^1) = d(\Gamma(M, \mathcal{O}_M)) + i_{\xi_M}(\Gamma(M, \Omega_M^2)).$$

Since V is normal, $g^{(0)} = 0$, i.e. $\Gamma(M, \mathcal{O}_M) = \pi^*(\Gamma(V, \mathcal{O}_V))$.

We now prove that ω is not contained in $\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle$. Consider $\eta, \varphi \in \Gamma(M, \Omega_M^1)$ locally around b .

Suppose $\eta = \eta_1 + \eta_2$ and $\varphi = \varphi_1 + \varphi_2$, where $\eta_1, \varphi_1 \in d(\Gamma(M, \mathcal{O}_M))$, $\eta_2, \varphi_2 \in i_{\xi_M}(\Gamma(M, \Omega_M^2))$. Let

$$\eta_2 = i_{\xi_M}(\zeta), \quad \zeta \doteq x_1^u g dx_1 \wedge dx_2, \quad g(0, x_2) \neq 0,$$

and

$$\varphi_2 = i_{\xi_M}(\varsigma), \quad \varsigma \doteq x_1^v h dx_1 \wedge dx_2, \quad h(0, x_2) \neq 0.$$

Then

$$\eta \wedge \varphi = \eta_1 \wedge \varphi_1 + (\eta_1 \wedge \varphi_2 + \eta_2 \wedge \varphi_1) + \eta_2 \wedge \varphi_2.$$

Since

$$d\pi^*(z_i) \wedge d\pi^*(z_j) = (r_i x_1^{r_i+r_j-1} f_i \frac{\partial f_j}{\partial x_2} - r_j x_1^{r_i+r_j-1} f_j \frac{\partial f_i}{\partial x_2}) dx_1 \wedge dx_2,$$

$Ord_{A_k} \eta_1 \wedge \varphi_1 \geq 2 \cdot r_1 - 1 > r$.

Write η_2 and φ_2 locally around b :

$$\eta_2 \doteq -x_1^{u+a_2} qgdx_1 + x_1^{u+a_1} pgdx_2,$$

$$\varphi_2 \doteq -x_1^{v+a_2} qhdx_1 + x_1^{v+a_1} phdx_2.$$

So $\eta_2 \wedge \varphi_2 \doteq 0$.

Also notice that

$$d\pi^*(z_j) = r_j x_1^{r_j-1} f_j dx_1 + x_1^{r_j} \frac{\partial f_j}{\partial x_2} dx_2$$

So

$$Ord_{A_k} \eta_1 \wedge \varphi_2 \geq r_1 + v > r$$

and

$$Ord_{A_k} \eta_2 \wedge \varphi_1 \geq r_1 + u > r.$$

From the discussion above, we can get $Ord_{A_k} \eta \wedge \varphi > r$.

Therefore ω is not a linear combination of elements in $\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle$

If $p_g = 0$, from [Ya3], the canonical bundle K_M is generated by its global sections in a neighborhood of the exceptional set. So there exists $\omega \in \Gamma(M, \Omega_M^2)$ such that ω does not vanish along some irreducible component A_k of A . The rest of the argument is same as those in the case of $p_g > 0$, with r is 0. Q.E.D.

The following theorem is the crucial part for the solution of the classical complex Plateau problem.

Theorem 2.14. ([Du-Ya]) *Let V be a 2-dimensional Stein space with 0 as its only normal singular point with \mathbb{C}^* -action, then $g^{(1,1)} \geq 1$.*

In the next section we will show that this bound is sharp.

3. EXPLICIT CALCULATION OF NEW INVARIANTS FOR SPECIAL RATIONAL SINGULARITIES

In this section. we suppose that V is a 2-dimensional Stein space with 0 as its only normal singularity and V is contractible to 0. It is well known that the singularities of type A_n , D_n , E_6 , E_7 , E_8 may be given by the following equations in \mathbb{C}^3 , with singularities at the origin.

$$\begin{aligned} A_n & \quad F(x, y, z) = xy - z^{n+1} = 0, & n \geq 1, \\ D_n & \quad F(x, y, z) = x^2z + y^2 - z^{n-1} = 0, & n \text{ even} \geq 4, \\ & \quad F(x, y, z) = x^2 + y^2z - z^{n-1} = 0, & n \text{ odd} \geq 5, \\ E_6 & \quad F(x, y, z) = x^2 - y^3 - z^4 = 0, \end{aligned}$$

$$E_7 \quad F(x, y, z) = x^2 + y^3 - yz^3 = 0,$$

$$E_8 \quad F(x, y, z) = x^2 - y^3 + z^5 = 0.$$

And cyclic quotient singularities are also well understood (see [La2]). In the following computation, we shall use explicit resolutions $\pi : M \rightarrow V$ of A_n, D_n, E_6, E_7, E_8 (see [Mo], where he splits Type D_n into two cases for calculation) and cyclic quotient singularities to compute our new invariants.

Proposition 3.1. ([Ca-F1]) If $(V, 0)$ is rational isolated singularity of dimension $n \geq 2$, then any closed holomorphic p -form η on $V \setminus \{0\}$ with $1 \leq p \leq 2$ is exact, i.e. after shrinking V as a neighborhood of 0 , there exists a $p - 1$ -form ξ on $V \setminus \{0\}$ with $d(\xi) = \eta$.

Corollary 3.2. If $(V, 0)$ is rational isolated singularity of dimension 2, M is a resolution of the singularity, then $H_h^1(M) = H_h^2(M) = 0$.

Note that for these rational singularities of dimension 2, the irregularity $q = 0$ (cf. [Ya6]), so $f^{(1,1)} = g^{(1,1)}$. In order to calculate our new invariants for these rational singularities, we must know all the holomorphic 1-forms and holomorphic 2-forms on M . Usually, it is not easy to calculate holomorphic 1-forms, but for rational singularities, we have the following lemma.

Lemma 3.3. If $(V, 0)$ is rational isolated singularity of dimension 2 and $\pi : M \rightarrow V$ is a resolution then $\Gamma(M, \Omega_M^1) = d(\Gamma(M, \mathcal{O}_M)) + d^{-1}(\Gamma(M, \Omega_M^2))$, where d is the exterior differential operator.

Proof. From the corollary above, we have the following exact sequence:

$$0 \rightarrow \Gamma(M, \mathcal{O}_M) \xrightarrow{d} \Gamma(M, \Omega_M^1) \xrightarrow{d} \Gamma(M, \Omega_M^2) \rightarrow 0.$$

For $\xi \in \Gamma(M, \Omega_M^1)$ and any $\zeta \in d^{-1}(d\xi)$, $d(\xi - \zeta) = 0$. So there exist $f \in \Gamma(M, \mathcal{O}_M)$ such that $\xi = \zeta + d(f)$. Q.E.D.

From the lemma above, we see that in order to get holomorphic 1-forms on M , we only need to calculate holomorphic functions and holomorphic 2-forms on M .

For rational double points, we may, without loss of generality, suppose V contains $\{(x, y, z) : F(x, y, z) = 0, |x|^2 + |y|^2 + |z|^2 < 1\}$, $F(x, y, z)$ as above. By abusing of notation, we denote local resolutions of the rational double points at 0 by $\pi : M \rightarrow V$.

