

MODULI SPACE OF BOUNDED COMPLETE REINHARDT DOMAINS
AND COMPLEX PLATEAU PROBLEM

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THESIS

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To my family:

My parents,

My wife: Yun GAO,

My newborn son: Andi DU.

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PREFACE

This Ph. D thesis is based on the papers (1), (2) and (3). Two main topics are stated in my thesis. The first topic is classification of domains in \mathbb{C}^n and special varieties which will be considered in Chapter 2. The second topic is classical complex Plateau problem which will be considered in Chapter 3.

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SUMMARY

One of the most fundamental problems in complex geometry is to determine when two bounded domains in \mathbb{C}^n are biholomorphically equivalent. Even for complete Reinhardt domains, this fundamental problem remained unsolved for many years. Using the Bergmann function theory, we construct an infinite family of numerical invariants from the Bergman functions for complete Reinhardt domains in \mathbb{C}^n . These infinite family of numerical invariants are actually a complete set of invariants if the domains are pseudoconvex with C^1 boundaries. For bounded complete Reinhardt domains with real analytic boundaries, the complete set of numerical invariants can be reduced dramatically although the set is still infinite. We shall also discuss the role of the Hilbert 14th problem in the construction of numerical biholomorphic invariants of complete Reinhardt domains in \mathbb{C}^n .

Moreover, for $n = 2$, we can construct an infinite family of numerical invariants from the Bergman functions for such domains in A_n -variety $\{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$. These infinite family of numerical invariants are actually a complete set of invariants for either the set of all bounded strictly pseudoconvex complete Reinhardt domain in A_n variety or the set of all bounded pseudoconvex complete Reinhardt domains with real analytic boundaries in A_n variety. In particular the moduli space of these domains in A_n variety is constructed explicitly as the image of this complete family of numerical invariants. It is well known that A_n variety is the quotient of cyclic group of order $n + 1$ on \mathbb{C}^2 . We prove that the moduli space of bounded

SUMMARY (Continued)

complete Reinhardt domains in A_n variety coincides with the moduli space of the corresponding bounded complete Reinhardt domains in \mathbb{C}^2 .

Another natural fundamental questions of complex geometry is to study the boundaries of complex varieties. For example, the famous classical complex Plateau problem asks which odd dimensional real sub-manifolds of \mathbb{C}^N are boundaries of complex sub-manifolds in \mathbb{C}^N . Let X be a compact connected strictly pseudoconvex CR manifold of real dimension $2n - 1$ in \mathbb{C}^{n+1} . For $n \geq 3$, Yau used Kohn-Rossi cohomology groups to solve the classical complex Plateau problem in 1981. For $n = 2$, the problem has remained unsolved for over a quarter of a century. In this paper, we introduce a new CR invariant of X to solve this problem completely.

CHAPTER 1

INTRODUCTION

1.1 History

One of the basic problems in complex geometry is to find a reasonable object which parameterizes all non-isomorphic complex manifolds. This is the well known moduli problem. The concept of moduli goes back to Riemann (4) who shows that the isomorphism class of a Riemann surface of genus $g \geq 2$ depends on $3g - 3$ parameters. The deformation of algebraic surfaces was first considered by Noether (5). It was Frölicher and Nijenhuis (6) who first studied deformation of higher dimensional complex manifolds by a differential geometric method. In a series of papers (7), Kodaira and Spencer systematically developed the theory of deformation of compact complex manifolds. The local moduli problem was solved by Kuranishi in his famous paper (8) in 1962. However, a precise formulation of global moduli problems in algebraic geometry, the definition of moduli spaces for curves and for certain higher dimensional manifolds have only been given recently (9), as well as solutions in some cases (10),(11). On the other hand, little is known about the moduli problem of open manifolds of complex dimension greater than one. This problem is substantially more difficult because it is infinite dimensional in nature.

Let D_1 and D_2 be two domains in \mathbb{C}^n . One of the most fundamental problems in complex geometry is to determine conditions which will imply that D_1 and D_2 are biholomorphically

equivalent. For $n = 1$, the celebrated Riemann mapping theorem states that any simply connected domains in \mathbb{C} are biholomorphically equivalent. For $n \geq 2$, it is well known that there are lots of domains which are topologically equivalent to the ball but not necessarily biholomorphically equivalent to the ball. This problem was first studied by Poincaré in 1907 (12). He worked on the perturbations of the unit ball in \mathbb{C}^2 of a particular kind, and found necessary and sufficient conditions on a first order perturbation that the perturbed domain be biholomorphically equivalent to the ball. More generally, Poincaré studied the invariance properties of a CR manifold X of real dimension $2n - 1$ which is a real hypersurface in \mathbb{C}^n with respect to the infinite pseudo-group of biholomorphic transformations. The systematic study of such properties for hypersurfaces with nondegenerate Levi form was made by Cartan (13) in 1932, and later by Chern and Moser (14). A main result of the theory is the existence of a complete system of local differential invariants for CR-structures on real hypersurfaces. In 1974, Fefferman (15) proved that a biholomorphic mapping between two strictly pseudoconvex domains is smooth up to the boundaries and the induced boundary mapping is a CR-equivalence on the boundary. Thus, by the fundamental theorem of Fefferman, the biholomorphically equivalent problem of bounded strictly pseudoconvex domains in \mathbb{C}^n is the same as the CR equivalent problem of strictly pseudoconvex compact CR manifolds of real dimension $2n - 1$ in \mathbb{C}^n .

In 1978, Burns, Shnider and Wells (16) used the Chern-Moser theory to distinguish generic perturbations of a given strictly pseudoconvex domain. They were able to construct perturbations with arbitrarily large parameters in such a way that the domains are biholomorphically

inequivalent if the parameter values of these domains are different. As a result the “number of moduli” of a “moduli space” of a bounded strictly pseudoconvex domain has to be infinite. On the other hand, Webster (17) gave a complete characterization when two ellipsoids in \mathbb{C}^n are CR equivalent by Chern-Moser’s theory and Cartan method of equivalence. In 1988, Lempert (18) made a significant progress in the subject. He considered smoothly bounded strictly convex domains containing the origin of \mathbb{C}^n . He called two such domains equivalent if there is an origin preserving biholomorphic mapping between them whose differential at the origin is the identity. With each smoothly bounded strictly convex domain D , Lempert associated a triple (I, H, Q) of invariants where I is the Kobayashi indicatrix of D at the origin, and the smooth hermitian form H and the smooth quadratic form Q are defined on the rank $n - 1$ vector bundle of $(1, 0)$ vectors tangent to the boundary of the Kobayashi indicatrix I . He showed that if two marked convex domains share the same invariants then they are equivalent. In dimension 2, the construction may be simplified and it is possible to reduce the number of invariants. This allows the explicit description of the moduli space of marked strictly convex domains in \mathbb{C}^2 as a subdomain of a suitable Fréchet bundle. Although the theory established by Lempert is extremely beautiful, the computation of his invariants is quite a hard problem.

Despite the success of the Chern-Moser theory and the Lempert theory, the fundamental question of distinguishing two strictly pseudoconvex CR manifolds remains unsolved. Let X be a compact connected strictly pseudoconvex CR manifold of real dimension $2n - 1$. In 1974 Boutel de Monvel (19) (also see Kohn (20)) proved that X is CR-embeddable in some \mathbb{C}^N if $\dim X \geq 5$. Throughout this paper, we shall only consider CR embeddable strictly pseudo-

convex CR manifolds. In view of a beautiful theorem of Harvey-Lawson (21), there exists a complex variety V in \mathbb{C}^N for N sufficiently large such that $\partial V = X$ and V has only normal isolated singularities. It is well known that one can use the structures of the singularities of V to distinguish the CR structure of X (see for example Theorem 3.1 of (22)). Thus if two strictly pseudoconvex CR manifolds bound non-isomorphic singularities, then their CR structures are different. The difficult unsolved CR equivalence problem is: how can one distinguish between strictly pseudoconvex CR manifolds X_1 and X_2 when they are lying in the same variety V . If V is \mathbb{C}^N , this difficult problem is just the classical problem discussed above and has been considered by many leading mathematicians Chern-Moser (14), Fefferman (23), Webster (17), Burns-Shnider-Wells (16), etc. Even in this case, it seems that the biholomorphically equivalence problem or moduli problem for complete Reinhardt domains remains open, although there is a beautiful theorem of Sunada (24) which relates two such domains by a special linear map. For example, consider the following natural family of complete Reinhardt domains $D(a_1, \dots, a_k; b_1, \dots, b_k; c_1, \dots, c_k) = \{(z_1, z_2) \in \mathbb{C}^2 : \sum_{i=1}^k [a_i |z_1|^{4i} + b_i |z_2|^{4i} + c_i |z_1 z_2|^{2i}] < d, \text{ where } a_1, \dots, a_k; b_1, \dots, b_k; c_1, \dots, c_k \text{ and } d \text{ are positive real numbers}\}$. It is not known how to solve the problem of biholomorphic equivalence of this family of complete Reinhardt domains by Sunada's theorem. On the other hand, when V is a singular variety, the CR equivalence problems is wide open. In (22), Yau discovered a novel technique to attack CR equivalence problem. He constructed a new biholomorphically invariant called Bergman function. The Bergman functions put a lot of restriction on biholomorphic maps between bounded complete Reinhardt domains, from which new holomorphic invariants can be con-

structed and the automorphism groups of the bounded complete Reinhardt domains can be determined. He illustrated how his new technique works in a concrete example of A_1 -variety $V = \{(x, y, z) \in \mathbb{C}^3 : xy = z^2\}$. He constructed a fundamental holomorphic numerical invariant ν_D (cf. Theorem 3.6 of (22)), for bounded complete Reinhardt domains D in V . He also computed the automorphism group of D . For a one parameter family of complete Reinhardt domains $D_a = \{(x, y, z) \in \mathbb{C}^3 : xy = z^2, a|x|^2 + |y|^2 + |z|^2 < \varepsilon_0, \text{ where } a > 0 \text{ and } \varepsilon_0 \text{ is a fixed positive constant}\}$, he showed that the holomorphic numerical invariant ν_{D_a} is a complete invariant in the sense that D_{a_1} is biholomorphically equivalent to D_{a_2} if and only if $\nu_{D_{a_1}} = \nu_{D_{a_2}}$.

