Moduli space of complete Reinhardt domains and complex Plateau problem

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Main Purpose

Two fundamental problems in complex geometry:

- Biholomorphically (resp. CR) equivalent problem for (resp. boundary of) domains in singular varieties and in $\mathbb{C}^n$.

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- Biholomorphically (resp. CR) equivalent problem for (resp. boundary of) domains in singular varieties and in $\mathbb{C}^n$.
- The classical complex Plateau problem.

Moduli space of bounded complete Reinhardt domains

D_1, D_2 are two domains in $\mathbb{C}^n$: *when are D_1 and D_2 (resp. ∂D_1 and ∂D_2) biholomorphically equivalent (resp. CR equivalent)?*

History

- $n = 1$ Riemann Mapping Theorem: Any simply connected domains in $\mathbb{C}$ are biholomorphically equivalent.

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History

- $n = 1$ Riemann Mapping Theorem: Any simply connected domains in $\mathbb{C}$ are biholomorphically equivalent.
- $n \geq 2$ There are many domains which are topologically equivalent to the ball but not biholomorphically equivalent to the ball.

- **Poincaré (1907)**:

  Found necessary and sufficient conditions on a first order perturbation of the unit ball in $\mathbb{C}^2$ that the perturbed domain is biholomorphic to the ball.

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  Systematic study of invariance properties for real hypersurfaces. The main result is the existence of complete system of local differential invariants for CR structures on real hypersurface.

- **Fefferman (1974)**:
  A biholomorphic mapping between two strictly pseudoconvex domains is smooth up to the boundaries and the induced boundary mapping gives a CR-equivalence between the boundaries.

- **Webster (1978)**:
  Gave a complete characterization when two ellipsoids in $\mathbb{C}^n$ are biholomorphically equivalent.

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- **Burns, Schnider and Wells (1978)**:
  The "number of moduli" of a "moduli space" of a strictly pseudoconvex bounded domain has to be infinite.
- **Lempert (1988)**:
  Constructed moduli space of bounded strictly convex domains of $\mathbb{C}^n$ with marking at the origin. Although the theory is beautiful, the computation of Lempert's invariants is a hard problem.

- **Sunada (1978)**:
  Let D_1, D_2 be two complete Reinhardt domains. If D_1 is biholomorphically equivalent to D_2, then there exists a permutation σ of n letters and a biholomorphic map

$$\psi(z_1, \cdots, z_n) = (a_1 z_{\sigma(1)}, \cdots, a_n z_{\sigma(n)}), \qquad a_i > 0$$

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- **Yau (2004)**:
  Introduced the new biholomorphic invariant Bergman function defined on pseudoconvex domains in a variety with only isolated singularities.

Preliminaries

Definition Let $D \subset \mathbb{C}^n$ be a domain (open connected subset). We say D is pseudoconvex if there exists a continuous plurisubharmonic function φ on D such that the sets $\{z \in D \mid \varphi(z) < x\}$ are relatively compact subsets of D for all $x \in \mathbb{R}$.

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$\Leftrightarrow$ Hessian of φ is positive semi-definite (resp. definite).

Definition If φ is strictly plurisubharmonic, then domain D is called strictly pseudoconvex.

Definition Let X be a compact connected orientable manifold of real dimension $2n-1$, $n \geq 2$. A **CR structure** on X is a rank $n-1$ subbundle S of the complexified tangent bundle $\mathbb{C}T(X)$ such that

(1) $S \cap \overline{S} = \{0\}$

(2) If L, L' are local sections of S, then so is $[L, L']$.

The manifold X, together with the CR structure S, is called a **CR manifold**.

Definition Let $L_1, \ldots, L_{n-1}$ be a local frame of the CR structure S on X so the $\overline{L}_1, \ldots, \overline{L}_{n-1}$ is a local frame of $\overline{S}$. Since $S \oplus \overline{S}$ has complex codimension one in $\mathbb{C}T(X)$, we may choose a local section N of $\mathbb{C}T(X)$ such that $L_1, \ldots, L_{n-1}$, $\overline{L}_1, \ldots, \overline{L}_{n-1}, N$ span $\mathbb{C}T(X)$. We may assume that N is purely imaginary. Then the matrix (c_{ij}) defined by

$$[L_i, \overline{L}_j] = \sum_k a_{ij}^k L_k + \sum_k b_{ij}^k \overline{L}_k + c_{ij} N$$

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Definition An open subset $D \subseteq \mathbb{C}^n$ is a complete Reinhardt domain if, whenever $(z_1, \cdots, z_n) \in D$ then $(\xi_1 z_1, \cdots, \xi_n z_n) \in D$ for all complex numbers ξ_j with $|\xi_j| \leqslant 1$.

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Let X be an embeddable compact CR manifold. Then there exists a complex variety V in $\mathbb{C}^N$ such that $\partial V = X$ and V has only normal isolated singularities.