Type A_n :

An explicit resolution $\pi : M \rightarrow V$ can be given in terms of coordinates and transition functions on M as follows:

Coordinates charts: $W_k = \{(u_k, v_k)\}$, $k = 0, 1, \dots, n$

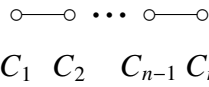
Transition functions :

$$\begin{cases} u_{k+1} = \frac{1}{v_k} \\ v_{k+1} = u_k v_k^2 \end{cases} \quad \text{or} \quad \begin{cases} u_k = u_{k+1}^2 v_{k+1} \\ v_k = \frac{1}{u_{k+1}} \end{cases}$$

Projection map: $\pi(u_k, v_k) = (u_k^{k+1} v_k^k, u_k^{n-k} v_k^{n-k+1}, u_k v_k)$

Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \dots \cup C_n$, where $C_k = \{u_{k-1} = 0\} \cup \{v_k = 0\}$

Dual graph:



Holomorphic functions on M :

Any holomorphic function on M has power series expansion $\sum_{\alpha, \beta \geq 0} c_{\alpha\beta} u_0^\alpha v_0^\beta$ which converges for all $(u_0, v_0) \in W_0$. Under changes of charts,

$$\begin{aligned} \sum_{\alpha, \beta \geq 0} c_{\alpha\beta} u_0^\alpha v_0^\beta &= \dots = \sum_{\alpha, \beta \geq 0} c_{\alpha\beta} u_k^{(k+1)\alpha - k\beta} v_k^{k\alpha - (k-1)\beta} = \dots \\ &= \sum_{\alpha, \beta \geq 0} c_{\alpha\beta} u_n^{(n+1)\alpha - n\beta} v_n^{n\alpha - (n-1)\beta}. \end{aligned}$$

The k -th power series has to converge for all $(u_k, v_k) \in W_k$. This occurs for all k if only if the indices α, β in each sum satisfy $(n+1)\alpha - n\beta \geq 0$. Thus any holomorphic function on M can be given by a convergent power series

$$\sum_{\substack{\alpha, \beta \geq 0 \\ (n+1)\alpha - n\beta \geq 0}} c_{\alpha\beta} u_0^\alpha v_0^\beta$$

on W_0 .

Conversely, any such convergent power series in the (u_0, v_0) chart defines a holomorphic function on M .

Holomorphic 2-forms on M :

The holomorphic 2-form $du_0 \wedge dv_0 = \dots = du_n \wedge dv_n (= \pi^*(\frac{dx \wedge dy}{F_z}))$ is nowhere vanishing on M . It follows that any holomorphic 2-form on M can be given in the (u_0, v_0) chart by a 2-form

$$\sum_{\substack{\alpha, \beta \geq 0 \\ (n+1)\alpha - n\beta \geq 0}} c_{\alpha\beta} u_0^\alpha v_0^\beta du_0 \wedge dv_0$$

whose power series coefficient converges on W_0 .

Conversely, any such 2-forms in the (u_0, v_0) chart defines a holomorphic 2-form on M .

Proposition 3.4. With the above notation for A_n singularities, $f^{(1,1)} = 1$.

Proof. From the above calculation, we know that the holomorphic functions on M are generated by a base $\{u_0^\alpha v_0^\beta\}_{(n+1)\alpha-n\beta \geq 0}$ and holomorphic 2-forms are generated by a base $\{u_0^\alpha v_0^\beta du_0 \wedge dv_0\}_{(n+1)\alpha-n\beta \geq 0}$. For every holomorphic 2-form $\omega = u_0^\alpha v_0^\beta du_0 \wedge dv_0$ on M , we consider $\xi = -\frac{u_0^\alpha v_0^{\beta+1}}{\beta+1} du_0$. ξ defines a holomorphic 1-form on W_0 and $d\xi = \omega$. It remains to check that under all changes of charts, ξ transforms to define a holomorphic 1-form in each coordinate chart. In fact, changed to (u_k, v_k) chart, for $k = 1, \dots, n$,

$$\xi = -\frac{1}{\beta+1} u_k^{(k+1)\alpha-k\beta} v_k^{k\alpha-(k-1)\beta} ((k+1)v_k du_k + k u_k dv_k) \quad \text{on } W_k$$

which is holomorphic. So

$$\{u_0^\alpha v_0^{\beta+1} du_0\}_{(n+1)\alpha-n\beta \geq 0}$$

generates $d^{-1}(\Gamma(M, \Omega_M^2))$.

We also know that $d(\Gamma(M, \mathcal{O}_M))$ is generated by

$$\{\alpha u_0^{\alpha-1} v_0^\beta du_0 + \beta u_0^\alpha v_0^{\beta-1} dv_0\}_{(n+1)\alpha-n\beta \geq 0, \alpha \geq 1}.$$

By Lemma 3.3, $\Gamma(M, \Omega_M^1) = d(\Gamma(M, \mathcal{O}_M)) + d^{-1}(\Gamma(M, \Omega_M^2))$. So by easy calculation $\Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1)$ is generated by

$$\{u_0^\alpha v_0^\beta du_0 \wedge dv_0\}_{(n+1)\alpha-n\beta \geq 0, \alpha \geq 1}.$$

Therefore

$$\frac{\Gamma(M, \Omega_M^2)}{\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle} = \langle u_0 \wedge v_0 \rangle,$$

and

$$f^{(1,1)} = \dim \frac{\Gamma(M, \Omega_M^2)}{\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle} = 1.$$

Q.E.D.

Type D_n , $n \geq 4$ and even:

An explicit resolution $\pi : M \rightarrow V$ can be given in terms of coordinate charts and transition functions on M as follows.

Coordinate charts:

$$W_k = \{(u_k, v_k) : u_k^{n-k-3} v_k^{n-k-2} \neq 1\}, \quad 0 \leq k \leq n-4$$

$$W_k = \{(u_k, v_k)\}, \quad k = n-3, n-2$$

$$W_k = \{(u_k, v_k) : u_k^2 v_k \neq -1\}, \quad k = n-1, n$$

Transition functions:

$$\begin{cases} u_k = u_{k+1}^2 v_{k+1} \\ v_k = \frac{1}{u_{k+1}} \end{cases} \quad 0 \leq k \leq n-3$$

$$\begin{cases} u_{n-2} = \frac{1}{1 + u_{n-1}^2 v_{n-1}} \\ v_{n-2} = \frac{1 + u_{n-1}^2 v_{n-1}}{u_{n-1}} \end{cases} \quad \text{and} \quad \begin{cases} u_{n-2} = \frac{u_n^2 v_n}{1 + u_n^2 v_n} \\ v_{n-2} = \frac{1 + u_n^2 v_n}{u_n} \end{cases}$$

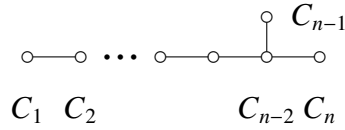
Projection map: $\pi(u_k, v_k) = (x, y, z)$, where

$$\begin{aligned} x &= u_0(u_0^{n-3} v_0^{n-2} - 1)^{\frac{n}{2}-1} (u_0^{n-3} v_0^{n-2} - 2) = \dots \\ &= u_{n-2}^{\frac{n}{2}-1} v_{n-2}^{n-2} (1 - u_{n-2})^{\frac{n}{2}-1} (1 - 2u_{n-2}) = \frac{v_{n-1}^{\frac{n}{2}-1} (u_{n-1}^2 v_{n-1} - 1)}{1 + u_{n-1}^2 v_{n-1}} = \frac{v_n^{\frac{n}{2}-1} (1 - u_n^2 v_n)}{1 + u_n^2 v_n} \\ y &= 2u_0^2 v_0 (u_0^{n-3} v_0^{n-2} - 1)^{\frac{n}{2}} = \dots \\ &= 2u_{n-2}^{\frac{n}{2}} v_{n-2}^{n-1} (1 - u_{n-2})^{\frac{n}{2}} = \frac{2u_{n-1} v_{n-1}^{\frac{n}{2}}}{1 + u_{n-1}^2 v_{n-1}} = \frac{2u_n v_n^{\frac{n}{2}}}{1 + u_n^2 v_n} \\ z &= u_0^2 v_0^2 (u_0^{n-3} v_0^{n-2} - 1) = \dots = u_{n-2} v_{n-2}^2 (1 - u_{n-2}) = v_{n-1} = v_n. \end{aligned}$$

Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \dots \cup C_k$, where

$$\begin{aligned} C_k &= \{u_{k-1} = 0\} \cup \{v_k = 0\} \quad 1 \leq k \leq n-2 \\ C_{n-1} &= \{v_{n-3} = 1\} \cup \{u_{n-2} = 1\} \cup \{v_{n-1} = 0\} \\ C_n &= \{u_{n-2} = 0\} \cup \{v_n = 0\} \end{aligned}$$

Dual graph:



Holomorphic functions on M :

Any holomorphic function f on M has a series expansion of the form $\sum_{\alpha \geq 0} f_\alpha(u_{n-2}) v_{n-2}^\alpha$ on W_{n-2} , where

$$f_\alpha(u_{n-2}) = \frac{1}{2\pi i} \int_{|v|=r} \frac{f(u_{n-2}, v)}{v^{\alpha+1}} dv,$$

provided $\{u_{n-2}\} \times \{|v_{n-3}| \leq r\} \subset W_{n-2}$. $\sum_{\alpha \geq 0} f_\alpha(u_{n-2}) v_{n-2}^\alpha$ converges absolutely and uniformly on any subset of W_{n-2} of the form (compact set) \times (closed disc centered at 0), while $f_\alpha(u_{n-2})$ is holomorphic for all u_2 . Then $f_\alpha(u_{n-2})$ has an expansion $f_\alpha(u_{n-2}) = \sum_{\beta \geq 0} c_{\alpha\beta} u_{n-2}^\beta$ on \mathbb{C} and f has an expansion

$\sum_{\alpha \geq 0} (\sum_{\beta \geq 0} c_{\alpha\beta} u_{n-2}^\beta v_{n-2}^\alpha)$ on W_{n-2} . Note that the expansion rearranges into a convergent power series near $(0, 0)$, but not necessarily on all W_{n-2} .