Another natural fundamental questions of complex geometry is to study the boundaries of complex varieties. For example, the famous classical complex Plateau problem asks which odd dimensional real sub-manifolds of \mathbb{C}^N are boundaries of complex sub-manifolds in \mathbb{C}^N . In 1975, Harvey and Lawson (21) showed that for any compact connected CR manifold X in \mathbb{C}^N , there is a unique complex variety V in \mathbb{C}^N such that the boundary of V is X .

If X is a strictly pseudoconvex CR manifold of dimension $2n - 1$, $n \geq 2$, contained in the boundary of a bounded strictly pseudoconvex domain D in \mathbb{C}^N , then V has boundary regularity at every point of X and V has only isolated singularities in $V - X$ (cf. (25)). The next fundamental question is to determine when X is a boundary of a complex sub-manifold in \mathbb{C}^N , i.e., when V is smooth. In 1981, Yau (26) solved the classical complex Plateau problem for the case $n \geq 3$ by calculation of Kohn-Rossi cohomology groups $H_{KR}^{p,q}(X)$. More precisely, if X is a compact connected strictly pseudoconvex CR manifold of real dimension $2n - 1$, $n \geq 3$, in

the boundary of a bounded strictly pseudoconvex domain D in \mathbb{C}^{n+1} . Then X is a boundary of the complex sub-manifold $V \subset D - X$ if and only if Kohn-Rossi cohomology groups $H_{KR}^{p,q}(X)$ are zeros for $1 \leq q \leq n - 2$ (see Theorem 3.3.0.1).

For $n = 2$, i.e. X is a 3-dimensional CR manifold, the classical complex Plateau problem remains unsolved for over a quarter of a century. The main difficulty is that the Kohn-Rossi cohomology groups are infinite dimensional in this case. Let V be a complex variety with X as its boundary. Then the singularities of V are surface singularities. In (27), the holomorphic De Rham cohomology, which is derived from Kohn-Rossi cohomology, is considered to determine what kind of singularities can happen in V . In fact, in (28), Tanaka introduced a spectral sequence $E_r^{p,q}(X)$ with $E_1^{p,q}(X)$ being the Kohn-Rossi cohomology group and $E_2^{k,0}(X)$ being the holomorphic De Rham cohomology denoted by $H_h^k(X)$. So consideration of De Rham cohomology is natural in the case of $n = 2$. Luk and Yau introduced s -invariant (cf. Definition 3.2.1.2 below) for isolated singularity $(V, 0)$ and proved a theorem in (27) that if $(V, 0)$ is a Gorenstein surface singularity with vanishing s -invariant, then $(V, 0)$ is a quasihomogeneous singularity whose link is rational homology sphere. In (27) they proved that if X is a strictly pseudoconvex compact Calabi-Yau CR manifold of dimension 3 contained in the boundary of a strictly pseudoconvex bounded domain D in \mathbb{C}^N and the holomorphic De Rham cohomology $H_h^2(X)$ vanishes, then X is a boundary of a complex variety V in D with boundary regularity and V has only isolated singularities in the interior and the normalizations of these singularities are Gorenstein surface singularities with vanishing s -invariant (see Theorem 3.3.0.4). As a corollary of this theorem, they get that if $N = 3$, the variety V bounded by X has only isolated quasi-

homogeneous singularities such that the dual graphs of the exceptional sets in the resolution are star shaped and all the curves are rational (see Corollary 3.3.0.5). Even though, one cannot judge when X is a boundary of a complex manifold with the vanishing of $H_h^2(X)$, it is a fundamental step toward the solution of the complex Plateau problem. Since there is a unique complex variety V such that the boundary of V is X , we can define the s -invariant $s(X)$ as the sum of all the s -invariants of the singularities of V . It is easy to see that $s(X)$ is a CR invariant. In this paper we introduce two new CR invariants $f^{(1,1)}(X)$ and $g^{(1,1)}(X)$ which have independent interest besides their application to complex Plateau problem. These new invariants together with s -invariant will have enough information to give a necessary and sufficient conditions for the variety V bounded by X being smooth.

1.2 Main results

In the first part of this paper, we introduce a higher order Bergman functions on domains in varieties which are global invariants. These Bergman functions are used to prove that biholomorphisms between two bounded complete Reinhardt domains are necessarily special linear maps. We construct an infinite family of numerical invariants for complete Reinhardt domains in A_n -variety. Our numerical invariants are able to distinguish any two bounded complete Reinhardt pseudoconvex domains with real analytic boundaries or any two bounded complete Reinhardt strictly pseudoconvex domains in A_n -variety. Thus the moduli spaces of bounded complete Reinhardt pseudoconvex domains with real analytic boundaries or any two bounded complete Reinhardt strictly pseudoconvex domains in A_n -variety are constructed explicitly as the image of this complete family of numerical invariants. Because each bounded complete

Reinhardt domain in A_n -variety corresponds to a unique bounded complete Reinhardt domain in \mathbb{C}^2 , we have also constructed the moduli space of a large class of bounded complete Reinhardt pseudoconvex domains in \mathbb{C}^2 .

Before stating our result, let us recall some notations.

An open subset $D \subseteq \mathbb{C}^n$ is a complete Reinhardt domain if, whenever $(z_1, \dots, z_n) \in D$ then $(\xi_1 z_1, \dots, \xi_n z_n) \in D$ for all complex numbers ξ_j with $|\xi_j| \leq 1$.

Let $\widetilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$. It is well known that \widetilde{V}_n is the quotient of \mathbb{C}^2 by a cyclic group of order $n+1$, i.e. $\delta.(z_1, z_2) = (\delta z_1, \delta^n z_2)$, where δ is a primitive $(n+1)$ -th root of unit. The quotient map $\pi : \mathbb{C}^2 \rightarrow \widetilde{V}$ is given by $\pi(z_1, z_2) = (z_1^{n+1}, z_2^{n+1}, z_1 z_2)$.

Definition 1.2.0.1. *An open set V in the A_n -variety $\widetilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ is called a complete Reinhardt domain if $\pi^{-1}(V)$ is a complete Reinhardt domain in \mathbb{C}^2 .*

Recall that the minimal resolution \widetilde{M}_n of \widetilde{V}_n consists of $n+1$ coordinate charts $\widetilde{W}_k = \mathbb{C}^2 = \{(u_k, v_k)\}, k = 0, 1, \dots, n$. The space of holomorphic two forms on \widetilde{M}_n has a basis $\{\phi_{\alpha\beta} = u_0^\alpha v_0^\beta du_0 \wedge dv_0 : \alpha \geq \frac{n}{n+1}\beta\}$. Let $M (\subseteq \widetilde{M}_n)$ be the resolution of complete Reinhardt domain V in \widetilde{V}_n . In what follows, we shall use notation $\|\phi_{\alpha\beta}\|_M^2$ for $\int_M \phi_{\alpha\beta} \wedge \overline{\phi_{\alpha\beta}}$.

Let $g^{(\alpha, \beta)} = \frac{\|\phi_{10}\|^{\alpha - \frac{n}{n+1}\beta} \|\phi_{n, n+1}\|^{\frac{\beta}{n+1}}}{\|\phi_{\alpha\beta}\| \|\phi_{00}\|^{\alpha - \frac{n-1}{n+1}\beta - 1}}$ and $[T]$ be the maximal integer which is less than $T \in \mathbb{R}$.

Theorem A. *Let $V_i, i = 1, 2$, be two bounded complete Reinhardt domains in A_n -variety $\widetilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$. If V_1 is biholomorphic to V_2 , then*

$$\xi^{(\alpha, \beta)} := g^{(\alpha, \beta)} \cdot g^{(n\alpha - (n-1)\beta, (n+1)\alpha - n\beta)},$$

$$\zeta^{(\alpha, \beta)} := g^{(\alpha, \beta)} + g^{(n\alpha - (n-1)\beta, (n+1)\alpha - n\beta)},$$

$$\eta^{(\alpha, p, q)} := (g^{(\alpha, p)} - g^{(n\alpha - (n-1)p, (n+1)\alpha - np}) \cdot (g^{(\alpha, q)} - g^{(n\alpha - (n-1)q, (n+1)\alpha - nq}))$$

and

$$\begin{aligned} \omega^{(\alpha_1, \alpha_2, p_1, p_2)} := & (g^{(\alpha_1, p_1)} - g^{(n\alpha_1 - (n-1)p_1, (n+1)\alpha_1 - np_1}) \cdot \\ & (g^{(\alpha_2, p_2)} - g^{(n\alpha_2 - (n-1)p_2, (n+1)\alpha_2 - np_2})), \end{aligned}$$

where

$$\begin{aligned} \alpha \geq 1, \alpha \geq \frac{n}{n+1}\beta, 0 \leq p, q \leq \left\lceil \frac{n+1}{n}\alpha \right\rceil, p \neq q, \\ 0 \leq p_i \leq \left\lceil \frac{n+1}{n}\alpha_i \right\rceil, \alpha_i \geq 1, \alpha_1 \neq \alpha_2, i = 1, 2, \end{aligned}$$

are all invariants, i.e.

$$\xi_{V_1}^{(\alpha, \beta)} = \xi_{V_2}^{(\alpha, \beta)}, \zeta_{V_1}^{(\alpha, \beta)} = \zeta_{V_2}^{(\alpha, \beta)}, \eta_{V_1}^{(\alpha, p, q)} = \eta_{V_2}^{(\alpha, p, q)},$$

$$\omega_{V_1}^{(\alpha_1, \alpha_2, p_1, p_2)} = \omega_{V_2}^{(\alpha_1, \alpha_2, p_1, p_2)},$$

where

$$\begin{aligned} \alpha \geq 1, \alpha \geq \frac{n}{n+1}\beta, 0 \leq p, q \leq \left\lceil \frac{n+1}{n}\alpha \right\rceil, p \neq q, \\ 0 \leq p_i \leq \left\lceil \frac{n+1}{n}\alpha_i \right\rceil, \alpha_i \geq 1, \alpha_1 \neq \alpha_2, i = 1, 2. \end{aligned}$$

The invariants in Theorem A determine completely the Bergman function up to automorphisms of A_n -variety.