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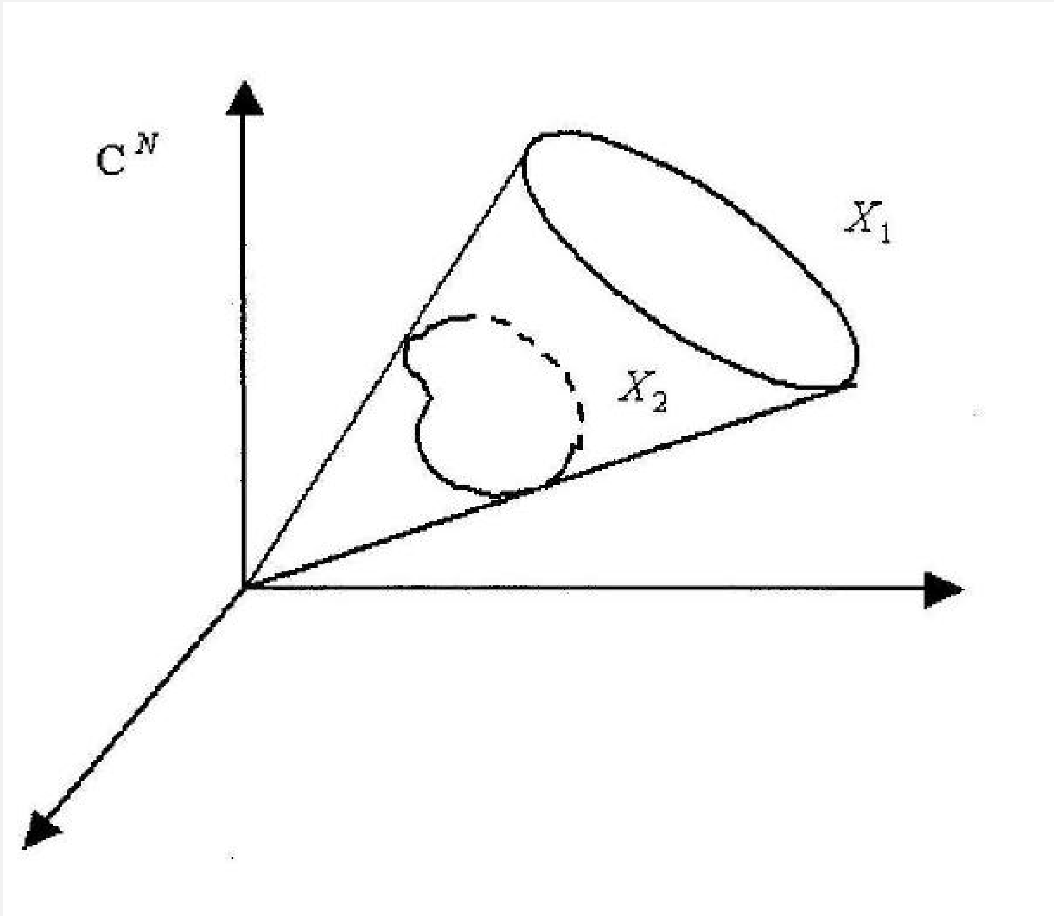
Theorem [Harvey–Lawson]
Let X be an embeddable compact CR manifold. Then there exists a complex variety V in $\mathbb{C}^N$ such that $\partial V = X$ and V has only normal isolated singularities.

Theorem [Yau] Let X_1, X_2 be two strictly pseudoconvex CR manifolds of dimension $2n-1$ which bound varieties V_1, V_2 respectively in $\mathbb{C}^N$ with only isolated normal singularities. If $\phi\colon X_1 \to X_2$ is a CR isomorphism, then ϕ can be extended to a biholomorphic map from V_1 to V_2.

Difficult unsolved problem If the strictly pseudoconvex CR manifolds X_1, X_2 are lying in the same variety V, how can one distinguish them?



Let M be a pseudoconvex complex manifold and A be a compact complex analytic variety in the interior of M.

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Definition Let F_M (respectively, $F_{M,A}$) be the space of all L^2-integrable holomorphic n-form on M (respectively, vanishing at the compact analytic subset A in M). Let $\{\omega_j\}$ (respectively, $\{\omega_j^A\}$) be a complete orthonormal basis of F_M (respectively, $F_{M,A}$). The Bergman kernel (respectively, Bergman kernel vanishing at A) is defined to be $K_M(z) = \sum_j \omega_j(z) \wedge \overline{\omega_j(z)}$ (respectively, $K_{M,A}(z) = \sum_j \omega_j^A(z) \wedge \overline{\omega_j^A(z)}$).

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Lemma (a) Bergman kernel $K_{M,A}(z)$ vanishing at the compact analytic subset A is independent of the choice of the complete orthonormal basis of $F_{M,A}$.

(b) Let $\Phi\colon (M_1, A_1) \to (M_2, A_2)$ be a biholomorphic map such that $\Phi(A_1) = A_2$. Then $K_{M_1,A_1}(z) = \Phi^* K_{M_2,A_2}(z)$.

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Theorem Let A_1 (respectively A_2) be compact analytic variety in complex manifold M_1 (respectively M_2). If $\Phi\colon (M_1, A_1) \to (M_2, A_2)$ is a biholomorphic map, then $B_{M_1,A_1}(z) = B_{M_2,A_2}(\Phi(z))$.

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For a special case, let V be a Stein variety of dimension $n \geqslant 2$ in $\mathbb{C}^N$ with only irreducible isolated singularities. We assume that ∂V is a smooth CR manifold. Let $\pi\colon M \to V$ be a resolution of singularity with E as an exceptional set.

Bounded complete Reinhardt domains in A_n-variety

$V =$ Stein variety of dimension $n \geq 2$ in $\mathbb{C}^N$

with only isolated normal singularities.

$\pi\colon M \longrightarrow V$ resolution of singularities.

$E =$ exceptional set in M

$= \pi^{-1}$ (singular set of V)

$F = \{\phi\colon \phi$ is L^2-holomorphic n-form on $M\}$

a separable Hilbert space with inner product

$$\langle \phi_1, \phi_2 \rangle = \left(\sqrt{-1}\right)^{n^2} \int_M \phi_1 \wedge \overline{\phi_2}.$$

Definition Let V be a Stein variety in $\mathbb{C}^N$ with only irreducible isolated singularities. Let $\pi\colon M \to V$ be a resolution of singularities of V such that the canonical bundle is generated by its global sections in a neighborhood of the exceptional set. Define the k-th order Bergman function $B_V^{(k)}$ on V to be the push forward of the k-th order Bergman function $B_M^{(k)}$ by the map π.