Under changes of charts, the expressions of f take on the following forms:

$$\begin{aligned} \sum_{\alpha} \left(\sum_{\beta} c_{\alpha\beta} u_{n-2}^\beta v_{n-2}^\alpha \right) &= \sum_{\alpha} \left(\sum_{\beta} c_{\alpha\beta} u_{n-3}^\alpha v_{n-3}^{2\alpha-\beta} \right) = \dots \\ &= \sum_{\alpha} \left(\sum_{\beta} c_{\alpha\beta} u_0^{(n-2)\alpha-(n-3)\beta} v_0^{(n-1)\alpha-(n-2)\beta} \right) \end{aligned}$$

Since all these series have to be convergent power series in respective neighborhoods of $(0, 0)$, the indices α, β restrict to $(n-1)\alpha \geq (n-2)\beta$. Thus

$$f = \sum_{\alpha \geq 0} \left(\sum_{0 \leq \beta \leq \frac{n-1}{n-2}\alpha} c_{\alpha\beta} u_{n-2}^\beta v_{n-2}^\alpha \right) \quad \text{on } W_{n-2}.$$

Lemma 3.5. With the above notation for D_n (n even), any holomorphic function f on M has the expansions

$$f = \sum_{\alpha \geq 0} \left(\sum_{\frac{1}{2}\alpha \leq \beta \leq \frac{n-1}{n-2}\alpha} c_{\alpha\beta} u_k^{(n-k-2)\alpha-(n-k-3)\beta} v_k^{(n-k-1)\alpha-(n-k-2)\beta} \right) \quad (3.1)$$

on W_k for $k = 0, 1, \dots, n-2$,

$$f = \sum_{\alpha \geq 0} \left(\sum_{\frac{1}{2}\alpha \leq \beta \leq \frac{n-1}{n-2}\alpha} c_{\alpha\beta} u_{n-1}^{-\alpha} (1 + u_{n-1}^2 v_{n-1})^{\alpha-\beta} \right) \quad \text{on } W_{n-1} \quad (3.2)$$

and

$$f = \sum_{\alpha \geq 0} \left(\sum_{\frac{1}{2}\alpha \leq \beta \leq \frac{n-1}{n-2}\alpha} c_{\alpha\beta} u_n^{2\beta-\alpha} v_n^\beta (1 + u_n^2 v_n)^{\alpha-\beta} \right) \quad \text{on } W_n \quad (3.3)$$

In each expansion, the sum over β is holomorphic on the corresponding W_k for each $\alpha \geq 0$, and the sum over α is absolutely convergent.

Proof. 1° With $f = \sum_{\alpha} (\sum_{\beta} c_{\alpha\beta} u_{n-2}^\beta v_{n-2}^\alpha)$ on W_{n-2} , changing from (u_{n-2}, v_{n-2}) to (u_{n-3}, v_{n-3}) gives $f = \sum_{\alpha} (\sum_{\beta} c_{\alpha\beta} u_{n-3}^\alpha v_{n-3}^{2\alpha-\beta})$ on $W_{n-3} \setminus \{v_{n-3} = 0\}$. For each $\alpha \geq 0$, the sum over β is a polynomial, say $\varphi_\alpha(u_{n-3}, v_{n-3})$. To show that the expansion of f also holds on $v_{n-3} = 0$, consider any $(c, 0) \in W_{n-3}$ and take any $\epsilon > 0$ such that $\{c\} \times \{|v_{n-3}| \leq \epsilon\} \subset W_{n-3}$. Then, on the circle $\{c\} \times \{|v_{n-3}| = \epsilon\}$,

$$\sum_{\alpha} \varphi_\alpha(c, v_{n-3}) = \sum_{\alpha} \left(\sum_{\beta} c_{\alpha\beta} u_{n-2}^\beta v_{n-2}^\alpha \right), \quad (3.4)$$

where $(u_{n-2}, v_{n-2}) = (v_{n-3}^{-1}, c v_{n-3}^2)$ lies on $\{|u_{n-2}| = \epsilon^{-1}\} \times \{|v_{n-2}| = |c|\epsilon^2\}$ in W_{n-2} . Since the right hand side of (3.4) converges uniformly on the indicated subset of W_{n-2} , so does the left hand side on $\{c\} \times \{|v_{n-3}| = \epsilon\}$. Then $\sum_{\alpha} \varphi_\alpha(c, v_{n-3})$ is holomorphic on $\{c\} \times \{|v_{n-3}| \leq \epsilon\}$, hence coincides

with f , in particular at $(c, 0)$. The absolute convergence of $\sum_{\alpha} \varphi_{\alpha}(u_{n-3}, v_{n-3})$ follows from that of $\sum_{\alpha} (\sum_{\beta} c_{\alpha\beta} u_{n-2}^{\beta} v_{n-2}^{\alpha})$ over α .

By the same argument, changing from (u_{n-2}, v_{n-2}) to (u_k, v_k) via

$$u_{n-2} = \frac{1}{u_k^{n-k-3} v_k^{n-k-2}}, \quad v_{n-2} = u_k^{n-k-2} v_k^{n-k-1}$$

gives (3.1) first on $W_k \setminus \{u_k v_k = 0\}$ and then on W_k , for $k = 0, 1, \dots, n-4$. It suffices to remark that for $c \neq 0$, small circle $\{c\} \times \{|v_k| = \epsilon\}$ (resp. $\{|u_k| = \epsilon\} \times \{c\}$) in W_k correspond to (u_{n-2}, v_{n-2}) with $|u_{n-2}| = \frac{1}{|c|^{n-k-3} \epsilon^{n-k-2}}$, $|v_{n-2}| = |c|^{n-k-2} \epsilon^{n-k-1}$ (resp. $|u_{n-2}| = \frac{1}{\epsilon^{n-k-3} |c|^{n-k-2}}$, $|v_{n-2}| = \epsilon^{n-k-2} |c|^{n-k-1}$) in W_{n-2} .

2° Changing from (u_{n-2}, v_{n-2}) to (u_{n-1}, v_{n-1}) gives (3.2) on $W_{n-1} \setminus \{u_{n-1} = 0\}$, where $1 + u_{n-1}^2 v_{n-1} \neq 0$. We need a trick to ensure that for each α , the sum over β is holomorphic on W_{n-1} . Since $(V, 0)$ is normal, f is the pullback under π of some power series $\sum_{i,j,k \geq 0} \tilde{c}_{ijk} x^i y^j z^k$ which converges in some neighborhood \tilde{U} of 0 in \mathbb{C}^3 . On $\pi^{-1}(\tilde{U}) \cap W_{n-2}$,

$$\begin{aligned} \sum_{\alpha} \left(\sum_{\beta} c_{\alpha\beta} u_{n-2}^{\beta} v_{n-2}^{\alpha} \right) &= \sum \tilde{c}_{ijk} (u_{n-2}^{\frac{n}{2}-1} v_{n-2}^{n-2} (1 - u_{n-2})^{\frac{n}{2}-1} (1 - 2u_{n-2}))^i \\ &\quad \cdot (2u_{n-2}^{\frac{n}{2}} v_{n-2}^{n-1} (1 - u_{n-2})^{\frac{n}{2}})^j (u_{n-2} v_{n-2}^2 (1 - u_{n-2}))^k \end{aligned} \quad (3.5)$$

which implies that for each $\alpha \geq 0$,

$$\begin{aligned} \sum_{0 \leq \beta \leq \frac{n-1}{n-2} \alpha} c_{\alpha\beta} u_{n-2}^{\beta} v_{n-2}^{\alpha} &= \sum_{(n-2)i + (n-1)j + 2k = \alpha} \tilde{c}_{ijk} (u_{n-2}^{\frac{n}{2}-1} v_{n-2}^{n-2} (1 - u_{n-2})^{\frac{n}{2}-1} (1 - 2u_{n-2}))^i \\ &\quad \cdot (2u_{n-2}^{\frac{n}{2}} v_{n-2}^{n-1} (1 - u_{n-2})^{\frac{n}{2}})^j (u_{n-2} v_{n-2}^2 (1 - u_{n-2}))^k \end{aligned} \quad (3.6)$$