Theorem B. *Let V_i , $i = 1, 2$, be two bounded complete Reinhardt strictly pseudoconvex (respectively C^ω -smooth pseudoconvex) domains in $\widetilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$. If*

$$\xi_{V_1}^{(\alpha, \beta)} = \xi_{V_2}^{(\alpha, \beta)}, \zeta_{V_1}^{(\alpha, \beta)} = \zeta_{V_2}^{(\alpha, \beta)}, \eta_{V_1}^{(\alpha, p, q)} = \eta_{V_2}^{(\alpha, p, q)},$$

$$\omega_{V_1}^{(\alpha_1, \alpha_2, p_1, p_2)} = \omega_{V_2}^{(\alpha_1, \alpha_2, p_1, p_2)},$$

where

$$\alpha \geq 1, \alpha \geq \frac{n}{n+1}\beta, 0 \leq p, q \leq \left[\frac{n+1}{n}\alpha \right], p \neq q,$$

$$0 \leq p_i \leq \left[\frac{n+1}{n}\alpha_i \right], \alpha_i \geq 1, \alpha_1 \neq \alpha_2, i = 1, 2,$$

then there exists an automorphism $\Psi = (\psi_1, \psi_2, \psi_3)$ of A_n -variety $\widetilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ given by either

$$(\psi_1, \psi_2, \psi_3) =$$

$$\left(\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}} x, \frac{\|\phi_{n, n+1}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{n, n+1}\|_{M_1}} y, \frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}} z \right),$$

or

$$(\psi_1, \psi_2, \psi_3) =$$

$$\left(\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{n, n+1}\|_{M_1}} y, \frac{\|\phi_{n, n+1}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}} x, \frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}} z \right).$$

such that Ψ sends V_1 to V_2 .

As an immediate corollary of Theorem B above, we have the following theorem.

Theorem C. *The moduli space of bounded complete Reinhardt strictly pseudoconvex (respectively C^ω -smooth pseudoconvex) domains in A_n -variety $\widetilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ is given by the image of the map $\Phi : \{V : V \text{ a bounded complete Reinhardt strictly pseudoconvex (respectively } C^\omega\text{-smooth pseudoconvex) domain in } \widetilde{V}_n\} \rightarrow \mathbb{R}^\infty$, where the component function of Φ are the invariant functions*

$$\xi^{(\alpha, \beta)}, \zeta^{(\alpha, \beta)}, \eta^{(\alpha, p, q)}, \omega^{(\alpha_1, \alpha_2, p_1, p_2)},$$

$$\alpha \geq 1, \alpha \geq \frac{n}{n+1}\beta, 0 \leq p, q \leq \left\lceil \frac{n+1}{n}\alpha \right\rceil, p \neq q,$$

$$0 \leq p_i \leq \left\lceil \frac{n+1}{n}\alpha_i \right\rceil, \alpha_i \geq 1, \alpha_1 \neq \alpha_2, i = 1, 2.$$

defined in Theorem A.

We are now ready to study a large class of complete Reinhardt domains in \mathbb{C}^2 . The following theorem says that the biholomorphic equivalence problem for bounded complete Reinhardt domains in A_n -variety \widetilde{V}_n is the same as the biholomorphic equivalence problem for the corresponding bounded complete Reinhardt domains in \mathbb{C}^2 .

Theorem D. *Let $\pi : \mathbb{C}^2 \rightarrow \widetilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ be the quotient map given by $\pi(z_1, z_2) = (z_1^{n+1}, z_2^{n+1}, z_1 z_2)$. Let $V_i, i = 1, 2$, be bounded complete Reinhardt domains in \widetilde{V}_n such that $W_i := \pi^{-1}(V_i), i = 1, 2$, are bounded complete Reinhardt domain in \mathbb{C}^2 .*

Then V_1 is biholomorphic to V_2 if and only if W_1 is biholomorphic to W_2 . In particular, V_1 is biholomorphic to V_2 if and only if there exists a biholomorphism $\Phi : V_1 \rightarrow V_2$ given by $\Phi(x, y, z) = (a^{n+1}x, b^{n+1}y, abz)$ or $\Phi(x, y, z) = (a^{n+1}y, b^{n+1}x, abz)$ where $a, b > 0$.

As a corollary of Theorem C and Theorem D, we have the following theorem .

Theorem E. (1) Let $\mathcal{W} = \{W : W = \pi^{-1}(V) \text{ where } V \text{ is a bounded complete Reinhardt domain in } A_n\text{-variety}\}$ be the space of bounded complete Reinhardt domains in \mathbb{C}^2 which are invariant under the action of the cyclic group of order $n + 1$ on \mathbb{C}^2 . Then

$$\xi^{(\alpha, \beta)}, \zeta^{(\alpha, \beta)}, \eta^{(\alpha, p, q)}, \omega^{(\alpha_1, \alpha_2, p_1, p_2)},$$

$$\alpha \geq 1, \alpha \geq \frac{n}{n+1}\beta, 0 \leq p, q \leq \left[\frac{n+1}{n}\alpha \right], p \neq q,$$

$$0 \leq p_i \leq \left[\frac{n+1}{n}\alpha_i \right], \alpha_i \geq 1, \alpha_1 \neq \alpha_2, i = 1, 2,$$

defined in Theorem A are invariants of \mathcal{W} .

(2) Let $\mathcal{W}_P = \{W : W = \pi^{-1}(V) \text{ where } V \text{ is a complete Reinhardt pseudoconvex } C^\omega\text{-smooth domain in } A_n\text{-variety}\}$ and $\mathcal{W}_{SP} = \{W : W = \pi^{-1}(V) \text{ where } V \text{ is a complete Reinhardt strictly pseudoconvex domain in } A_n\text{-variety}\}$. Then the moduli space of \mathcal{W}_P (respectively \mathcal{W}_{SP}) is given by the image of the map $\widetilde{\Phi}_P : \mathcal{W}_P \rightarrow \mathbb{R}^\infty$ (respectively $\widetilde{\Phi}_{SP} : \mathcal{W}_{SP} \rightarrow \mathbb{R}^\infty$), where the component functions of $\widetilde{\Phi}_P$ (respectively $\widetilde{\Phi}_{SP}$) are the invariant functions

$$\xi^{(\alpha, \beta)}, \zeta^{(\alpha, \beta)}, \eta^{(\alpha, p, q)}, \omega^{(\alpha_1, \alpha_2, p_1, p_2)},$$

$$\alpha \geq 1, \alpha \geq \frac{n}{n+1}\beta, 0 \leq p, q \leq \left\lfloor \frac{n+1}{n}\alpha \right\rfloor, p \neq q,$$

$$0 \leq p_i \leq \left\lfloor \frac{n+1}{n}\alpha_i \right\rfloor, \alpha_i \geq 1, \alpha_1 \neq \alpha_2, i = 1, 2,$$

defined in Theorem A. In particular, the moduli space of \mathcal{W}_P (respectively \mathcal{W}_{SP}) is the same as the moduli space of bounded complete Reinhardt pseudoconvex C^ω -smooth domains (respectively bounded complete Reinhardt strictly pseudoconvex domains) in A_n -variety $\widetilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$.

We also show that Yau's Bergman function theory can also solve the biholomorphic equivalence problem or moduli problem for complete Reinhardt pseudoconvex domains in \mathbb{C}^n for all $n \geq 2$. In order to describe the complete biholomorphic invariants of bounded complete Reinhardt domains in \mathbb{C}^n , we introduce the following notations. Let S_n be the symmetric group of degree n . Recall that the group ring $\mathbb{R}[S_n]$ is a ring of the form $\mathbb{R}[\tau_1, \tau_2, \dots, \tau_n!]$ with $\tau_i \in S_n$ for $1 \leq i \leq n!$. Let $\sum_i x_i \tau_i$ and $\sum_j y_j \tau_j$, where x_i, y_j are in \mathbb{R} , be two elements in $\mathbb{R}[S_n]$. Then $(\sum_i x_i \tau_i)(\sum_j y_j \tau_j) := \sum_{i,j} x_i y_j (\tau_i \cdot \tau_j)$, where $\tau_i \cdot \tau_j$ is the product in the group S_n . We shall consider $\mathbb{R}[S_n] \times \dots \times \mathbb{R}[S_n]$ the product of the group ring with itself. Such a product has a natural S_n -module structure in the following manner. Let $\sigma \in S_n$ and $(\sum_i x_i \tau_i, \dots, \sum_j y_j \tau_j) \in \mathbb{R}[S_n] \times \dots \times \mathbb{R}[S_n]$. Then

$$\sigma \left(\sum_i x_i \tau_i, \dots, \sum_j y_j \tau_j \right) := \left(\sum_i x_i (\tau_i \sigma), \dots, \sum_j y_j (\tau_j \sigma) \right). \quad (1.2.1)$$

Definition 1.2.0.2. Two elements f, g in $\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n]$ are said to be equivalent and denoted by $f \sim g$ if there exists a $\sigma \in S_n$ such that $\sigma(f) = g$.

Before we can describe our main results, we need the following notations. Let $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$ be an n -tuple of nonnegative integers. Denote $\phi_{\vec{\alpha}} = \left(\prod_{i=1}^n z_i^{\alpha_i} \right) dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$. For a domain D in \mathbb{C}^n . We shall use notation $\|\phi_{\vec{\alpha}}\|_D^2 = \int_D \phi_{\vec{\alpha}} \wedge \phi_{\vec{\alpha}}$. In this paper, we show that all the biholomorphic invariants of a bounded complete Reinhardt domain are contained in $(\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n]) / \sim$ where there are $n!$ copies of $\mathbb{R}[S_n]$ and \sim is the equivalent relation defined in Definition 1.2.0.2.