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Theorem Let V be a Stein variety in $\mathbb{C}^N$ with only irreducible isolated singularities. Assume that there exists a resolution M of singularities of V such that the canonical bundle is generated by its global sections in a neighborhood of the exceptional set. Then the k-th order Bergman function $B_V^{(k)}$ on V is invariant under biholomorphic maps and $B_V^{(k)}$ vanishes precisely on the singular set of V.

Definition An open set V in the A_n-variety $\tilde{V_n} = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ is called a complete Reinhardt domain if $p^{-1}(V)$ is a complete Reinhardt domain in $\mathbb{C}^2$, where $p : \mathbb{C}^2 \to \tilde{V_n}$ is given by $p(z_1, z_2) = (z_1^{n+1}, z_2^{n+1}, z_1 z_2)$.

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Recall that the minimal resolution $\tilde{M_n}$ of $\tilde{V_n}$ consists of $n+1$ coordinate charts $\tilde{W_k} = \mathbb{C}^2 = \{(u_k, v_k)\}, k = 0, 1, \cdots, n$. The space of holomorphic two forms on $\tilde{M_n}$ has a basis $\{\phi_{\alpha\beta} = u_0^\alpha v_0^\beta du_0 \wedge dv_0 : \alpha \geqslant \frac{n}{n+1}\beta\}$. Let M $(\subseteq \tilde{M_n})$ be the resolution of complete Reinhardt domain V in $\tilde{V_n}$. In what follows, we shall use notation $\|\phi_{\alpha\beta}\|_M^2$ for $\int_M \phi_{\alpha\beta} \wedge \overline{\phi_{\alpha\beta}}$

Let

$$g^{(\alpha,\beta)} := \frac{\|\phi_{10}\|^{\alpha-\frac{n}{n+1}\beta}\|\phi_{n,n+1}\|^{\frac{\beta}{n+1}}}{\|\phi_{\alpha\beta}\|\|\phi_{00}\|^{\alpha-\frac{n-1}{n+1}\beta-1}}.$$

$$\xi^{(\alpha,\beta)} := g^{(\alpha,\beta)} \cdot g^{(n\alpha-(n-1)\beta,(n+1)\alpha-n\beta)},$$

$$\zeta^{(\alpha,\beta)} := g^{(\alpha,\beta)} + g^{(n\alpha-(n-1)\beta,(n+1)\alpha-n\beta)},$$

$$\eta^{(\alpha,p,q)} := (g^{(\alpha,p)} - g^{(n\alpha-(n-1)p,(n+1)\alpha-np)}) \cdot (g^{(\alpha,q)} - g^{(n\alpha-(n-1)q,(n+1)\alpha-nq)})$$

and

$$\omega^{(\alpha_1,\alpha_2,p_1,p_2)} := (g^{(\alpha_1,p_1)} - g^{(n\alpha_1-(n-1)p_1,(n+1)\alpha_1-np_1)}) \cdot (g^{(\alpha_2,p_2)} - g^{(n\alpha_2-(n-1)p_2,(n+1)\alpha_2-np_2)}),$$

where

$$\alpha \geqslant 1, \alpha \geqslant \frac{n}{n+1}\beta, 0 \leqslant p,\ q \leqslant \left[\frac{n+1}{n}\alpha\right], p \neq q,$$

$$0 \leqslant p_i \leqslant \left[\frac{n+1}{n}\alpha_i\right], \alpha_i \geqslant 1, \alpha_1 \neq \alpha_2, i = 1, 2,$$

Theorem A Let V_i, $i = 1, 2$, be two bounded complete Reinhardt domains in A_n-variety $\tilde{V_n} = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$. If V_1 is biholomorphic to V_2, then

$$\xi_{V_1}^{(\alpha,\beta)} = \xi_{V_2}^{(\alpha,\beta)}, \zeta_{V_1}^{(\alpha,\beta)} = \zeta_{V_2}^{(\alpha,\beta)}, \eta_{V_1}^{(\alpha,p,q)} = \eta_{V_2}^{(\alpha,p,q)},$$

$$\omega_{V_1}^{(\alpha_1,\alpha_2,p_1,p_2)} = \omega_{V_2}^{(\alpha_1,\alpha_2,p_1,p_2)}.$$

Theorem B Let V_i, $i=1,2$, be two bounded complete Reinhardt strictly pseudoconvex (respectively C^ω-smooth pseudoconvex) domains in $\tilde{V_n}=\{(x,y,z)\in\mathbb{C}^3\colon xy=z^{n+1}\}$. If

$$\xi_{V_1}^{(\alpha,\beta)}=\xi_{V_2}^{(\alpha,\beta)},\zeta_{V_1}^{(\alpha,\beta)}=\zeta_{V_2}^{(\alpha,\beta)},\eta_{V_1}^{(\alpha,p,q)}=\eta_{V_2}^{(\alpha,p,q)},$$

$$\omega_{V_1}^{(\alpha_1,\alpha_2,p_1,p_2)}=\omega_{V_2}^{(\alpha_1,\alpha_2,p_1,p_2)},$$

then there exists an automorphism $\Psi=(\psi_1,\psi_2,\psi_3)$ of A_n-variety $\tilde{V_n}=\{(x,y,z)\in\mathbb{C}^3: xy=z^{n+1}\}$ given by either

$$(\psi_1,\psi_2,\psi_3)=$$

$$\left(\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}}x,\ \frac{\|\phi_{n,n+1}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{n,n+1}\|_{M_1}}y,\frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}}z\right),$$

or

$$(\psi_1,\psi_2,\psi_3)=$$

$$\left(\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{n,n+1}\|_{M_1}}y,\ \frac{\|\phi_{n,n+1}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}}x,\frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}}z\right).$$

such that Ψ sends V_1 to V_2.