Take a neighborhood U_{n-1} of $(0, 0)$ in W_{n-1} such that $U_{n-1} \subset \pi^{-1}(\tilde{U}) \cap W_{n-1}$. Changing the finite sums on both side of (3.6) to (u_{n-1}, v_{n-1}) gives, on $U_{n-1} \setminus \{u_{n-1} = 0\}$,

$$\begin{aligned} &\sum_{0 \leq \beta \leq \frac{n-1}{n-2} \alpha} c_{\alpha\beta} u_{n-1}^{-\alpha} (1 + u_{n-1}^2 v_{n-1})^{\alpha-\beta} \\ &= \sum_{(n-2)i + (n-1)j + 2k = \alpha} \tilde{c}_{ijk} \left(\frac{v_{n-1}^{\frac{n}{2}-1} (u_{n-1}^2 v_{n-1} - 1)}{1 + u_{n-1}^2 v_{n-1}} \right)^i \left(\frac{2u_{n-1} v_{n-1}^{\frac{n}{2}}}{1 + u_{n-1}^2 v_{n-1}} \right)^j v_{n-1}^k. \end{aligned} \quad (3.7)$$

The two sides of (3.7) being rational functions of (u_{n-1}, v_{n-1}) , they must be identical. Since the right hand side is holomorphic on W_{n-1} , so is the left hand side.

Denote $\sum_{0 \leq \beta \leq \frac{n-1}{n-2} \alpha} c_{\alpha\beta} u_{n-1}^{-\alpha} (1 + u_{n-1}^2 v_{n-1})^{\alpha-\beta}$ by $\psi_{\alpha}(u_{n-1}, v_{n-1})$. Since all ψ_{α} are holomorphic, to prove (3.2) also at $u_{n-1} = 0$, say at any $(0, c) \in W_{n-1}$, it suffices, as before, to find some $\epsilon > 0$ such that (i) $\{|u_{n-1}| \leq \epsilon\} \times \{c\} \subset W_{n-1}$

and (ii) $\sum_{\alpha} \psi_{\alpha}(u_{n-1}, c)$ converges uniformly on the circle $\{|u_{n-1}| = \epsilon\} \times \{c\}$. We can clearly take $\epsilon > 0$ satisfying (i). Then $(u_{n-1}, v_{n-1}) = (\epsilon e^{i\theta}, c)$, $\theta \in \mathbb{R}$, corresponds to $(u_{n-2}, v_{n-2}) = (\frac{1}{1+c\epsilon^2 e^{2i\theta}}, \frac{1+c\epsilon^2 e^{2i\theta}}{\epsilon e^{i\theta}}) \in W_{n-2}$. If we can bound these (u_{n-2}, v_{n-2}) within some $S' = (\text{compact set}) \times (\text{closed disc centered at } 0) \subset W_{n-2}$, then the uniform convergence of $\sum_{\alpha} (\sum_{\beta} c_{\alpha\beta} u_{n-2}^{\beta} v_{n-2}^{\alpha})$ on S' gives (ii). To get S' , it suffices to choose a sufficiently smaller $\epsilon > 0$ such that for all θ , $(\frac{1}{1+c\epsilon^2 e^{2i\theta}}, \frac{\max_{\phi} |1+c\epsilon^2 e^{2i\phi}|}{\epsilon}) \in W_{n-2}$.

Since $(u_{n-1}, v_{n-1}) = (\epsilon e^{i\theta}, c) \in W_{n-1}$ for all θ , $|x|^2 + |y|^2 + |z|^2 < 1$ on W_{n-1} gives

$$|c|^{n-2} \left| \frac{1 - c\epsilon^2 e^{2i\theta}}{1 + c\epsilon^2 e^{2i\theta}} \right|^2 + \frac{4|c|^n \epsilon^2}{|1 + c\epsilon^2 e^{2i\theta}|^2} + |c|^2 < 1 \quad \text{for all } \theta. \quad (3.8)$$

Let $|1 + c\epsilon^2 e^{2i\theta_0}| = \max_{\theta} |1 + c\epsilon^2 e^{2i\theta}|$. Then $(\frac{1}{1+c\epsilon^2 e^{2i\theta_0}}, \frac{|1+c\epsilon^2 e^{2i\theta_0}|}{\epsilon}) \in W_{n-2}$ if the corresponding inequality $|x|^2 + |y|^2 + |z|^2 < 1$ is satisfied, namely,

$$\begin{aligned} |c|^{n-2} \left| \frac{1 - c\epsilon^2 e^{2i\theta}}{1 + c\epsilon^2 e^{2i\theta}} \right|^2 \left| \frac{1 + c\epsilon^2 e^{2i\theta_0}}{1 + c\epsilon^2 e^{2i\theta}} \right|^{2n-4} + \frac{4|c|^n \epsilon^2}{|1 + c\epsilon^2 e^{2i\theta}|^2} \left| \frac{1 + c\epsilon^2 e^{2i\theta_0}}{1 + c\epsilon^2 e^{2i\theta}} \right|^{2n-2} \\ + |c|^2 \left| \frac{1 + c\epsilon^2 e^{2i\theta_0}}{1 + c\epsilon^2 e^{2i\theta}} \right|^4 < 1 \end{aligned} \quad (3.9)$$

In view of (3.8), a sufficiently smaller $\epsilon > 0$ can be chosen such that (3.9) holds for all θ . With such choice of ϵ , the lemma for $k = n - 1$ follows.

3° Changing from (u_{n-2}, v_{n-2}) to (u_n, v_n) gives (3.3) on $W_n \setminus \{u_n = 0\}$, where $1 + u_n^2 v_n \neq 0$. To see that for each α , the sum over β is holomorphic on W_n , it suffices to check that $2\beta - \alpha \geq 0$. By (3.6),

$$\begin{aligned} \sum_{0 \leq \beta \leq \frac{n-1}{2}\alpha} c_{\alpha\beta} u_{n-2}^{\beta} &= \sum_{(n-2)i + (n-1)j + 2k = \alpha} \tilde{c}_{ijk} (u_{n-2}^{\frac{n}{2}-1} (1 - u_{n-2})^{\frac{n}{2}-1} (1 - 2u_{n-2}))^i \\ &\quad \cdot (2u_{n-2}^{\frac{n}{2}} (1 - u_{n-2})^{\frac{n}{2}})^j (u_{n-2} (1 - u_{n-2}))^k. \end{aligned} \quad (3.10)$$

(3.10) implies that for $(n-2)i + (n-1)j + 2k = \alpha$, $\beta \geq (\frac{n}{2}-1)i + \frac{n}{2}j + k$. Hence $2\beta - \alpha \geq 0$.

To check (3.3) holds at any $(0, c) \in W_n$, we repeat the argument for W_{n-1} . Then it suffices to find some $\epsilon > 0$ such that (i)' $(u_n, v_n) = (\epsilon e^{i\theta}, c)$ and (ii)' $(u_{n-2}, v_{n-2}) = (\frac{c\epsilon^2 e^{2i\theta}}{1+c\epsilon^2 e^{2i\theta}}, \frac{1+c\epsilon^2 e^{2i\theta}}{\epsilon e^{i\theta}}) \in W_{n-2}$ lies in some $S' = (\text{compact set}) \times (\text{closed disc centered at } 0) \subset W_{n-2}$, for all $\theta \in \mathbb{R}$. We first take $\epsilon > 0$ satisfying (i)'. Then, letting $|1 + c\epsilon^2 e^{2i\theta_0}| = \max_{\theta} |1 + c\epsilon^2 e^{2i\theta}|$, we compare the inequality $|x|^2 + |y|^2 + |z|^2 < 1$ on W_n guaranteed by (i)' with the inequality $|x|^2 + |y|^2 + |z|^2 < 1$ on W_{n-2} required for $(\frac{c\epsilon^2 e^{2i\theta}}{1+c\epsilon^2 e^{2i\theta}}, \frac{1+c\epsilon^2 e^{2i\theta}}{\epsilon e^{i\theta}}) \in W_{n-2}$, for all $\theta \in \mathbb{R}$. It turns out that the inequalities are exactly the same as (3.8) and (3.9) respectively. Thus we can choose ϵ , S' and finish the proof. We remark that the first factor of S' is a compact neighborhood of 0 while the first factor of S in the previous case is a compact neighborhood of 1. Q.E.D.