Theorem F. *Let D be a bounded complete Reinhardt domain in \mathbb{C}^n . Let $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$ be a n -tuple of nonnegative integers. For any $\tau \in S_n$, denote*

$$g_D^\tau(\vec{\alpha}) = \frac{\|\phi_{\vec{0}}\|_D^{\sum \alpha_i - 1} \|\phi_{\tau(\vec{\alpha})}\|_D}{\prod_{i=1}^n \|\phi_{\vec{e}_i}\|_D^{\alpha_{\tau(i)}}} \quad (1.2.2)$$

where $\tau(\vec{\alpha}) = (\alpha_{\tau(1)}, \dots, \alpha_{\tau(n)})$ and $\vec{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the i th component. Then for all n -tuples of nonnegative integers $\vec{\beta}_1, \dots, \vec{\beta}_{n!}$, $\xi_D^{\vec{\beta}_1, \dots, \vec{\beta}_{n!}} = \left(\sum_{\tau \in S_n} g_D^\tau(\vec{\beta}_1)\tau, \dots, \sum_{\tau \in S_n} g_D^\tau(\vec{\beta}_{n!})\tau \right)$ as an element in $(\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n]) / \sim$ is a biholomorphic invariant. In fact, if D_1 and D_2 are two such domains which are biholomorphically equivalent, then there exists a $\sigma \in S_n$ such that $g_{D_1}^\tau(\vec{\alpha}) = g_{D_2}^{\tau \circ \sigma}(\vec{\alpha}) \forall \tau \in S_n$ and $\forall \vec{\alpha}$ n -tuple of nonnegative integers.

The invariants in Theorem F are complete invariants for bounded complete Reinhardt pseudoconvex domains with C^1 boundaries.

Theorem G. Let $D_i, i = 1, 2$, be two bounded complete Reinhardt pseudoconvex domains in \mathbb{C}^n with C^1 boundaries. If for all $\vec{\alpha}, \dots, \vec{\alpha}_{n!}$ n -tuples of nonnegative integers, $\xi_{D_1}^{\vec{\alpha}, \dots, \vec{\alpha}_{n!}} = \xi_{D_2}^{\vec{\alpha}, \dots, \vec{\alpha}_{n!}}$ in $(\mathbb{R}[S_n] \times \dots \times \mathbb{R}[S_n]) / \sim$, where $\xi_D^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}} = (\sum_{\tau \in S_n} g_D^\tau(\vec{\alpha}_1)\tau, \dots, \sum_{\tau \in S_n} g_{D_i}^\tau(\vec{\alpha}_{n!})\tau)$, then there exists $\sigma \in S_n$ and a biholomorphic map

$$\Psi_\sigma(z_1, \dots, z_n) = (a_1 z_{\sigma(1)}, \dots, a_n z_{\sigma(n)}),$$

where $a_i = \frac{\|\phi_{\bar{0}}\|_{D_1} \|\phi_{\bar{e}_i}\|_{D_2}}{\|\phi_{\bar{e}_{\sigma(i)}}\|_{D_1} \|\phi_{\bar{0}}\|_{D_2}}$, such that Ψ_σ sends D_1 onto D_2 .

Theorem F and Theorem G above give a complete characterization of two bounded complete Reinhardt domains in \mathbb{C}^n to be biholomorphically equivalent in terms of the quotient of group ring $(\mathbb{R}[S_n] \times \dots \times \mathbb{R}[S_n]) / \sim$. In case $n = 2$, we can actually write down the complete numerical invariants for two bounded complete Reinhardt in \mathbb{C}^2 to be biholomorphically equivalent.

Theorem H. Let D_1, D_2 be two bounded complete Reinhardt pseudoconvex domains in \mathbb{C}^2 with C^1 boundaries. Then D_1 is biholomorphic to D_2 if and only if

$$g_{D_1}(\alpha_1, \alpha_2) + g_{D_1}(\alpha_2, \alpha_1) = g_{D_2}(\alpha_1, \alpha_2) + g_{D_2}(\alpha_2, \alpha_1) \quad (1.2.3)$$

$$g_{D_1}(\alpha_1, \alpha_2)g_{D_1}(\alpha_2, \alpha_1) = g_{D_2}(\alpha_1, \alpha_2)g_{D_2}(\alpha_2, \alpha_1) \quad (1.2.4)$$

$$\begin{aligned} & (g_{D_1}(\alpha_1, \alpha_2) - g_{D_1}(\alpha_2, \alpha_1))(g_{D_1}(\beta_1, \beta_2) - g_{D_1}(\beta_2, \beta_1)) \\ &= (g_{D_2}(\alpha_1, \alpha_2) - g_{D_2}(\alpha_2, \alpha_1))(g_{D_2}(\beta_1, \beta_2) - g_{D_2}(\beta_2, \beta_1)) \quad (1.2.5) \end{aligned}$$

for all nonnegative integers α_i, β_i , where

$$g_{D_i}(\alpha_1, \alpha_2) = \frac{\|\phi_{\vec{0}}\|_{D_i}^{\alpha_1 + \alpha_2 - 1} \|\phi_{(\alpha_1, \alpha_2)}\|_{D_i}}{\prod_{j=1}^2 \|\phi_{\vec{e}_j}\|_{D_i}^{\alpha_j}}.$$

Corollary I. The moduli space of bounded complete Reinhardt pseudoconvex domains with C^1 boundaries in \mathbb{C}^2 can be constructed explicitly as the image of the complete family of numerical invariants:

$$g_D(\alpha_1, \alpha_2) + g_D(\alpha_2, \alpha_1),$$

$$g_D(\alpha_1, \alpha_2)g_D(\alpha_2, \alpha_1),$$

and

$$(g_D(\alpha_1, \alpha_2) - g_D(\alpha_2, \alpha_1)) \cdot (g_D(\beta_1, \beta_2) - g_D(\beta_2, \beta_1))$$

$\forall \alpha_i, \beta_i$ nonnegative integers.

In order to find the complete numerical biholomorphic invariants of bounded complete Reinhardt domain in \mathbb{C}^n for $n \geq 3$, we need to consider the finite symmetric group $S_n = \{\sigma_1, \sigma_2, \dots, \sigma_n\}$ of degree n acting on the affine space $\mathbb{C}^{n!} = \mathbb{C}^n \times \dots \times \mathbb{C}^n$, which is the product of $n!$ copies of \mathbb{C}^n , in the following manner. Let $\tau \in S_n$ and

$$(x_{\sigma_1}, \dots, x_{\sigma_n}; \dots; y_{\sigma_1}, \dots, y_{\sigma_n}) \in \mathbb{C}^n \times \dots \times \mathbb{C}^n = \mathbb{C}^{n!}.$$

Then $\tau(x_{\sigma_1}, \dots, x_{\sigma_n}; \dots; y_{\sigma_1}, \dots, y_{\sigma_n}) = (x_{\sigma_1\tau}, \dots, x_{\sigma_n\tau}; \dots; y_{\sigma_1\tau}, \dots, y_{\sigma_n\tau})$. Since S_n is linearly reductive, by Hilbert Theorem, the ring of invariants $\mathbb{C}[x_{\sigma_1}, \dots, x_{\sigma_n}; \dots; y_{\sigma_1}, \dots, y_{\sigma_n}]^{S_n}$ is finitely generated.

Theorem J. Let $f_1, \dots, f_N \in \mathbb{C}[x_{\sigma_1}, \dots, x_{\sigma_n}; \dots; y_{\sigma_1}, \dots, y_{\sigma_n}]^{S_n}$ be the generators of the ring of invariant polynomials. Let D be a bounded complete Reinhardt domain in \mathbb{C}^n . Then, for $\vec{\alpha}_1, \vec{\alpha}_2, \dots, \vec{\alpha}_{n!}$ n -tuples of nonnegative integers,

$$f_1(g_D^\sigma(\vec{\alpha}_1), \dots, g_D^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n}, \dots, f_N(g_D^\sigma(\vec{\alpha}_1), \dots, g_D^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n}$$

are biholomorphic invariants, where

$$g_D^\sigma(\vec{\beta}) = \frac{\|\phi_{\vec{0}}\|_D^{\sum \beta_i - 1} \|\phi_{\sigma(\vec{\beta})}\|_D}{\prod_{i=1}^n \|\phi_{\vec{e}_i}\|_D^{\beta_{\sigma(i)}}}, \quad \vec{\beta} = (\beta_1, \beta_2, \dots, \beta_n).$$

The following theorem says that the above invariants are actually complete in case the domain D is pseudoconvex.

Theorem K. Let D_i , $i = 1, 2$, be two bounded complete Reinhardt pseudoconvex domains in \mathbb{C}^n with C^1 boundaries. Let $f_1, \dots, f_N \in \mathbb{C}[x_{\sigma_1}, \dots, x_{\sigma_n}; \dots; y_{\sigma_1}, \dots, y_{\sigma_n}]^{S_n}$ be the generators of the ring of invariant polynomials. If for all $\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}$ n -tuples of nonnegative integers

$$f_i(g_{D_1}^\sigma(\vec{\alpha}_1), \dots, g_{D_1}^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n} = f_i(g_{D_2}^\sigma(\vec{\alpha}_1), \dots, g_{D_2}^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n},$$

$$i = 1, 2, \dots, N,$$

then there exists $\tau \in S_n$ and a biholomorphic map $\Psi_\tau: \mathbb{C}^n \rightarrow \mathbb{C}^n$,

$$\Psi_\tau(z_1, \dots, z_n) = (a_1 z_{\tau(1)}, \dots, a_n z_{\tau(n)}),$$

where

$$a_i = \frac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e}_i}\|_{D_2}}{\|\phi_{\vec{e}_{\tau(i)}}\|_{D_1} \|\phi_{\vec{0}}\|_{D_2}},$$

such that Ψ_τ sends D_1 onto D_2 .

Corollary L. The moduli space of bounded complete Reinhardt pseudoconvex domains with C^1 boundaries in \mathbb{C}^n can be constructed explicitly as the image of the complete family of numerical invariants:

$$f_i(g_D^\sigma(\vec{\alpha}_1), \dots, g_D^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n}, \quad 1 \leq i \leq N,$$

where $\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}$ are all possible n -tuples of nonnegative integers.

For complete Reinhardt pseudoconvex domains with real analytic boundaries, we can use fewer numerical invariants to characterize these domains. More precisely, we have the following theorems.