Theorem C The moduli space of bounded complete Reinhardt strictly pseudoconvex (respectively C^ω-smooth pseudoconvex) domains in A_n-variety $\tilde{V}_n = \{(x,y,z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ is given by the image of the map $\Phi : \{V : V$ a bounded complete Reinhardt strictly pseudoconvex (respectively C^ω-smooth pseudoconvex) domain in $\tilde{V}_n\} \to \mathbb{R}^\infty$, where the component function of Φ are the invariant functions

$$\xi^{(\alpha,\beta)}, \zeta^{(\alpha,\beta)}, \eta^{(\alpha,p,q)} \omega^{(\alpha_1,\alpha_2,p_1,p_2)}.$$

Relation between bounded complete Reinhardt domains in A_n-variety $\tilde{V_n}$ and the corresponding bounded complete Reinhardt domains in $\mathbb{C}^2$

Theorem D Let $\pi : \mathbb{C}^2 \to \tilde{V_n} = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ be the quotient map given by $\pi(z_1, z_2) = (z_1^{n+1}, z_2^{n+1}, z_1 z_2)$. Let V_i, $i = 1, 2$, be bounded complete Reinhardt domains in $\tilde{V_n}$ such that $W_i := \pi^{-1}(V_i), i = 1, 2$, are bounded complete Reinhardt domain in $\mathbb{C}^2$. Then V_1 is biholomorphic to V_2 if and only if W_1 is biholomorphic to W_2. In particular, V_1 is biholomorphic to V_2 if and only if there exists a biholomorphism $\Phi : V_1 \to V_2$ given by $\Phi(x, y, z) = (a^{n+1}x, b^{n+1}y, abz)$ or $\Phi(x, y, z) = (a^{n+1}y, b^{n+1}x, abz)$ where $a, b > 0$.

Theorem E (1) Let $\mathcal{W} = \{W : W = \pi^{-1}(V)$ where V is a bounded complete Reinhardt domain in A_n-variety$\}$. Then

$$\xi^{(\alpha,\beta)}, \zeta^{(\alpha,\beta)}, \eta^{(\alpha,p,q)}, \omega^{(\alpha_1,\alpha_2,p_1,p_2)}$$

are invariants of $\mathcal{W}$.

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are invariants of $\mathcal{W}$.

(2) Let $\mathcal{W}_P = \{W : W = \pi^{-1}(V)$ where V is a complete Reinhardt pseudoconvex C^ω-smooth domain in A_n-variety$\}$ and $\mathcal{W}_{SP} = \{W : W = \pi^{-1}(V)$ where V is a complete Reinhardt strictly pseudoconvex domain in A_n-variety$\}$. Then the moduli space of $\mathcal{W}_{\mathcal{P}}$ (respectively $\mathcal{W}_{SP}$) is given by the image of the map $\tilde{\Phi_P} : \mathcal{W}_P \to \mathbb{R}^\infty$ (respectively $\tilde{\Phi_{SP}} : \mathcal{W}_{SP} \to \mathbb{R}^\infty$), where the component functions of $\tilde{\Phi_P}$ (respectively $\tilde{\Phi_{SP}}$) are the invariant functions

$$\xi^{(\alpha,\beta)}, \zeta^{(\alpha,\beta)}, \eta^{(\alpha,p,q)}, \omega^{(\alpha_1,\alpha_2,p_1,p_2)}.$$

In particular, the moduli space of $\mathcal{W}_P$ (respectively $\mathcal{W}_{SP}$) is the same as the moduli space of bounded complete Reinhardt pseudoconvex C^ω-smooth domains (respectively bounded complete Reinhardt strictly pseudoconvex domains) in A_n-variety $\tilde{V_n} = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$.

Let

$$\nu^{(\alpha,\beta)} = \frac{\sqrt{\xi^{(\alpha,\beta)} \cdot \xi^{(\alpha,2\alpha-\beta)}}}{\sqrt{\xi^{(\alpha,\alpha)}}}, \qquad .$$

Example

Let

$$V^{(d)}_{(a,b,c)} = \{(x,y,z) \in \mathbb{C}^3 \colon xy = z^2, a|x|^{2d} + b|y|^{2d} + c|z|^{2d} < \varepsilon_0\}.$$

Let $\sim$ denote the biholomorphic equivalence. Then the map

$$\varphi \colon \{V^{(d)}_{(a,b,c)}\} \to \mathbb{R}_+,\ V^{(d)}_{(a,b,c)} \mapsto \nu^{(2d-1,d-1)}$$

is injective up to a biholomorphism $\sim$. More precisely the induced map

$$\tilde{\varphi} \colon \{V^{(d)}_{(a,b,c)}\}/\sim\ \to \mathbb{R}_+$$

is one-to-one map from $\{V^{(d)}_{(a,b,c)}\}/\sim$ onto $\left(0, \frac{2}{\pi}\right)$. So the moduli space of $\{V^{(d)}_{(a,b,c)}\}$ is an open interval $\left(0, \frac{2}{\pi}\right)$.

Bounded complete Reinhardt domains in $\mathbb{C}^n$

Let S_n be the symmetric group of degree n. Recall that the group ring $\mathbb{R}[S_n]$ is a ring of the form $\mathbb{R}[\tau_1, \tau_2, \dots, \tau_{n!}]$ with $\tau_i \in S_n$ for $1 \leqslant i \leqslant n!$. Let $\sigma \in S_n$ and $(\sum_i x_i\tau_i, \dots, \sum y_i\tau_i) \in \mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n]$. Then

$$\sigma\left(\sum_i x_i\tau_i, \dots, \sum y_i\tau_i\right) := \left(\sum_i x_i(\tau_i\sigma), \dots, \sum y_i(\tau_i\sigma)\right). \tag{1}$$

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Definition Two elements f, g in $\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n]$ are said to be equivalent and denoted by $f \sim g$ if there exists a $\sigma \in S_n$ such that $\sigma(f) = g$.

Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$ be an n-tuple of nonnegative integers.

Denote $\phi_{\vec{\alpha}} = \left(\prod_{i=1}^{n} z_i^{\alpha_i}\right) dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$.

For a domain D in $\mathbb{C}^n$, $\|\phi_{\vec{\alpha}}\|_D^2 := \int_D \phi_{\vec{\alpha}} \wedge \phi_{\vec{\alpha}}$.

Theorem F Let D be a bounded complete Reinhardt domain in $\mathbb{C}^n$. Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$ be a n-tuple of nonnegative integers. For any $\tau \in S_n$, denote

$$g_D^\tau(\vec{\alpha}) = \frac{\|\phi_{\vec{0}}\|_D^{\Sigma\alpha_i - 1} \|\phi_{\tau(\vec{\alpha})}\|_D}{\prod_{i=1}^{n} \|\phi_{\vec{e}_i}\|_D^{\alpha_{\tau(i)}}} \tag{2}$$

where $\tau(\vec{\alpha}) = (\alpha_{\tau(1)}, \ldots, \alpha_{\tau(n)})$ and $\vec{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the ith component. Then for all n-tuples of nonnegative integers $\vec{\beta}_1, \ldots, \vec{\beta}_{n!}$, $\xi_D^{\vec{\beta}_1, \ldots, \vec{\beta}_{n!}} = \left(\sum_{\tau \in S_n} g_D^\tau(\vec{\beta}_1)\tau, \ldots, \sum_{\tau \in S_n} g_D^\tau(\vec{\beta}_{n!})\tau\right)$ as an element in $(\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n])/\sim$ is a biholomorphic invariant.

Theorem G Let D_i, $i=1,2$, be two bounded complete Reinhardt pseudoconvex domains in $\mathbb{C}^n$ with C^1 boundaries. If for all $\vec{\alpha},\ldots,\vec{\alpha}_{n!}$ n-tuples of nonnegative integers, $\xi_{D_1}^{\vec{\alpha},\ldots,\vec{\alpha}_{n!}} = \xi_{D_2}^{\vec{\alpha}_1,\ldots,\vec{\alpha}_{n!}}$ in $(\mathbb{R}[S_n]\times\cdots\times\mathbb{R}[S_n])/\sim$, where $\xi_D^{\vec{\alpha}_1,\ldots,\vec{\alpha}_{n!}} = (\sum_{\tau\in S_n} g_D^\tau(\vec{\alpha}_1)\tau,\ldots,\sum_{\tau\in S_n} g_{D_i}^\tau(\vec{\alpha}_{n!})\tau)$, then there exists $\sigma\in S_n$ and a biholomorphic map

$$\Psi_\sigma(z_1,\ldots,z_n) = \left(a_1 z_{\sigma(1)},\ldots,a_n z_{\sigma(n)}\right),$$

where $a_i = \frac{\|\phi_{\vec{0}}\|_{D_1}\|\phi_{\vec{e}_i}\|_{D_2}}{\|\phi_{\vec{e}_{\sigma(i)}}\|_{D_1}\|\phi_{\vec{0}}\|_{D_2}}$, such that Ψ_σ sends D_1 onto D_2.

In case $n = 2$, we can actually write down the complete numerical invariants for two bounded complete Reinhardt in $\mathbb{C}^2$ to be biholomorphically equivalent.

Theorem H Let D_1, D_2 be two bounded complete Reinhardt pseudoconvex domains in $\mathbb{C}^2$ with C^1 boundaries. Then D_1 is biholomorphic to D_2 if and only if

$$g_{D_1}(\alpha_1, \alpha_2) + g_{D_1}(\alpha_2, \alpha_1) = g_{D_2}(\alpha_1, \alpha_2) + g_{D_2}(\alpha_2, \alpha_1) \tag{3}$$

$$g_{D_1}(\alpha_1, \alpha_2) g_{D_1}(\alpha_2, \alpha_1) = g_{D_2}(\alpha_1, \alpha_2) g_{D_2}(\alpha_2, \alpha_1) \tag{4}$$

$$\begin{aligned}&\left(g_{D_1}(\alpha_1, \alpha_2) - g_{D_1}(\alpha_2, \alpha_1)\right)\left(g_{D_1}(\beta_1, \beta_2) - g_{D_1}(\beta_2, \beta_1)\right) \\ &\qquad = \left(g_{D_2}(\alpha_1, \alpha_2) - g_{D_2}(\alpha_2, \alpha_1)\right)\left(g_{D_2}(\beta_1, \beta_2) - g_{D_2}(\beta_2, \beta_1)\right)\end{aligned} \tag{5}$$

for all nonnegative integers α_i, β_i, where

$$g_{D_i}(\alpha_1, \alpha_2) = \frac{\|\phi_{\vec{0}}\|_{D_i}^{\alpha_1+\alpha_2-1} \|\phi_{(\alpha_1,\alpha_2)}\|_{D_i}}{\prod_{j=1}^{2} \|\phi_{\vec{e}_j}\|_{D_i}^{\alpha_j}}.$$

Corollary I The moduli space of bounded complete Reinhardt pseudoconvex domains with C^1 boundaries in $\mathbb{C}^2$ can be constructed explicitly as the image of the complete family of numerical invariants:

$$g_D(\alpha_1, \alpha_2) + g_D(\alpha_2, \alpha_1),$$

$$g_D(\alpha_1, \alpha_2) g_D(\alpha_2, \alpha_1),$$

and

$$(g_D(\alpha_1, \alpha_2) - g_D(\alpha_2, \alpha_1)) \cdot (g_D(\beta_1, \beta_2) - g_2(\beta_2, \beta_1))$$

$\forall\ \alpha_i, \beta_i$ nonnegative integers.