Holomorphic 2-forms on M :

The holomorphic 2-form $\varphi_0 = du_0 \wedge dv_0 = du_1 \wedge dv_1 = \cdots = du_{n-2} \wedge dv_{n-2} = -\frac{du_{n-1} \wedge dv_{n-1}}{1+u_{n-1}^2 v_{n-1}} = \frac{du_n \wedge dv_n}{1+u_n^2 v_n} (= \pi^* (\frac{dx \wedge dy}{F_z}))$ is nowhere zero on M . Hence any holomorphic 2-form on M is of the form $f\varphi_0$, where f is a holomorphic function on M .

Type D_n , $n \geq 5$ and odd:

A resolution $\pi : M \rightarrow V$ can be given by the same charts and transition functions as for even n , but with a different projection

$$\begin{aligned} x &= u_0^2 v_0 (u_0^{n-3} v_0^{n-2} - 1)^{\frac{n-1}{2}} (u_0^{n-3} v_0^{n-2} - 2) = \cdots \\ &= u_{n-2}^{\frac{n-1}{2}} v_{n-2}^{n-1} (1 - u_{n-2})^{\frac{n-1}{2}} (1 - 2u_{n-2}) = \frac{v_{n-1}^{\frac{n-1}{2}} (u_{n-1}^2 v_{n-1} - 1)}{1 + u_{n-1}^2 v_{n-1}} = \frac{v_n^{\frac{n-1}{2}} (1 - u_n^2 v_n)}{1 + u_n^2 v_n} \end{aligned}$$

$$\begin{aligned} y &= 2u_0 (u_0^{n-3} v_0^{n-2} - 1)^{\frac{n-1}{2}} = \cdots \\ &= 2u_{n-2}^{\frac{n-1}{2}} v_{n-2}^{n-2} (1 - u_{n-2})^{\frac{n-1}{2}} = \frac{2u_{n-1} v_{n-1}^{\frac{n-1}{2}}}{1 + u_{n-1}^2 v_{n-1}} = \frac{2u_n v_n^{\frac{n-1}{2}}}{1 + u_n^2 v_n} \end{aligned}$$

$$z = u_0^2 v_0^2 (u_0^{n-3} v_0^{n-2} - 1) = \cdots = u_{n-2} v_{n-2}^2 (1 - u_{n-2}) = v_{n-1} = v_n.$$

The exceptional set is given in the same way as for even n .

Holomorphic function on M is again given by (3.1), (3.2), (3.3). The proof is similar as for odd n .

Any holomorphic 2-form on M also has form $f\varphi_0$ where f is a holomorphic function on M and φ_0 is the same as for even n .

Proposition 3.6. With the above notation for D_n singularities, $f^{(1,1)} = 1$.

Proof. From the above calculation, we know that the holomorphic functions on M are generated by a base $\{u_{n-2}^\beta v_{n-2}^\alpha\}_{\frac{1}{2}\alpha \leq \beta \leq \frac{n-1}{2}\alpha}$ and holomorphic 2-forms are generated by a base $\{u_{n-2}^\beta v_{n-2}^\alpha du_{n-2} \wedge dv_{n-2}\}_{\frac{1}{2}\alpha \leq \beta \leq \frac{n-1}{2}\alpha}$. For every holomorphic 2-form $\omega = u_{n-2}^\beta v_{n-2}^\alpha du_{n-2} \wedge dv_{n-2}$ on M , we consider $\xi = -\frac{u_{n-2}^\beta v_{n-2}^{\alpha+1}}{\alpha+1} du_{n-2}$. ξ defines a holomorphic 1-form on W_{n-2} and $d\xi = \omega$. It remains to check that under all changes of charts, ξ transforms to define a holomorphic 1-form in each coordinate chart. In fact, changed to (u_k, v_k) chart, for $k = 1, \cdots, n-2$, on W_k ,

$$\xi = -\frac{1}{\alpha+1} u_k^{(n-k-2)\alpha - (n-k-3)\beta} v_k^{(n-k-1)\alpha - (n-k-2)\beta} ((n-k-3)v_k du_k + (n-k-2)u_k dv_k)$$

which is holomorphic. And changing charts to (u_{n-2}, v_{n-3}) , we see by Lemma 3.5 that

$$\xi = -\frac{v_{n-3}^{2\alpha-\beta} u_{n-3}^{\alpha+1}}{\alpha+1} dv_{n-3}$$

defines a holomorphic 1-form on W_{n-3} . Finally, changed to (u_{n-1}, v_{n-1}) and (u_n, v_n) charts

$$\xi = \left(\frac{u_{n-1}^{-\alpha} (1 + u_{n-1}^2 v_{n-1})^{\alpha-\beta}}{\alpha+1} \right) \left(\frac{2v_{n-1} du_{n-1} + u_{n-1} dv_{n-1}}{1 + u_{n-1}^2 v_{n-1}} \right) \quad \text{on } W_{n-1}$$

and

$$\xi = \left(\frac{u_n^{2\beta-\alpha} v_n^\beta (1 + u_n^2 v_n)^{\alpha-\beta}}{\alpha+1} \right) \left(\frac{2v_n du_n + u_n dv_n}{1 + u_n^2 v_n} \right) \quad \text{on } W_n.$$

Again by Lemma 3.5, ξ is holomorphic on W_{n-1} and W_n respectively. So

$$\{u_{n-2}^\beta v_{n-2}^{\alpha+1} du_{n-2}\}_{\frac{1}{2}\alpha \leq \beta \leq \frac{n-1}{n-2}\alpha}$$

generates $d^{-1}(\Gamma(M, \Omega_M^2))$.

We also know that $d(\Gamma(M, \mathcal{O}_M))$ is generated by

$$\{\beta u_{n-2}^{\beta-1} v_{n-2}^\alpha du_{n-2} + \alpha u_{n-2}^\beta v_{n-2}^{\alpha-1} dv_{n-2}\}_{\frac{1}{2}\alpha \leq \beta \leq \frac{n-1}{n-2}\alpha}.$$

By Lemma 3.3, $\Gamma(M, \Omega_M^1) = d(\Gamma(M, \mathcal{O}_M)) + d^{-1}(\Gamma(M, \Omega_M^2))$. So by easy calculation $\Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1)$ is generated by

$$\{u_{n-2}^\beta v_{n-2}^\alpha du_{n-2} \wedge dv_{n-2}\}_{\frac{1}{2}\alpha \leq \beta \leq \frac{n-1}{n-2}\alpha, \alpha \geq 1}.$$

Therefore

$$\frac{\Gamma(M, \Omega_M^2)}{\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle} = \langle u_{n-2} \wedge v_{n-2} \rangle,$$

and

$$f^{(1,1)} = \dim \frac{\Gamma(M, \Omega_M^2)}{\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle} = 1.$$

Q.E.D.

Type E_n , $n = 6, 7, 8$:

Resolutions $\pi : M \rightarrow V$ for E_n , $n = 6, 7, 8$, can be given as follows.