Theorem G'. Let D_i , $i = 1, 2$, be two bounded complete Reinhardt pseudoconvex domains in \mathbb{C}^n with real analytic boundaries. Then D_1 is biholomorphically equivalent to D_2 if and only if

for all $\vec{\alpha}$ n -tuple of nonnegative integers, $\xi_{D_1}^{\vec{\alpha}} = \xi_{D_2}^{\vec{\alpha}}$ in $\mathbb{R}[S_n]/\sim$, where $\xi_{D_i}^{\vec{\alpha}} = \sum_{\tau \in S_n} g_{D_i}^{\tau}(\vec{\alpha})\tau$. In this case, there exists $\sigma \in S_n$ and a biholomorphic map

$$\Psi_{\sigma}(z_1, \dots, z_n) = (a_1 z_{\sigma(1)}, \dots, a_n z_{\sigma(n)}),$$

where $a_i = \frac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e}_i}\|_{D_2}}{\|\phi_{\vec{e}_{\sigma(i)}}\|_{D_1} \|\phi_{\vec{0}}\|_{D_2}}$, such that Ψ_{σ} sends D_1 onto D_2 .

Theorem H'. Let D_1, D_2 be two bounded complete Reinhardt pseudoconvex domains in \mathbb{C}^2 with real analytic boundaries. Then D_1 is biholomorphic to D_2 if and only if

$$g_{D_1}(\alpha_1, \alpha_2) + g_{D_1}(\alpha_2, \alpha_1) = g_{D_2}(\alpha_1, \alpha_2) + g_{D_2}(\alpha_2, \alpha_1)$$

$$g_{D_1}(\alpha_1, \alpha_2)g_{D_1}(\alpha_2, \alpha_1) = g_{D_2}(\alpha_1, \alpha_2)g_{D_2}(\alpha_2, \alpha_1)$$

for all nonnegative integers α_1, α_2 , where

$$g_{D_i}(\alpha_1, \alpha_2) = \frac{\|\phi_{\vec{0}}\|_{D_i}^{\alpha_1 + \alpha_2 - 1} \|\phi_{(\alpha_1, \alpha_2)}\|_{D_i}}{\prod_{j=1}^2 \|\phi_{\vec{e}_j}\|_{D_i}^{\alpha_j}}.$$

Theorem K'. Let $D_i, i = 1, 2$, be two bounded complete Reinhardt pseudoconvex domains in \mathbb{C}^n with real analytic boundaries. Let

$$f_1, \dots, f_N \in \mathbb{C}[x_{\sigma_1}, \dots, x_{\sigma_n}]^{S_n}$$

be the generators of the ring of invariant polynomials. Then D_1 is biholomorphically equivalent to D_2 if and only if for all $\vec{\alpha}$ n -tuples of nonnegative integers

$$f_i(g_{D_1}^\sigma(\vec{\alpha})_{\sigma \in S_n}) = f_i(g_{D_2}^\sigma(\vec{\alpha})_{\sigma \in S_n}),$$

$$i = 1, 2, \dots, N.$$

In this case, there exists $\tau \in S_n$ and a biholomorphic map $\Psi_\tau: \mathbb{C}^n \rightarrow \mathbb{C}^n$ $\Psi_\tau(z_1, \dots, z_n) = (a_1 z_{\tau(1)}, \dots, a_n z_{\tau(n)})$, where $a_i = \frac{\|\phi_{\bar{0}}\|_{D_1} \|\phi_{\bar{e}_i}\|_{D_2}}{\|\phi_{e_{\sigma(i)}}\|_{D_1} \|\phi_{\bar{0}}\|_{D_2}}$, such that Ψ_τ sends D_1 onto D_2 .

In the second part of this paper, 2-dimensional complex Plateau problem is solved. We introduce two new CR invariants $f^{(1,1)}(X)$ and $g^{(1,1)}(X)$ which have independent interest besides their application to complex Plateau problem. These new invariants together with s -invariant will have enough information to give a necessary and sufficient conditions for the variety V bounded by X being smooth.

Theorem M. *Let X be a strictly pseudoconvex compact Calabi-Yau CR manifold of dimension 3. Suppose that X is contained in the boundary of a strictly pseudoconvex bounded domain D in \mathbb{C}^N . Then X is a boundary of the complex sub-manifold $V \subset D - X$ with boundary regularity if and only if s -invariant $s(X)$ and $f^{(1,1)}(X)$ or $g^{(1,1)}(X)$ vanish.*

Corollary N. *Let X be a strictly pseudoconvex compact CR manifold of dimension 3. Suppose that X is contained in the boundary of a strictly pseudoconvex bounded domain D in \mathbb{C}^3 . Then*

X is a boundary of the complex sub-manifold $V \subset D - X$ if and only if s -invariant $s(X)$ and $f^{(1,1)}(X)$ or $g^{(1,1)}(X)$ vanish.

Although $s(X)$ and $f^{(1,1)}(X)$ are CR invariants, it is not clear how to compute them directly on X . Corollary O and Corollary P can be used to determine if X is a boundary of the complex sub-manifold of $D - X$ by using information solely on X .

Corollary O. *Let X be a strictly pseudoconvex compact Calabi-Yau CR manifold of dimension 3. Suppose that X is contained in the boundary of a strictly pseudoconvex bounded domain D in \mathbb{C}^N with $H_h^2(X) = 0$. Then X is a boundary of the complex sub-manifold $V \subset D - X$ with boundary regularity and the variety is smooth if and only if $g^{(1,1)}(X) = 0$.*

Corollary P. *Let X be a strictly pseudoconvex compact CR manifold of dimension 3. Suppose that X is contained in the boundary of a strictly pseudoconvex bounded domain D in \mathbb{C}^3 with $H_h^2(X) = 0$. Then X is a boundary of the complex sub-manifold $V \subset D - X$ if and only if $g^{(1,1)}(X) = 0$.*

1.3 Organization

Chapter 2 is about moduli space of bounded complete Reinhardt domains.

In section 2.1, we introduce the higher order Bergman functions which are biholomorphic invariants. In section 2.2.1, we show how to write down the k -th order Bergman functions for domains on A_n -variety and recall the fundamental CR-invariant $\nu_V^{(1,0)}$ which we need to use later. In section 2.2.2, we determines all possible biholomorphisms between two domains in A_n -variety. In section 2.2.3, we use higher order Bergman functions to construct numerical

invariants in Theorem A. Moreover, in Theorem B, we prove that the invariants in the Theorem A determine the Bergman function up to automorphism of A_n -variety. Theorem C, Theorem D and Theorem E are proved also in this section. In section 2.2.4, we calculate some concrete examples. In section 2.3.1, we construct continuous invariants of bounded complete Reinhardt domains in \mathbb{C}^n and prove Theorem F and Theorem G (G'). In section 2.3.2, the Hilbert 14th problem is discussed and complete continuous numerical invariants of bounded complete Reinhardt pseudoconvex domains are constructed. Theorem H (H'), Theorem J and Theorem K (K') are proved in this section. In section 2.3.3, we give applications to some concrete examples.

Chapter 3 is about complex Plateau problem. In section 3.1, we shall recall the definition of holomorphic De Rham cohomology for a CR manifold. In section 3.2, after recalling several local invariants of isolated singularity, we introduce some new invariants of singularities and new CR invariants for CR manifolds. In section 3.3, we prove the main theorems in chapter 3.

CHAPTER 2

MODULI SPACE OF BOUNDED COMPLETE REINHARDT DOMAINS

2.1 Preliminaries

In this section, we shall recall some basic definitions and results in the paper (22) which will facilitate our subsequent discussion. We also take this opportunity to correct some small mistakes in (22).

Recall that a complex manifold M is called pseudoconvex if there is a compact set B in M , and a continuous real valued function ϕ on M , which is plurisubharmonic outside B and such that for each $c \in \mathbb{R}$, the set $M_c = \{x \in M : \phi(x) < c\}$ is relatively compact in M . If ϕ is C^2 , “plurisubharmonic” is equivalent to that Hessian of ϕ is positive semi-definite. We call M strictly pseudoconvex if Hessian of ϕ is positive definite and then ϕ is called strictly plurisubharmonic. Note that a strictly pseudoconvex complex manifold is a modification of a Stein space at a finite many points.

Let M be a pseudoconvex complex manifold and A be a compact complex analytic variety in the interior of M .

Definition 2.1.0.3. Let F_M (respectively, $F_{M,A}$) be the space of all L^2 -integrable holomorphic n -form on M (respectively, vanishing at the compact analytic subset A in M). Let $\{\omega_j\}$ (respectively, $\{\omega_j^A\}$) be a complete orthonormal basis of F_M (respectively, $F_{M,A}$). The Bergman

kernel (respectively, Bergman kernel vanishing at A) is defined to be $K_M(z) = \sum_j \omega_j(z) \wedge \overline{\omega_j(z)}$ (respectively, $K_{M,A}(z) = \sum_j \omega_j^A(z) \wedge \overline{\omega_j^A(z)}$).

Lemma 2.1.0.4. (a) Bergman kernel $K_{M,A}(z)$ vanishing at the compact analytic subset A is independent of the choice of the complete orthonormal basis of $F_{M,A}$.

(b) Let $\Phi: (M_1, A_1) \rightarrow (M_2, A_2)$ be a biholomorphic map such that $\Phi(A_1) = A_2$. Then $K_{M_1, A_1}(z) = \Phi^* K_{M_2, A_2}(z)$.

Definition 2.1.0.5. The Bergman function $B_{M,A}$ on M is defined to be $K_{M,A}(z)/K_M(z)$.

The following Theorem 2.1.0.6 can be found in (22).

Theorem 2.1.0.6. Let A_1 (respectively A_2) be compact analytic variety in complex manifold M_1 (respectively M_2). If $\Phi: (M_1, A_1) \rightarrow (M_2, A_2)$ is a biholomorphic map, then $B_{M_1, A_1}(z) = B_{M_2, A_2}(\Phi(z))$.

For a special case, let V be a Stein variety of dimension $n \geq 2$ in \mathbb{C}^N with only irreducible isolated singularities. We assume that ∂V is a smooth CR manifold. Let $\pi: M \rightarrow V$ be a resolution of singularity with E as an exceptional set. We shall define the k -th order Bergman function $B_M^{(k)}(z)$ on M which is a biholomorphic invariant of M .

Definition 2.1.0.7. Let F (respectively, F_k) be the set of all L^2 integrable holomorphic n -forms Ψ on M (respectively, vanishing at least the k -th order on the exception set E of M). Let $\{w_j\}$ (respectively, $\{w_j^{(k)}\}$) be a complete orthonormal basis of F (respectively, F_k). The Bergman kernel (respectively Bergman kernel vanishing on the exceptional set of k -th order) is defined to be $K(z) = \sum w_j(z) \wedge \overline{w_j(z)}$ (respectively, $K^{(k)}(z) = \sum w_j^{(k)}(z) \wedge \overline{w_j^{(k)}(z)}$).