For $n \geq 3$, the problem is related to Hilbert 14^{th} problem.

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Theorem [Hilbert]

$R = \mathbb{C}[x_1, \ldots, x_{n!}, \ldots, y_1, \ldots, y_{n!}]^{S_n}$ is finitely generated.

Theorem J Let $f_1, \dots, f_N \in \mathbb{C}[x_{\sigma_1}, \dots x_{\sigma_{n!}}; \dots ; y_{\sigma_1}, \dots y_{\sigma_{n!}}]^{S_n}$ be the generators of the ring of invariant polynomials. Let D be a bounded complete Reinhardt domain in $\mathbb{C}^n$. Then, for $\vec{\alpha}_1, \vec{\alpha}_2, \dots, \vec{\alpha}_{n!}$ n-tuples of nonnegative integers,

$$f_1(g_D^\sigma(\vec{\alpha}_1), \dots, g_D^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n}, \dots, f_N(g_D^\sigma(\vec{\alpha}_1), \dots, g_D^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n}$$

are biholomorphic invariants, where

$$g_D^\sigma(\vec{\beta}) = \frac{\|\phi_{\vec{0}}\|_D^{\Sigma\beta_i - 1} \|\phi_{\sigma(\vec{\beta})}\|_D}{\prod_{i=1}^{n} \|\phi_{\vec{e}_i}\|_D^{\beta_{\sigma(i)}}}, \quad \vec{\beta} = (\beta_1, \beta_2, \dots, \beta_n).$$

Theorem K Let D_i, $i = 1, 2$, be two bounded complete Reinhardt pseudoconvex domains in $\mathbb{C}^n$ with C^1 boundaries. Let $f_1, \ldots, f_N \in \mathbb{C}[x_{\sigma_1}, \ldots, x_{\sigma_{n!}}; \ldots; y_{\sigma_i}, \ldots, y_{\sigma_{n!}}]^{S_n}$ be the generators of the ring of invariant polynomials. If for all $\vec{\alpha}_1, \ldots, \vec{\alpha}_{n!}$ n-tuples of nonnegative integers

$$f_i(g^{\sigma}_{D_1}(\vec{\alpha}_1), \ldots, g^{\sigma}_{D_1}(\vec{\alpha}_{n!}))_{\sigma \in S_n} = f_i\left(g^{\sigma}_{D_2}(\vec{\alpha}_1), \ldots, g^{\sigma}_{D_2}(\vec{\alpha}_{n!})\right)_{\sigma \in S_n},$$
$$i = 1, 2, \ldots, N,$$

then there exists $\tau \in S_n$ and a biholomorphic map $\Psi_\tau \colon \mathbb{C}^n \to \mathbb{C}^n$,

$$\Psi_\tau(z_1, \ldots, z_n) = (a_1 z_{\tau(1)}, \ldots, a_n z_{\tau(n)}),$$

where

$$a_i = \frac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e}_i}\|_{D_2}}{\|\phi_{e_{\tau(i)}}\|_{D_1} \|\phi_{\vec{0}}\|_{D_2}},$$

such that Ψ_τ sends D_1 onto D_2.

Corollary L The moduli space of bounded complete Reinhardt pseudoconvex domains with C^1 boundaries in $\mathbb{C}^n$ can be constructed explicitly as the image of the complete family of numerical invariants:

$$f_i(g_D^\sigma(\vec{\alpha}_1), \ldots, g_D^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n}, \quad 1 \leqslant i \leqslant N,$$

where $\vec{\alpha}_1, \ldots, \vec{\alpha}_{n!}$ are all possible n-tuples of nonnegative integers.

For complete Reinhardt pseudoconvex domains with real analytic boundaries, we can use fewer numerical invariants to characterize these domains.

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Theorem G' Let D_i, $i = 1, 2$, be two bounded complete Reinhardt pseudoconvex domains in $\mathbb{C}^n$ with real analytic boundaries. Then D_1 is biholomorphically equivalent to D_2 if and only if for all $\vec{\alpha}$ n-tuple of nonnegative integers, $\xi^{\vec{\alpha}}_{D_1} = \xi^{\vec{\alpha}}_{D_2}$ in $\mathbb{R}[S_n]/\sim$, where $\xi^{\vec{\alpha}}_{D_i} = \sum_{\tau \in S_n} g^{\tau}_{D_i}(\vec{\alpha})\tau$. In this case, there exists $\sigma \in S_n$ and a biholomorphic map

$$\Psi_\sigma(z_1, \ldots, z_n) = (a_1 z_{\sigma(1)}, \ldots, a_n z_{\sigma(n)}),$$

where $a_i = \dfrac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e}}\|_{D_2}}{\|\phi_{\vec{e}_{\sigma(i)}}\|_{D_1} \|\phi_{\vec{0}}\|_{D_2}}$, such that Ψ_σ sends D_1 onto D_2.