Coordinate charts:

$$W_k = \{(u_k, v_k) : u_k^{2-k} v_k^{3-k} \neq 1\}, \quad k = 0, 1$$

$$W_k = \{(u_k, v_k)\}, \quad k = 2, 3$$

$$W_4 = \{(u_4, v_4) : u_4^2 v_4 \neq 1\}$$

$$W_k = \{(u_k, v_k) : u_k^{k-3} v_k^{k-4} \neq -1\}, \quad 5 \leq k \leq n$$

Transition functions:

$$\begin{cases} u_k = u_{k+1}^2 v_{k+1} \\ v_k = \frac{1}{u_{k+1}} \end{cases} \quad 0 \leq k \leq 2 \text{ and } 5 \leq k \leq n-1$$

$$\begin{cases} u_3 = \frac{1}{1 + u_4^2 v_4} \\ v_3 = \frac{1 + u_4^2 v_4}{u_4} \end{cases} \quad \text{and} \quad \begin{cases} u_3 = \frac{u_5^2 v_5}{1 + u_5^2 v_5} \\ v_3 = \frac{1 + u_5^2 v_5}{u_5} \end{cases}$$

Projection map: $\pi(u_k, v_k) = (x, y, z)$, where

For E_6 :

$$\begin{cases} x = 4u_0^2(u_0^2 v_0^3 - 1)^3(u_0^2 v_0^3 + 1) = \dots \\ \quad = 4u_3^4 v_3^6 (1 - u_3)^3 (1 + u_3) = \dots = \frac{4v_6^2(1 + 2u_6^3 v_6^2)}{(1 + u_6^3 v_6^2)^2} \\ y = 4u_0^2 v_0 (u_0^2 v_0^3 - 1)^2 = \dots \\ \quad = 4u_3^3 v_3^4 (1 - u_3)^2 = \dots = \frac{4u_6 v_6^2}{1 + u_6^3 v_6^2} \\ z = 2u_0 (u_0^2 v_0^3 - 1)^2 = \dots = 2u_3^2 v_3^3 (1 - u_3)^2 = \dots = \frac{2v_6}{1 + u_6^3 v_6^2}. \end{cases}$$

For E_7 :

$$\begin{cases} x = u_0^3 (u_0^2 v_0^3 - 1)^5 = \dots \\ \quad = u_3^7 v_3^9 (1 - u_3)^5 = \dots = \frac{u_7 v_7^3}{(1 + u_7^4 v_7^3)^3} \\ y = u_0^2 (u_0^2 v_0^3 - 1)^3 = \dots \\ \quad = u_3^5 v_3^6 (1 - u_3)^3 = \dots = \frac{u_7^2 v_7^3}{(1 + u_7^4 v_7^3)^3} \\ z = u_0^2 v_0 (u_0^2 v_0^3 - 1)^2 = \dots = u_3^3 v_3^4 (1 - u_3)^2 = \dots = \frac{v_7}{(1 + u_7^4 v_7^3)^3}. \end{cases}$$

For E_8 :

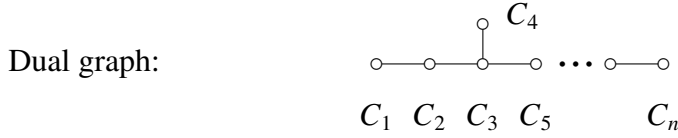
$$\left\{ \begin{array}{l} x = u_0^5(u_0^2v_0^3 - 1)^8 = \dots \\ \quad = u_3^{12}v_3^{15}(1 - u_3)^8 = \dots = \frac{v_8^3}{(1 + u_8^5v_8^4)^5} \\ y = u_0^4v_0(u_0^2v_0^3 - 1)^5 = \dots \\ \quad = u_3^8v_3^{10}(1 - u_3)^5 = \dots = \frac{v_8^2}{(1 + u_8^5v_8^4)^3} \\ z = u_0^2(u_0^2v_0^3 - 1)^3 = \dots = u_3^5v_3^6(1 - u_3)^3 = \dots = \frac{u_8v_8^2}{(1 + u_8^5v_8^4)^2}. \end{array} \right.$$

Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \dots \cup C_k$, where

$$C_k = \{u_{k-1} = 0\} \cup \{v_k = 0\} \quad 1 \leq k \leq 3 \text{ and } 6 \leq k \leq n$$

$$C_4 = \{v_2 = 1\} \cup \{u_3 = 1\} \cup \{v_4 = 0\}$$

$$C_5 = \{u_3 = 0\} \cup \{v_5 = 0\}$$



Holomorphic function on M :

W_3 (resp. W_2) has the property that it intersects each plane $\{u_3 = \text{constant}\}$ (resp. $\{v_2 = \text{constant}\}$) in a disc centered at the point $(u_3, 0)$ (resp. $(0, v_2)$) which belongs to A . Any holomorphic function of M has an expression $f = \sum_{\alpha \geq 0} (\sum_{\beta \geq 0} c_{\alpha\beta} u_3^\beta) v_3^\alpha$ on W_3 .

Lemma 3.7. With the above notation for E_n ($n = 6, 7, 8$), any holomorphic function f on M has the expansions

$$f = \sum_{\alpha \geq 0} \left(\sum_{\frac{n-4}{n-3}\alpha \leq \beta \leq \frac{4}{3}\alpha} c_{\alpha\beta} u_k^{(3-k)\alpha - (2-k)\beta} v_k^{(4-k)\alpha - (3-k)\beta} \right) \quad (3.11)$$

on W_k for $0 \leq k \leq 3$,

$$f = \sum_{\alpha \geq 0} \left(\sum_{\frac{n-4}{n-3}\alpha \leq \beta \leq \frac{4}{3}\alpha} c_{\alpha\beta} u_4^{-\alpha} (1 + u_4^2 v_4)^{\alpha - \beta} \right) \quad \text{on } W_4 \quad (3.12)$$

and

$$f = \sum_{\alpha \geq 0} \left(\sum_{\frac{n-4}{n-3}\alpha \leq \beta \leq \frac{4}{3}\alpha} c_{\alpha\beta} u_k^{(k-3)\beta - (k-4)\alpha} v_k^{(k-4)\beta - (k-5)\alpha} (1 + u_k^{k-3} v_k^{k-4})^{\alpha - \beta} \right) \quad (3.13)$$

on W_k , $5 \leq k \leq n$

In each expansion, the sum over β is holomorphic on the corresponding W_k for each $\alpha \geq 0$, and the sum over α is absolutely convergent.

Proof. The charts (u_k, v_k) , $0 \leq k \leq 5$, and their transition functions are the same as those of D_5 . The assertions for $0 \leq k \leq 5$ can be proved as in Lemma 3.5, except that we need to check the different projection.

Case E_6 . The formulas corresponding to (3.7), (3.10) are

$$\sum_{\beta} c_{\alpha\beta} u_4^{-\alpha} (1 + u_4^2 v_4)^{\alpha-\beta} = \sum_{6i+4j+3k=\alpha} \tilde{c}_{ijk} \left(\frac{4v_4^3(2 + u_4^2 v_4)}{(1 + u_4^2 v_4)^2} \right)^i \left(\frac{4v_4}{1 + u_4^2 v_4} \right)^j \left(\frac{2u_4 v_4^2}{1 + u_4^2 v_4} \right)^k, \quad (3.14)$$

$$\sum_{\beta} c_{\alpha\beta} u_3^{\beta} = \sum_{6i+4j+3k=\alpha} \tilde{c}_{ijk} (4u_3^4(1 - u_3)^3(1 + u_3))^i \cdot (4u_3^3(1 - u_3)^2)^j (2u_3^2(1 - u_3)^2)^k, \quad (3.15)$$

because v_3 appears as powers of 6, 4, 3 in x, y, z respectively.

(3.14) shows that the sum over β is holomorphic on W_4 . (3.15) shows that $3\beta - 2\alpha \geq 0$ hence the sums over β in (3.13) are holomorphic on each W_k .