The proofs of the following two Lemmas are exactly the same as those in (22).

Lemma 2.1.0.8. *F/F_k is a finite dimensional vector space.*

Lemma 2.1.0.9. *Bergman kernel vanishing on the exceptional set of k -th order $K^{(k)}(z)$ is independent of the choice of the complete orthonormal basis of F_k and $K^{(k)}(z)$ is invariant under biholomorphic maps.*

Definition 2.1.0.10. *Let M be a resolution of a Stein variety V of dimension $n \geq 2$ in \mathbb{C}^N with only irreducible isolated singularity at the origin. The k -th order Bergman function $B_M^{(k)}$ on M is defined to be $K_M^{(k)}/K_M$.*

The proof of the following Theorem 2.1.0.11 is the same as the proof in Theorem 2.5 of (22). However, the last statement of Theorem 2.5 of (22) is not true in general (for example for some special weakly elliptic singularities). It is true when the canonical bundle is generated by its global sections in a neighborhood of the exceptional set, which is automorphically satisfied if V has only rational surface singularities.

Theorem 2.1.0.11. *$B_M^{(k)}$ is a global function defined on M which is invariant under biholomorphic maps. Moreover, $B_M^{(k)}$ is nowhere vanishing outside the exceptional set of M . If the canonical bundle is generated by its global sections in a neighborhood of the exceptional set, then the zero set of the k -th order Bergman function $B_M^{(k)}$ is precisely the exceptional set of M .*

The same argument of the proof of Theorem 1 in (29) will prove the following theorem.

Theorem 2.1.0.12. *Let M be a strictly pseudoconvex complex manifold of dimension $n \geq 2$ with exceptional set E . Let A be a compact submanifold contained in E . Let $\pi: M_1 \rightarrow M$*

be the blow up of M along A . Then we have $K_{M_1}^{(k)}(z) = \pi^* K_M^{(k)}(z)$ and $K_{M_1}(z) = \pi^* K_M(z)$.

Consequently $B_{M_1}^{(k)}(z) = \pi^* B_M^{(k)}(z)$.

Let $\pi_i: M_i \rightarrow V$, $i = 1, 2$, be two resolutions of singularities of V . By Hironaka's theorem (30), there exists a resolution $\tilde{\pi}: \tilde{M} \rightarrow V$ of singularities of V such that \tilde{M} can be obtained from M_i , $i = 1, 2$, by successive blowing up along submanifolds in exceptional set. In view of Theorem 2.1.0.11 and Theorem 2.1.0.12, the following definition is well defined if the canonical bundle is generated by its global sections in a neighborhood of the exceptional set. Moreover we can get Theorem 2.8 easily.

Definition 2.1.0.13. *Let V be a Stein variety in \mathbb{C}^N with only irreducible isolated singularities.*

Let $\pi: M \rightarrow V$ be a resolution of singularities of V such that the canonical bundle is generated by its global sections in a neighborhood of the exceptional set. Define the k -th order Bergman function $B_V^{(k)}$ on V to be the push forward of the k -th order Bergman function $B_M^{(k)}$ by the map π .

Theorem 2.1.0.14. *Let V be a Stein variety in \mathbb{C}^N with only irreducible isolated singularities.*

Assume that there exists a resolution M of singularities of V such that the canonical bundle is generated by its global sections in a neighborhood of the exceptional set. Then the k -th order Bergman function $B_V^{(k)}$ on V is invariant under biholomorphic maps and $B_V^{(k)}$ vanishes precisely on the singular set of V .

For the convenience of the readers, we recall the following three important theorems.

Theorem 2.1.0.15. ((23)) *A biholomorphic mapping between two strictly pseudoconvex domains is smooth up to boundary and the induced boundary mapping gives a CR-equivalence between the boundaries.*

Theorem 2.1.0.16. ((24)) *Two n -dimensional bounded Reinhardt domains D_1 and D_2 are mutually equivalent if and only if there exists a transformation $\phi : \mathbb{C}^n \rightarrow \mathbb{C}^n$ given by $z_i \mapsto r_i z_{\sigma(i)}$ ($r_i > 0, i = 1, \dots, n$ and σ being a permutation of the indices i) such that $\phi(D_1) = D_2$.*

Proposition 2.1.0.17. ((31)) *The Bergman kernel blows up at every boundary point in a pseudoconvex domain with C^1 -bounding in \mathbb{C}^n .*

2.2 Moduli space of bounded complete Reinhardt domains in A_n -variety

In this section, we will consider bounded complete Reinhardt domains in A_n -variety.

Let us recall some notations first. An open subset $D \subseteq \mathbb{C}^n$ is a complete Reinhardt domain if, whenever $(z_1, \dots, z_n) \in D$ then $(\xi_1 z_1, \dots, \xi_n z_n) \in D$ for all complex numbers ξ_j with $|\xi_j| \leq 1$.

Let $\widetilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$. It is well known that \widetilde{V}_n is the quotient of \mathbb{C}^2 by a cyclic group of order $n + 1$, i.e. $\delta.(z_1, z_2) = (\delta z_1, \delta^n z_2)$, where δ is a primitive $(n + 1)$ -th root of unit. The quotient map $\pi : \mathbb{C}^2 \rightarrow \widetilde{V}$ is given by $\pi(z_1, z_2) = (z_1^{n+1}, z_2^{n+1}, z_1 z_2)$.

Definition 2.2.0.18. *An open set V in the A_n -variety $\widetilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ is called a complete Reinhardt domain if $\pi^{-1}(V)$ is a complete Reinhardt domain in \mathbb{C}^2 .*

2.2.1 Continuous invariant k^{th} order Bergman function

Let X be a strictly pseudoconvex CR manifold of real dimension $2n - 1$. It is well known (19) that X can be CR embedded into \mathbb{C}^N if $n \geq 3$. For any embeddable strictly pseudo-

convex CR manifold of real dimension at least 3, the famous theorem of Harvey and Lawson (21) implies that X is a boundary of a variety V in \mathbb{C}^N for some N such that V has only isolated normal singularities.

Proposition 2.2.1.1. ((22)) *Let X_1, X_2 be two strictly pseudoconvex CR manifolds of dimension $2n - 1$ which bound varieties V_1, V_2 respectively in \mathbb{C}^N with only isolated normal singularities. If $\Phi: X_1 \rightarrow X_2$ is a CR-isomorphism, then Φ can be extended to a biholomorphic map from V_1 to V_2 .*

In view of the above Proposition 2.2.1.1, if X_1 and X_2 are two strictly pseudoconvex CR manifolds which bound varieties V_1 and V_2 respectively with non-isomorphic singularities, then X_1 and X_2 are not CR equivalent. Therefore to study the CR equivalence of two strictly pseudoconvex CR manifolds X_1 and X_2 , it remains to consider the case when X_1 and X_2 are lying on the same variety V . The purpose of this section is to show that our global invariant Bergman function of k -th order defined in section 2.1 can be used to study the CR equivalence problem of smooth CR manifolds lying on the same variety. As an example, we shall show explicitly how CR manifolds varies in the A_n -variety $\widetilde{V}_n = \{(x, y, z) \in \mathbb{C}^3: f(x, y, z) = xy - z^{n+1} = 0\}$. An explicit resolution $\widetilde{\pi}: \widetilde{M}_n \rightarrow \widetilde{V}_n$ can be given in terms of coordinate charts and transition functions as follows:

$$\text{Coordinate charts: } \widetilde{W}_k = \mathbb{C}^2 = \{(u_k, v_k)\}, \quad k = 0, 1, \dots, n.$$

$$\text{Transition functions: } \begin{cases} u_{k+1} = \frac{1}{v_k} \\ v_{k+1} = u_k v_k^2 \end{cases} \quad \text{or} \quad \begin{cases} u_k = u_{k+1}^2 v_{k+1} \\ v_k = \frac{1}{u_{k+1}} \end{cases}$$

$$\begin{aligned} \text{Resolution map: } \tilde{\pi}(u_k, v_k) &= (u_k^{k+1} v_k^k, u_k^{n-k} v_k^{n+1-k}, u_k v_k) \quad \text{or} \\ (x, y, z) &= (u_0, u_0^n v_0^{n+1}, u_0 v_0) = \cdots = (u_n^{n+1} v_n^n, v_n, u_n v_n) \end{aligned}$$

$$\begin{aligned} \text{Exceptional set: } E &= \tilde{\pi}^{-1}(0) = C_k = \{u_{k-1} = 0\} \cup \{v_k = 0\}, \\ k &= 1, \dots, n. \end{aligned}$$

From now on, we suppose V to be a bounded complete Reinhardt domain in \widetilde{V}_n . Then let $M = \tilde{\pi}^{-1}(V) = \cup_{k=0}^n W_k$, where $W_k = \tilde{\pi}^{-1}(V) \cap \widetilde{W}_k$, $k = 0, 1, \dots, n$. Observe that under $\pi := \tilde{\pi}|_M: M \rightarrow V$, $W_0 \setminus C_1$ is mapped biholomorphically onto $V \setminus y$ -axis. In particular $M \setminus W_0$ is of measure zero in the obvious sense. Hence, we may compute integrals on M using the (u_0, v_0) coordinate on the chart W_0 alone.

The following proposition is a general consequence of the proof of Proposition 3.2 of (22) (also cf. Proposition 8 in (29)).