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$$g_{D_1}(\alpha_1,\alpha_2)+g_{D_1}(\alpha_2,\alpha_1)=g_{D_2}(\alpha_1,\alpha_2)+g_{D_2}(\alpha_2,\alpha_1)$$

$$g_{D_1}(\alpha_1,\alpha_2)g_{D_1}(\alpha_2,\alpha_1)=g_{D_2}(\alpha_1,\alpha_2)g_{D_2}(\alpha_2,\alpha_1)$$

for all nonnegative integers α_1,α_2, where

$$g_{D_i}(\alpha_1,\alpha_2)=\frac{\|\phi_{\vec{0}}\|_{D_i}^{\alpha_1+\alpha_2-1}\|\phi_{(\alpha_1,\alpha_2)}\|_{D_i}}{\prod_{j=1}^{2}\|\phi_{\vec{e}_j}\|_{D_i}^{\alpha_j}}.$$

Theorem K' Let D_i, $i = 1, 2$, be two bounded complete Reinhardt pseudoconvex domains in $\mathbb{C}^n$ with real analytic boundaries. Let

$$f_1, \ldots, f_N \in \mathbb{C}[x_{\sigma_1}, \ldots, x_{\sigma_{n!}}]^{S_n}$$

be the generators of the ring of invariant polynomials. Then D_1 is biholomorphically equivalent to D_2 if and only if for all $\vec{\alpha}$ n-tuples of nonnegative integers

$$f_i(g^{\sigma}_{D_1}(\vec{\alpha})_{\sigma \in S_n}) = f_i(g^{\sigma}_{D_2}(\vec{\alpha}))_{\sigma \in S_n},$$
$$i = 1, 2, \ldots, N.$$

In this case, there exists $\tau \in S_n$ and a biholomorphic map $\Psi_\tau \colon \mathbb{C}^n \to \mathbb{C}^n$ $\Psi_\tau(z_1, \ldots, z_n) = (a_1 z_{\tau(1)}, \ldots, a_n z_{\tau_{(n)}})$, where $a_i = \dfrac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e}_i}\|_{D_2}}{\|\phi_{e_{\sigma(i)}}\|_{D_1} \|\phi_{\vec{0}}\|_{D_2}}$, such that Ψ_τ sends D_1 onto D_2.

Complex Plateau problem

Background

Classical comple Plateau problem: What kind of odd dimensional $(2n-1,\, n \geq 2)$ real sub-manifolds in $\mathbb{C}^N$ are boundaries of complex sub-manifolds in $\mathbb{C}^N$.

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- **Harvey and Lawson (1975)**
- **Yau (1981)**: Solved the classical complex Plateau problem for the case $n \geq 3$ by calculation of Kohn-Rossi cohomology groups $H^{p,q}_{KR}(X)$.
- For $n = 2$, i.e. X is a 3-dimensional CR manifold, the classical complex Plateau problem remains unsolved for over a quarter of a century.

Invariants of singularities

Let V be a n-dimensional complex analytic subvariety in $\mathbb{C}^N$ with only isolated singularities.

Four kinds of coherent sheaves of germs of holomorphic p-forms:

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- $\Omega_V^p := \Omega^p_{\mathbb{C}^N}/\mathscr{K}^p$, where $\mathscr{K}^p = \{f\alpha + dg \wedge \beta : \alpha \in \Omega^p_{\mathbb{C}^N}; \beta \in \Omega^{p-1}_{\mathbb{C}^N}; f, g \in \mathscr{I}\}$ and $\mathscr{I}$ is the ideal sheaf of V in $\mathbb{C}^N$.

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- $\bar{\bar{\Omega}}_V^p := \theta_* \Omega_{V \backslash V_{sing}}^p$ where $\theta : V \backslash V_{sing} \longrightarrow V$ is the inclusion map and V_{sing} is the singular set of V.
- $\Omega_V^p := \Omega_{\mathbb{C}^N}^p / \mathscr{K}^p$, where $\mathscr{K}^p = \{ f\alpha + dg \wedge \beta : \alpha \in \Omega_{\mathbb{C}^N}^p ; \beta \in \Omega_{\mathbb{C}^N}^{p-1} ; f, g \in \mathscr{I} \}$ and $\mathscr{I}$ is the ideal sheaf of V in $\mathbb{C}^N$.
- $\widetilde{\Omega}_V^p := \Omega_{\mathbb{C}^N}^p / \mathscr{H}^p$, where $\mathscr{H}^p = \{ \omega \in \Omega_{\mathbb{C}^N}^p : \omega|_{V \backslash V_{sing}} = 0 \}$.

Definition Let V be a n-dimensional Stein space with 0 as its only singular point. Let $\pi : (M, A) \to (V, 0)$ be a resolution of the singularity with A as exceptional set. The geometric genus p_g, the irregularity q and $g^{(p)}$ invariant of the singularity are defined as follows:

$$p_g := dim\Gamma(M\backslash A, \Omega^n)/\Gamma(M, \Omega^n), \tag{6}$$

$$q := dim\Gamma(M\backslash A, \Omega^{n-1})/\Gamma(M, \Omega^{n-1}), \tag{7}$$

$$g^{(p)} := dim\Gamma(M, \Omega^p_M)/\pi^*\Gamma(V, \Omega^p_V). \tag{8}$$

The s-invariant of the singularity is defined as follows

$$s := dim\Gamma(M\backslash A, \Omega^n)/[\Gamma(M, \Omega^n) + d\Gamma(M\backslash A, \Omega^{n-1})]. \tag{9}$$

New invariants

Definition Let $(V,0)$ be a Stein germ of a 2-dimensional analytic space with an isolated singularity at 0. Let $\pi : (M,A) \to (V,0)$ be a resolution of the singularity with A as exceptional set. Define a sheaf of germs $\Omega_M^{1,1}$ by the sheaf associated to the presheaf

$$U \mapsto < \Gamma(U, \Omega_M^1) \wedge \Gamma(U, \Omega_M^1) >,$$

where U is an open set of M. Let $\bar{\Omega}_V^{1,1} := \pi_* \Omega_M^{1,1}$.

Lemma Let $(V, 0)$ be a Stein germ of a 2-dimensional analytic space with an isolated singularity at 0. Let $\pi : (M, A) \rightarrow (V, 0)$ be a resolution of the singularity with A as exceptional set. Then $\bar{\Omega}_V^{1,1}$ is coherent and there is a short exact sequence

$$0 \longrightarrow \bar{\Omega}_V^{1,1} \longrightarrow \bar{\Omega}_V^2 \longrightarrow \mathscr{F}^{(1,1)} \longrightarrow 0 \tag{10}$$

where $\mathscr{F}^{(1,1)}$ is a sheaf supported on the singular point of V. Let

$$F^{(1,1)}(M) := \Gamma(M, \Omega_M^2) / < \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) >, \tag{11}$$

then $dim \mathscr{F}_0^{(1,1)} = dim F^{(1,1)}(M)$.