To prove (3.12) at $(0, c) \in W_4$ (resp. $(0, c) \in W_5$, $(0, c)$ and $(c, 0) \in W_6$), we assume the inequality $|x|^2 + |y|^2 + |z|^2 < 1$ at $(u_4, v_4) = (\epsilon e^{i\theta}, c)$ (resp. $(u_5, v_5) = (\epsilon e^{i\theta}, c)$, $(u_6, v_6) = (\epsilon e^{i\theta}, c)$ and $(c, \epsilon e^{i\theta})$) for all $\theta \in \mathbb{R}$. Under coordinate changes, we would get the same inequality for $(u_3, v_3) = \left(\frac{1}{1+c\epsilon^2 e^{2i\theta}}, \frac{1+c\epsilon^2 e^{2i\theta}}{\epsilon e^{i\theta}} \right)$ (resp. $\left(\frac{c\epsilon^2 e^{2i\theta}}{1+c\epsilon^2 e^{2i\theta}}, \frac{1+c\epsilon^2 e^{2i\theta}}{\epsilon e^{i\theta}} \right)$, $\left(\frac{c^2 \epsilon^3 e^{3i\theta}}{1+c^2 \epsilon^3 e^{3i\theta}}, \frac{1+c^2 \epsilon^3 e^{3i\theta}}{c\epsilon^2 e^{2i\theta}} \right)$ and $\left(\frac{c^3 \epsilon^2 e^{2i\theta}}{1+c^3 \epsilon^2 e^{2i\theta}}, \frac{1+c^3 \epsilon^2 e^{2i\theta}}{c^2 \epsilon e^{i\theta}} \right)$). If for all θ , we change v_3 to $v_3^\circ = \frac{\max_{\theta} |1+c\epsilon^2 e^{2i\theta}|}{\epsilon}$ (resp. $\frac{\max_{\theta} |1+c\epsilon^2 e^{2i\theta}|}{\epsilon}$, $\frac{\max_{\theta} |1+c^2 \epsilon^3 e^{3i\theta}|}{|c|^2 \epsilon}$ and $\frac{\max_{\theta} |1+c^3 \epsilon^2 e^{2i\theta}|}{|c|^2 \epsilon}$), then the inequality corresponding to (3.9) becomes

$$|x|^2 \left| \frac{v_3^\circ}{v_3} \right|^{12} + |y|^2 \left| \frac{v_3^\circ}{v_3} \right|^8 + |z|^2 \left| \frac{v_3^\circ}{v_3} \right|^6 < 1 \quad (3.16)$$

in all cases, because of the way v_3 appears in x, y, z . It is clear that $\epsilon > 0$ can be found such that (3.16) holds for all $\theta \in \mathbb{R}$. Then the proof for E_6 can be finished as for Lemma 3.5.

Case E_7 . The formulas corresponding to (3.7), (3.10) are

$$\sum_{\beta} c_{\alpha\beta} u_4^{-\alpha} (1 + u_4^2 v_4)^{\alpha-\beta} = \sum_{9i+6j+4k=\alpha} \tilde{c}_{ijk} \left(\frac{u_4 v_4^5}{(1 + u_4^2 v_4)^3} \right)^i \left(\frac{v_4^3}{(1 + u_4^2 v_4)^2} \right)^j \left(\frac{v_4^2}{1 + u_4^2 v_4} \right)^k, \quad (3.17)$$

$$\sum_{\beta} c_{\alpha\beta} u_3^{\beta} = \sum_{9i+6j+4k=\alpha} \tilde{c}_{ijk} (u_3^7(1-u_3)^5)^i (u_3^5(1-u_3))^j (u_3^3(1-u_3)^2)^k. \quad (3.18)$$

Again 3.18, implies $4\beta - 3\alpha(\geq i + 2j) \geq 0$.

The inequality corresponding to (3.9) for proving the various cases of (3.12), (3.13) is

$$|x|^2 \left| \frac{v_3^{\circ}}{v_3} \right|^{18} + |y|^2 \left| \frac{v_3^{\circ}}{v_3} \right|^{12} + |z|^2 \left| \frac{v_3^{\circ}}{v_3} \right|^8 < 1 \quad (3.19)$$

under a corresponding assumption $|x|^2 + |y|^2 + |z|^2 < 1$. The rest of the proof is similar.

Case E_8 . The formulas corresponding to (3.7), (3.10) are

$$\sum_{\beta} c_{\alpha\beta} u_4^{-\alpha} (1 + u_4^2 v_4)^{\alpha-\beta} = \sum_{15i+10j+6k=\alpha} \tilde{c}_{ijk} \left(\frac{u_4 v_4^8}{(1 + u_4^2 v_4)^5} \right)^i \left(\frac{v_4^5}{(1 + u_4^2 v_4)^3} \right)^j \left(\frac{v_4^3}{(1 + u_4^2 v_4)^2} \right)^k, \quad (3.20)$$

$$\sum_{\beta} c_{\alpha\beta} u_3^{\beta} = \sum_{15i+10j+6k=\alpha} \tilde{c}_{ijk} (u_3^{12}(1-u_3)^8)^i (u_3^8(1-u_3)^5)^j (u_3^5(1-u_3)^3)^k. \quad (3.21)$$

Again 3.21, implies $5\beta - 4\alpha(\geq k) \geq 0$.

The inequality corresponding to (3.9) takes the form

$$|x|^2 \left| \frac{v_3^{\circ}}{v_3} \right|^{30} + |y|^2 \left| \frac{v_3^{\circ}}{v_3} \right|^{20} + |z|^2 \left| \frac{v_3^{\circ}}{v_3} \right|^{12} < 1. \quad (3.22)$$

The proof is then clear.

Q.E.D.

Holomorphic 2-forms on M :

The holomorphic 2-form $\varphi_0 = \pi^{-1} \left(\frac{dx \wedge dy}{F_z} \right) = du_0 \wedge dv_0 = \cdots = du_3 \wedge dv_3 = du_{n-2} \wedge dv_{n-2} = -\frac{du_4 \wedge dv_4}{1+u_4^2 v_4} = \frac{du_k \wedge dv_k}{1+u_k^{k-3} v_k^{k-4}}$, $5 \leq k \leq n$, is nowhere zero on M . Hence any holomorphic 2-form on M is of the form $f\varphi_0$, where f is a holomorphic function on M .

Proposition 3.8. With the above notation for E_n singularities, $n = 6, 7, 8$, $f^{(1,1)} = 1$.

Proof. From the above calculation, we know that the holomorphic functions on M are generated by a base $\{u_3^{\beta} v_3^{\alpha}\}_{\frac{n-4}{n-3} \alpha \leq \beta \leq \frac{4}{3} \alpha}$ and holomorphic 2-forms are generated by a base $\{u_3^{\beta} v_3^{\alpha} du_3 \wedge dv_3\}_{\frac{n-4}{n-3} \alpha \leq \beta \leq \frac{4}{3} \alpha}$. For every holomorphic 2-form $\omega = u_3^{\beta} v_3^{\alpha} du_3 \wedge dv_3$ on M , we consider $\xi = -\frac{u_3^{\beta} v_3^{\alpha+1}}{\alpha+1} du_3$. ξ defines a holomorphic 1-form on W_3 and $d\xi = \omega$. It remains to check that under all changes of charts, ξ transforms to define a holomorphic 1-form in each coordinate chart. In fact, changed to (u_k, v_k) chart, for $k = 0, 1, 2$, on W_k ,

$$\xi = -\frac{1}{\alpha+1} u_k^{(3-k)\alpha-(2-k)\beta} v_k^{(4-k)\alpha-(3-k)\beta} ((2-k)v_k du_k + (3-k)u_k dv_k)$$

which is holomorphic. And changing charts to (u_4, v_4) chart, we see by Lemma 3.7 that

$$\xi = -\frac{u_4^{-\alpha}}{\alpha+1} (1+u_4^2 v_4)^{\alpha-\beta} \left(\frac{2v_4 du_4 + u_4 dv_4}{1+u_4^2 v_4} \right)$$

defines a holomorphic 1-form on W_4 . Finally, changed to (u_k, v_k) chart, for $5 \leq k \leq n$, charts

$$\xi = -\frac{u_k^{(k-3)\beta-(k-4)\alpha} v_k^{(k-4)\beta-(k-5)\alpha}}{\alpha+1} (1+u_k^{k-3} v_k^{k-4})^{\alpha-\beta} \left(\frac{(k-3)v_k du_k + (k-4)u_k dv_k}{1+u_k^{k-3} v_k^{k-4}} \right)$$

Again by Lemma 3.7, ξ is holomorphic on W_k , for $5 \leq k \leq n$. So

$$\{u_3^\beta v_3^{\alpha+1} du_3\}_{\frac{n-4}{n-3}\alpha \leq \beta \leq \frac{4}{3}\alpha}$$

generates $d^{-1}(\Gamma(M, \Omega_M^2))$.