Proposition 2.2.1.2. *In the above notations, let $\phi_{\alpha\beta} = u_0^\alpha v_0^\beta du_0 \wedge dv_0$, $\alpha, \beta = 0, 1, 2, \dots$. Then $\left\{ \frac{\phi_{\alpha\beta}}{\|\phi_{\alpha\beta}\|_M} : \alpha \geq \frac{n}{n+1}\beta \right\}$ is a complete orthonormal base of F and $\left\{ \frac{\phi_{\alpha\beta}}{\|\phi_{\alpha\beta}\|_M} : \alpha \geq \frac{n}{n+1}\beta \text{ and}$*

$\alpha \geq k$ is a complete orthonormal base of F_k . Therefore the Bergman kernel vanishing on the exceptional set of k -th order $K_M^{(k)}$ and the Bergman kernel K_M are given respectively by:

$$K_M^{(k)}(u_0, v_0) = \Theta_M^{(k)} du_0 \wedge dv_0 \wedge d\bar{u}_0 \wedge d\bar{v}_0$$

where

$$\Theta_M^{(k)} = \sum_{\substack{\alpha \geq \frac{n}{n+1}\beta \\ \alpha \geq k}} \frac{|u_0|^{2\alpha} |v_0|^{2\beta}}{\|\phi_{\alpha\beta}\|_M^2},$$

and

$$K_M(u_0, v_0) = \left(\frac{1}{\|\phi_{00}\|_M^2} + \sum_{\substack{\alpha \geq \frac{n}{n+1}\beta \\ 1 \leq \alpha \leq k-1}} \frac{|u_0|^{2\alpha} |v_0|^{2\beta}}{\|\phi_{\alpha\beta}\|_M^2} + \Theta_M^{(k)} \right) du_0 \wedge dv_0 \wedge d\bar{u}_0 \wedge d\bar{v}_0 \quad (2.2.1)$$

The following results generalize Theorem 3.3 in (22).

Theorem 2.2.1.3. *In the above notations, the k -th order Bergman function for the strictly pseudoconvex complex manifold M is given by*

$$B_M^{(k)}(u_0, v_0) = \frac{\Theta_M^{(k)}}{\left(\frac{1}{\|\phi_{00}\|_M^2} + \sum_{\substack{\alpha \geq \frac{n}{n+1}\beta \\ \alpha \geq 1}} \frac{|u_0|^{2\alpha} |v_0|^{2\beta}}{\|\phi_{\alpha\beta}\|_M^2} \right)} \quad (2.2.2)$$

The k -th order Bergman function for the variety is given by

$$B_V^{(k)}(x, y) = \frac{\Theta_V^{(k)}}{\left(\frac{1}{\|\phi_{00}\|_M^2} + \Theta_V^{(1)}\right)}, \quad (2.2.3)$$

where

$$\Theta_V^{(k)} = \sum_{\substack{\alpha \geq \frac{n}{n+1}\beta \\ \alpha \geq k}} \frac{|x|^{2\alpha - \frac{2n\beta}{n+1}} |y|^{\frac{2\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_M^2} \quad (2.2.4)$$

Proof: $B_M^{(k)}(u_0, v_0) = \frac{K_M^{(k)}}{K_M} = \frac{\|\phi_{00}\|_M^2 \Theta_M^{(k)}}{1 + \|\phi_{00}\|_M^2 \Theta_M^{(1)}}$, so Equation 2.2.2 follows immediately. Recall that the resolution map is given by $(x, y, z) = (u_0, u_0^n v_0^{n+1}, u_0 v_0)$. Then Equation 2.2.3 and Equation 2.2.4 follow from Equation 2.2.2. Q. E. D.

2.2.2 Biholomorphic maps between two bounded complete Reinhardt domains in A_n -variety

Lemma 2.2.2.1. *Let V be a complete Reinhardt domain in the A_n -variety \widetilde{V}_n . Any biholomorphism $\Psi = (\psi_1, \psi_2, \psi_3): V \rightarrow V$ has the following representation*

$$\begin{pmatrix} \psi_1(x, y, z) \\ \psi_2(x, y, z) \\ \psi_3(x, y, z) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

+ higher order terms in x, y and z .

If $n = 1$, the constants satisfy the following equations

$$a_{11}a_{21} - a_{31}^2 = 0 \quad (2.2.5)$$

$$a_{12}a_{22} - a_{32}^2 = 0 \quad (2.2.6)$$

$$a_{13}a_{23} - a_{33}^2 + a_{11}a_{22} + a_{12}a_{21} - 2a_{31}a_{32} = 0 \quad (2.2.7)$$

$$a_{11}a_{23} + a_{13}a_{21} - 2a_{31}a_{33} = 0 \quad (2.2.8)$$

$$a_{12}a_{23} + a_{13}a_{22} - 2a_{32}a_{33} = 0 \quad (2.2.9)$$

$$\det(a_{ij}) \neq 0 \quad (2.2.10)$$

If $n > 1$, the constants satisfy the following equations

$$a_{11}a_{21} = 0 \quad (2.2.11)$$

$$a_{12}a_{22} = 0 \quad (2.2.12)$$

$$a_{13}a_{23} = 0 \quad (2.2.13)$$

$$a_{11}a_{23} + a_{13}a_{21} = 0 \quad (2.2.14)$$

$$a_{12}a_{23} + a_{13}a_{22} = 0 \quad (2.2.15)$$

$$a_{11}a_{22} + a_{12}a_{21} - a_{33}^{n+1} = 0 \quad (2.2.16)$$

$$\det(a_{ij}) \neq 0 \quad (2.2.17)$$

Proof: The case of $n = 1$ has been proved in (22). For $n > 1$, since $\Psi: V \rightarrow V$, we have $\psi_1(x, y, z)\psi_2(x, y, z) - \psi_3^{n+1}(x, y, z) = 0$. By looking at the quadratic part of this equation and

the fact that $xy = z^{n+1}$, we obtain $(a_{11}x + a_{12}y + a_{13}z)(a_{21}x + a_{22}y + a_{23}z) - a_{33}^{n+1}xy = 0$. Then

the lemma follows easily.

Q. E. D.

Proposition 2.2.2.2. *Let V_i , $i=1,2$, be two complete Reinhardt domains in $\widetilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$. Let $M_i = \widetilde{\pi}^{-1}(V_i)$, $i=1,2$. Suppose that $\Psi: V_1 \rightarrow V_2$ is a biholomorphic map given by $\Psi(x, y, z) = (a_{11}x + a_{12}y + a_{13}z, a_{21}x + a_{22}y + a_{23}z, a_{31}x + a_{32}y + a_{33}z) + \text{higher order term}$. Then*

$$\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{10}\|_{M_2}^2} |a_{11}|^2 + \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2} |a_{31}|^2 + \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{n,n+1}\|_{M_2}^2} |a_{21}|^2 = \frac{\|\phi_{00}\|_{M_1}^2}{\|\phi_{10}\|_{M_1}^2} \quad (2.2.18)$$

$$\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{10}\|_{M_2}^2} |a_{12}|^2 + \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2} |a_{32}|^2 + \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{n,n+1}\|_{M_2}^2} |a_{22}|^2 = \frac{\|\phi_{00}\|_{M_1}^2}{\|\phi_{n,n+1}\|_{M_1}^2} \quad (2.2.19)$$

$$\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{10}\|_{M_2}^2} |a_{13}|^2 + \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2} |a_{33}|^2 + \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{n,n+1}\|_{M_2}^2} |a_{23}|^2 = \frac{\|\phi_{00}\|_{M_1}^2}{\|\phi_{11}\|_{M_1}^2} \quad (2.2.20)$$

$$\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{10}\|_{M_2}^2} a_{11} \overline{a_{12}} + \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2} a_{31} \overline{a_{32}} + \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{n,n+1}\|_{M_2}^2} a_{21} \overline{a_{22}} = 0 \quad (2.2.21)$$

$$\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{10}\|_{M_2}^2} a_{11} \overline{a_{13}} + \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2} a_{31} \overline{a_{33}} + \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{n,n+1}\|_{M_2}^2} a_{21} \overline{a_{23}} = 0 \quad (2.2.22)$$

$$\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{10}\|_{M_2}^2} a_{12} \overline{a_{13}} + \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2} a_{32} \overline{a_{33}} + \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{n,n+1}\|_{M_2}^2} a_{22} \overline{a_{23}} = 0 \quad (2.2.23)$$

Proof: Since $B_{V_1}(x, y, z) = B_{V_2}(\Psi(x, y, z))$, we have

$$\begin{aligned} & \frac{\|\phi_{00}\|_{M_1}^2}{\|\phi_{10}\|_{M_1}^2} |x|^2 + \frac{\|\phi_{00}\|_{M_1}^2}{\|\phi_{n,n+1}\|_{M_1}^2} |y|^2 + \frac{\|\phi_{00}\|_{M_1}^2}{\|\phi_{11}\|_{M_1}^2} |z|^2 \\ &= \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{10}\|_{M_2}^2} |a_{11}x + a_{12}y + a_{13}z|^2 + \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{n,n+1}\|_{M_2}^2} |a_{21}x + a_{22}y + a_{23}z|^2 + \end{aligned}$$

$$\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2} |a_{31}x + a_{32}y + a_{33}z|^2.$$

By comparing the coefficients of $|x|^2$, $|y|^2$, $|z|^2$, $x\bar{y}$, $x\bar{z}$, $y\bar{z}$, $\bar{x}y$, $\bar{x}z$ and $\bar{y}z$, we can get the identities immediately. Q. E. D.

Next we recall some results in (22).

Theorem 2.2.2.3. (22) *Let V be a bounded complete Reinhardt domain in $\widetilde{V}_1 = \{(x, y, z) \in \mathbb{C}^3 : xy = z^2\}$ such that ∂V is a smooth CR manifold. Let $\tilde{\pi} : \widetilde{M} \rightarrow \widetilde{V}_1$ be a resolution of \widetilde{V}_1 and $M = \tilde{\pi}^{-1}(V)$. With the notation in Proposition 2.2.1.2, $\nu_V^{(1,0)} := \frac{\|\phi_{11}\|_M^2}{\|\phi_{10}\|_M \|\phi_{12}\|_M}$ is a holomorphic invariant of V in \widetilde{V}_1 , i.e., if V_1 and V_2 are two such bounded complete Reinhardt domains in \widetilde{V}_1 which are biholomorphically equivalent, then $\frac{\|\phi_{11}\|_{M_1}^2}{\|\phi_{10}\|_{M_1} \|\phi_{12}\|_{M_1}} = \frac{\|\phi_{11}\|_{M_2}^2}{\|\phi_{10}\|_{M_2} \|\phi_{12}\|_{M_2}}$, where $M_i = \tilde{\pi}^{-1}(V_i)$, $i = 1, 2$.*

Corollary 2.2.2.4. (22) *Let V_i , $i = 1, 2$, be two bounded complete Reinhardt domains in $\widetilde{V}_1 = \{(x, y, z) \in \mathbb{C}^3 : xy = z^2\}$. If the holomorphic invariant $\nu_{V_1}^{(1,0)}$ or $\nu_{V_2}^{(1,0)}$ in Theorem 2.2.2.3 is not equal to $\frac{1}{2}$, then the biholomorphic map $\Psi = (\psi_1, \psi_2, \psi_3) : V_1 \rightarrow V_2$ must be one of the following forms:*

1. $(\psi_1, \psi_2, \psi_3) = (a_{11}x, a_{22}y, a_{33}z) + \text{higher order terms}$ and $a_{33}^2 = a_{11}a_{22}$, where $a_{11}a_{22}a_{33} \neq 0$.
2. $(\psi_1, \psi_2, \psi_3) = (a_{12}y, a_{21}x, a_{33}z) + \text{higher order terms}$ and $a_{33}^2 = a_{12}a_{21}$, where $a_{11}a_{22}a_{33} \neq 0$.