Definition Let V be a 2-dimensional Stein space with 0 as its only singular point. Let $\pi : (M, A) \to (V, 0)$ be a resolution of the singularity with A as exceptional set. Let

$$f^{(1,1)}(0) := dim\mathscr{F}_0^{(1,1)} = dimF^{(1,1)}(M).$$

Definition Let $(V, 0)$ be a Stein germ of a 2-dimensional analytic space with an isolated singularity at 0. Let $\theta : V\backslash\{0\} \hookrightarrow V$ be the embedding map. Define a sheaf of germs $\Omega^{1,1}_{V\backslash\{0\}}$ by the sheaf associated to the presheaf

$$U \mapsto < \Gamma(U, \Omega^1_V) \wedge \Gamma(U, \Omega^1_V) >,$$

where U is an open set of $V\backslash\{0\}$. Let $\bar{\bar{\Omega}}^{1,1}_V := \theta_*(\Omega^{1,1}_{V\backslash\{0\}})$.

Lemma Let V be a 2-dimensional Stein space with 0 as its only singular point in $\mathbb{C}^N$. Let $\pi : (M, A) \to (V, 0)$ be a resolution of the singularity with A as exceptional set. Then $\bar{\bar{\Omega}}_V^{1,1}$ is coherent and there is a short exact sequence

$$0 \longrightarrow \bar{\bar{\Omega}}_V^{1,1} \longrightarrow \bar{\bar{\Omega}}_V^2 \longrightarrow \mathscr{G}^{(1,1)} \longrightarrow 0 \tag{12}$$

where $\mathscr{G}^{(1,1)}$ is a sheaf supported on the singular point of V. Let

$$G^{(1,1)}(M \backslash A) := \Gamma(M \backslash A, \Omega_M^2) / < \Gamma(M \backslash A, \Omega_M^1) \wedge \Gamma(M \backslash A, \Omega_M^1) >, \tag{13}$$

then $dim \mathscr{G}_0^{(1,1)} = dim G^{(1,1)}(M \backslash A)$.

Definition Let V be a 2-dimensional Stein space with 0 as its only singular point. Let $\pi : (M, A) \to (V, 0)$ be a resolution of the singularity with A as exceptional set. Define

$$g^{(1,1)}(0) := dim \mathscr{G}_0^{(1,1)} = dim G^{(1,1)}(M \setminus A).$$

Proposition Let V be a 2-dimensional Stein space with 0 as its only singular point. Then $f^{(1,1)} \leq g^{(2)}$ and $g^{(1,1)} \leq p_g + g^{(2)}$.

Let X be a compact connected strictly pseudoconvex CR manifold of real dimension 3, in the boundary of a bounded strictly pseudoconvex domain D in $\mathbb{C}^N$. By Harvey and Lawson, there is a unique complex variety V in $\mathbb{C}^N$ such that the boundary of V is X.

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s-invariant, $f^{(1,1)}$ and $g^{(1,1)}$ are also CR invariants

Theorem Let V be a 2-dimensional Stein space with 0 as its only normal singular point with $\mathbb{C}^*$-action. Let $\pi : (M, A) \to (V, 0)$ be a minimal good resolution of the singularity with A as exceptional set, then $f^{(1,1)} \geq 1$ and $g^{(1,1)} \geq 1$.

Main results

Theorem M Let X be a strictly pseudoconvex compact Calabi-Yau CR manifold of dimension 3. Suppose that X is contained in the boundary of a strictly pseudoconvex bounded domain D in $\mathbb{C}^N$. Then X is a boundary of the complex sub-manifold $V \subset D - X$ with boundary regularity if and only if s-invariant $s(X)$ and $f^{(1,1)}(X)$ if and only if s-invariant $s(X)$ and $g^{(1,1)}(X)$ vanish.

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Corollary N Let X be a strictly pseudoconvex compact CR manifold of dimension 3. Suppose that X is contained in the boundary of a strictly pseudoconvex bounded domain D in $\mathbb{C}^3$. Then X is a boundary of the complex sub-manifold $V \subset D - X$ if and only if s-invariant $s(X)$ and $f^{(1,1)}(X)$ if and only if s-invariant $s(X)$ and $g^{(1,1)}(X)$ vanish.

Corollary O Let X be a strictly pseudoconvex compact Calabi-Yau CR manifold of dimension 3. Suppose that X is contained in the boundary of a strictly pseudoconvex bounded domain D in $\mathbb{C}^N$ with $H_h^2(X) = 0$. Then X is a boundary of the complex sub-manifold $V \subset D-X$ with boundary regularity and the variety is smooth if and only if $g^{(1,1)}(X) = 0$.

Corollary O Let X be a strictly pseudoconvex compact Calabi-Yau CR manifold of dimension 3. Suppose that X is contained in the boundary of a strictly pseudoconvex bounded domain D in $\mathbb{C}^N$ with $H_h^2(X) = 0$. Then X is a boundary of the complex sub-manifold $V \subset D - X$ with boundary regularity and the variety is smooth if and only if $g^{(1,1)}(X) = 0$.

Corollary P Let X be a strictly pseudoconvex compact CR manifold of dimension 3. Suppose that X is contained in the boundary of a strictly pseudoconvex bounded domain D in $\mathbb{C}^3$ with $H_h^2(X) = 0$. Then X is a boundary of the complex sub-manifold $V \subset D - X$ if and only if $g^{(1,1)}(X) = 0$.

Thank you!