We also know that $d(\Gamma(M, \mathcal{O}_M))$ is generated by

$$\{\beta u_3^{\beta-1} v_3^\alpha du_3 + \alpha u_3^\beta v_3^{\alpha-1} dv_3\}_{\frac{n-4}{n-3}\alpha \leq \beta \leq \frac{4}{3}\alpha}$$

By Lemma 3.3, $\Gamma(M, \Omega_M^1) = d(\Gamma(M, \mathcal{O}_M)) + d^{-1}(\Gamma(M, \Omega_M^2))$. So by easy calculation $\Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1)$ is generated by

$$\{u_3^\beta v_3^\alpha du_3 \wedge dv_3\}_{\frac{n-4}{n-3}\alpha \leq \beta \leq \frac{4}{3}\alpha, \alpha \geq 1}$$

Therefore

$$\frac{\Gamma(M, \Omega_M^2)}{\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle} = \langle u_3 \wedge v_3 \rangle,$$

and

$$f^{(1,1)} = \dim \frac{\Gamma(M, \Omega_M^2)}{\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle} = 1.$$

Q.E.D.

Cyclic quotient singularities:

An explicit resolution $\pi : M \rightarrow V$ can be given in terms of coordinates and transition functions on M as follows:

Coordinates charts: $W_k = \mathbb{C}^2 = \{(u_k, v_k)\}$, $k = 0, 1, \dots, n$

Transition functions :

$$\begin{cases} u_k = u_{k+1}^{e_{k+1}} v_{k+1} \\ v_k = \frac{1}{u_{k+1}} \end{cases}$$

Exceptional set: $A = \pi^{-1}(0) = C_1 \cup \dots \cup C_n$, where $C_k = \{u_{k-1} = 0\} \cup \{v_k = 0\}$

Dual graph:
$$\begin{array}{c} -e_1 -e_2 \qquad \qquad -e_n \\ \circ \text{---} \circ \dots \circ \text{---} \circ \\ C_1 \ C_2 \qquad \qquad C_n \end{array}$$

where $e_i \geq 2$ is the self-intersection number, $i = 1, 2, \dots, n$.

Holomorphic functions on M :

Let

$$\begin{aligned} q_0(\alpha, \beta) &= \alpha, & q_1(\alpha, \beta) &= e_1 q_0(\alpha, \beta) - \beta = e_1 \alpha - \beta, \\ q_i(\alpha, \beta) &= e_i q_{i-1}(\alpha, \beta) - q_{i-2}(\alpha, \beta) \quad i = 2, 3, \dots, n. \end{aligned}$$

Any holomorphic function on M has power series expansion $\sum_{\alpha, \beta \geq 0} c_{\alpha\beta} u_0^\alpha v_0^\beta$ which converges for all $(u_0, v_0) \in W_0$. Under changes of charts,

$$\begin{aligned} \sum_{\alpha, \beta \geq 0} c_{\alpha\beta} u_0^\alpha v_0^\beta &= \sum_{\alpha, \beta \geq 0} c_{\alpha\beta} u_1^{q_1(\alpha-\beta)} v_1^{q_0(\alpha-\beta)} = \sum_{\alpha, \beta \geq 0} c_{\alpha\beta} u_2^{q_2(\alpha-\beta)} v_2^{q_1(\alpha-\beta)} = \dots \\ &= \sum_{\alpha, \beta \geq 0} c_{\alpha\beta} u_n^{q_n(\alpha-\beta)} v_n^{q_{n-1}(\alpha-\beta)}. \end{aligned}$$

The k -th power series has to converge for all $(u_k, v_k) \in W_k$. This occurs for all k if only if the indices α, β in each sum satisfy $q_i(\alpha - \beta) \geq 0$, $\beta \geq 0$, $i = 0, 1, \dots, n$. Thus any holomorphic function on M can be generated by

$$\{u_0^\alpha v_0^\beta\}_{q_i(\alpha, \beta) \geq 0, \beta \geq 0, i=0,1,\dots,n}$$

on W_0 .

Conversely, any such convergent power series in the (u_0, v_0) chart defines a holomorphic function on M .

Holomorphic 2-forms on M :

We know

$$\begin{cases} du_i = e_{i+1} u_{i+1}^{e_{i+1}-1} v_{i+1} du_{i+1} + u_{i+1}^{e_{i+1}} dv_{i+1} \\ v_i = -\frac{1}{u_{i+1}^2} du_{i+1} \end{cases}$$

The holomorphic 2-form

$$\begin{aligned} du_0 \wedge dv_0 &= u_1^{q_1(1,1)-1} du_1 \wedge dv_1 = u_2^{q_2(1,1)-1} v_2^{q_1(1,1)-1} du_2 \wedge dv_2 = \dots \\ &= u_n^{q_n(1,1)-1} v_n^{q_{n-1}(1,1)-1} du_n \wedge dv_n. \end{aligned}$$

So

$$\begin{aligned} u_0^\alpha v_0^\beta du_0 \wedge dv_0 &= u_1^{q_1(\alpha+1, \beta+1)-1} du_1 \wedge dv_1 = u_2^{q_2(\alpha+1, \beta+1)-1} v_2^{q_1(\alpha+1, \beta+1)-1} du_2 \wedge dv_2 \\ &= \dots = u_n^{q_n(\alpha+1, \beta+1)-1} v_n^{q_{n-1}(\alpha+1, \beta+1)-1} du_n \wedge dv_n. \end{aligned}$$

It follows that any holomorphic 2-form on M can be generated in the (u_0, v_0) chart by 2-forms

$$\{u_0^\alpha v_0^\beta du_0 \wedge dv_0\}_{q_i(\alpha+1, \beta+1) \geq 1, \beta \geq 0, i=0,1,\dots,n}.$$

Proposition 3.9. With the above notation for cyclic quotient singularities, $f^{(1,1)} = 1$.

Proof. From the above calculation, we know that the holomorphic functions on M are generated by a base

$$\{u_0^\alpha v_0^\beta\}_{q_i(\alpha, \beta) \geq 0, \beta \geq 0, i=0,1,\dots,n}$$

and holomorphic 2-forms are generated by a base

$$\{u_0^\alpha v_0^\beta du_0 \wedge dv_0\}_{q_i(\alpha+1, \beta+1) \geq 1, \beta \geq 0, i=0,1,\dots,n}.$$

For every holomorphic 2-form $\omega = u_0^\alpha v_0^\beta du_0 \wedge dv_0$ on M , we consider $\xi = -\frac{u_0^\alpha v_0^{\beta+1}}{\beta+1} du_0$. ξ defines a holomorphic 1-form on W_0 and $d\xi = \omega$. It remains to check that under all changes of charts, ξ transforms to define a holomorphic 1-form in each coordinate chart. In fact, changed to (u_k, v_k) chart, for $k = 1, \dots, n$, on W_k ,

$$\xi = -\frac{1}{\beta+1} u_k^{q_k(\alpha+1, \beta+1)-1} v_k^{q_{k-1}(\alpha+1, \beta+1)-1} (q_k(0, -1)v_k du_k + q_{k-1}(0, -1)u_k dv_k)$$

which is holomorphic. So

$$\{u_0^\alpha v_0^{\beta+1} du_0\}_{q_i(\alpha+1, \beta+1) \geq 1, \beta \geq 0, i=0,1,\dots,n}$$

generates $d^{-1}(\Gamma(M, \Omega_M^2))$.

We also know that $d(\Gamma(M, \mathcal{O}_M))$ is generated by

$$\{u_0^\alpha v_0^\beta\}_{q_i(\alpha, \beta) \geq 0, \beta \geq 0, \alpha \geq 1, i=0,1,\dots,n}.$$

By Lemma 3.3, $\Gamma(M, \Omega_M^1) = d(\Gamma(M, \mathcal{O}_M)) + d^{-1}(\Gamma(M, \Omega_M^2))$. So by easy calculation $\Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1)$ is generated by

$$\{u_0^\alpha v_0^\beta du_0 \wedge dv_0\}_{q_i(\alpha+1, \beta+1) \geq 1, \beta \geq 0, \alpha \geq 1, i=0,1,\dots,n}.$$

Therefore

$$\frac{\Gamma(M, \Omega_M^2)}{\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle} = \langle u_0 \wedge v_0 \rangle,$$

and

$$f^{(1,1)} = \dim \frac{\Gamma(M, \Omega_M^2)}{\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle} = 1.$$

Q.E.D.

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