The following lemma generalizes Corollary 3.8 in (22) and can be proved easily.

Lemma 2.2.2.5. *Let $V_i, i = 1, 2$, be two complete Reinhardt domains in $\widetilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$, where $n > 1$. Then the biholomorphic map $\Psi = (\psi_1, \psi_2, \psi_3): V_1 \rightarrow V_2$ must be one of the following forms:*

1. $(\psi_1, \psi_2, \psi_3) = (a_{11}x, a_{22}y, a_{33}z) + \text{higher order terms}$ and $a_{33}^{n+1} = a_{11}a_{22}$, where $a_{11}a_{22}a_{33} \neq 0$.
2. $(\psi_1, \psi_2, \psi_3) = (a_{12}y, a_{21}x, a_{33}z) + \text{higher order terms}$ and $a_{33}^{n+1} = a_{12}a_{21}$, where $a_{11}a_{22}a_{33} \neq 0$.

Next we shall show that our Bergman function of order 1 can be used to determine the biholomorphisms between two bounded complete Reinhardt domains in A_n -variety.

Theorem 2.2.2.6. *Let $V_i, i = 1, 2$, be two bounded complete Reinhardt domains in A_n -variety $\widetilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$. If $n = 1$ and $\nu_{V_1}^{(1,0)}$ or $\nu_{V_2}^{(1,0)}$ the holomorphic invariant defined in the previous section is not equal to $\frac{1}{2}$ or $n > 1$, then the biholomorphism Ψ from V_1 to V_2 must be one of the following forms:*

1. $(\psi_1, \psi_2, \psi_3) = (a_{11}x, a_{22}y, a_{33}z)$ and $a_{33}^{n+1} = a_{11}a_{22}$,

where

$$|a_{11}| = \frac{\|\phi_{10}\|_{M_2} \|\phi_{00}\|_{M_1}}{\|\phi_{00}\|_{M_2} \|\phi_{10}\|_{M_1}}, \quad (2.2.24)$$

$$|a_{22}| = \frac{\|\phi_{n,n+1}\|_{M_2} \|\phi_{00}\|_{M_1}}{\|\phi_{00}\|_{M_2} \|\phi_{n,n+1}\|_{M_1}}, \quad (2.2.25)$$

$$|a_{33}| = \frac{\|\phi_{11}\|_{M_2} \|\phi_{00}\|_{M_1}}{\|\phi_{00}\|_{M_2} \|\phi_{11}\|_{M_1}}, \quad (2.2.26)$$

2. $(\psi_1, \psi_2, \psi_3) = (a_{12}y, a_{21}x, a_{33}z)$ and $a_{33}^{n+1} = a_{12}a_{21}$,

where

$$|a_{12}| = \frac{\|\phi_{10}\|_{M_2} \|\phi_{00}\|_{M_1}}{\|\phi_{00}\|_{M_2} \|\phi_{n,n+1}\|_{M_1}}, \quad (2.2.27)$$

$$|a_{21}| = \frac{\|\phi_{n,n+1}\|_{M_2} \|\phi_{00}\|_{M_1}}{\|\phi_{00}\|_{M_2} \|\phi_{10}\|_{M_1}}, \quad (2.2.28)$$

$$|a_{33}| = \frac{\|\phi_{11}\|_{M_2} \|\phi_{00}\|_{M_1}}{\|\phi_{00}\|_{M_2} \|\phi_{11}\|_{M_1}}. \quad (2.2.29)$$

Moreover, if V_1 is biholomorphic to V_2 , then

$$\frac{\|\phi_{11}\|_{M_1}^{n+1}}{\|\phi_{00}\|_{M_1}^{n-1} \|\phi_{10}\|_{M_1} \|\phi_{n,n+1}\|_{M_1}} = \frac{\|\phi_{11}\|_{M_2}^{n+1}}{\|\phi_{00}\|_{M_2}^{n-1} \|\phi_{10}\|_{M_2} \|\phi_{n,n+1}\|_{M_2}} \quad (2.2.30)$$

Proof: In view of Corollary 2.2.2.4 and Lemma 2.2.2.5, we know that $\Psi = (\psi_1, \psi_2, \psi_3)$ must be one of the following forms:

1. $(\psi_1, \psi_2, \psi_3) = (a_{11}x, a_{22}y, a_{33}z) + \text{higher order terms}$ and $a_{33}^{n+1} = a_{11}a_{22}$.
2. $(\psi_1, \psi_2, \psi_3) = (a_{12}y, a_{21}x, a_{33}z) + \text{higher order terms}$ and $a_{33}^{n+1} = a_{12}a_{21}$.

We only need to get rid of the higher order terms in the statement of Corollary 2.2.2.4 and Lemma 2.2.2.5. Recall that the Bergman function is of the form

$$B_V^{(1)}(x, y) = \frac{\Theta_V^{(1)}}{\left(\frac{1}{\|\phi_{00}\|_M^2} + \Theta_V^{(1)}\right)}.$$

Since

$$B_{V_1}(x, y, z) = B_{V_2}(\Psi(x, y, z)),$$

$$\|\phi_{00}\|_{M_1}^2 \Theta_{V_1}^{(1)} = \|\phi_{00}\|_{M_2}^2 \Theta_{V_2}^{(1)}. \quad (2.2.31)$$

Putting forms (1) and (2) in the equality above and comparing the 3rd order terms, we see that the 2nd order terms of (ψ_1, ψ_2, ψ_3) are zero. Repeating this argument, we see that (ψ_1, ψ_2, ψ_3) has only linear terms.

Except for Equation 2.2.30, the rest of the theorem follows from Proposition 2.2.2.2, Corollary 2.2.2.4 and Lemma 2.2.2.5. To see Equation 2.2.30, we have two cases.

In case (1), $a_{33}^{n+1} = a_{11}a_{22}$ and Equation 2.2.24, Equation 2.2.25, Equation 2.2.26 imply

$$\frac{\|\phi_{11}\|_{M_2}^{n+1}}{\|\phi_{00}\|_{M_2}^{n+1}} \cdot \frac{\|\phi_{00}\|_{M_1}^{n+1}}{\|\phi_{11}\|_{M_1}^{n+1}} = \frac{\|\phi_{n,n+1}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{n,n+1}\|_{M_1}} \cdot \frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}}.$$

So Equation 2.2.30 holds.

In case (2), $a_{33}^{n+1} = a_{12}a_{21}$ and Equation 2.2.27, Equation 2.2.28, Equation 2.2.29 imply

$$\frac{\|\phi_{11}\|_{M_2}^{n+1}}{\|\phi_{00}\|_{M_2}^{n+1}} \cdot \frac{\|\phi_{00}\|_{M_1}^{n+1}}{\|\phi_{11}\|_{M_1}^{n+1}} = \frac{\|\phi_{n,n+1}\|_{M_2} \|\phi_{00}\|_{M_1}}{\|\phi_{00}\|_{M_2} \|\phi_{10}\|_{M_1}} \cdot \frac{\|\phi_{00}\|_{M_1} \|\phi_{10}\|_{M_2}}{\|\phi_{n,n+1}\|_{M_1} \|\phi_{00}\|_{M_2}}.$$

So Equation 2.2.30 holds.

Q. E. D.

The following Corollary 2.2.2.7 corrects some misprints in Theorem 5.1 of (22) and generalizes the case to A_n type.

Corollary 2.2.2.7. *Let V be a bounded complete Reinhardt domain in A_n -variety $\widetilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$. Let $\nu_V^{(1,0)}$ be the CR invariant defined in the Theorem 3.6. Then the automorphism group of V for $n = 1$ and $\nu_V^{(1,0)} \neq \frac{1}{2}$ or $n > 1$ consists of biholomorphic map $\Psi = (\psi_1, \psi_2, \psi_3)$ of the following forms:*

1. $(\psi_1, \psi_2, \psi_3) = (a_{11}x, a_{22}y, a_{33}z)$, where $|a_{11}| = |a_{22}| = |a_{33}| = 1$
2. $(\psi_1, \psi_2, \psi_3) = (a_{12}y, a_{21}x, a_{33}z)$,

$$\text{where } |a_{11}| = \frac{\|\phi_{10}\|_M}{\|\phi_{n,n+1}\|_M}, |a_{12}| = \frac{\|\phi_{n,n+1}\|_M}{\|\phi_{10}\|_M}, |a_{21}| = 1$$

Now we are going to deal with the biholomorphism between complete Reinhardt domains with $\nu_V^{(1,0)} = \frac{1}{2}$.

Theorem 2.2.2.8. *Let V_i , $i = 1, 2$, be two bounded complete Reinhardt domains in A_1 -variety $\widetilde{V}_1 = \{(x, y, z) \in \mathbb{C}^3 : xy = z^2\}$. If $\nu_{V_1}^{(1,0)}$ or $\nu_{V_2}^{(1,0)}$ the holomorphic invariant defined in the previous section is equal to $\frac{1}{2}$, then the biholomorphism Ψ from V_1 to V_2 must be one of the following forms:*

$$\begin{aligned}
1. (\psi_1, \psi_2, \psi_3) &= \left(e^{i\theta_1} \frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}} x, e^{i\theta_2} \frac{\|\phi_{12}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{12}\|_{M_1}} y, \right. \\
&\quad \left. \pm e^{i\frac{\theta_1+\theta_2}{2}} \frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}} z \right) \\
2. (\psi_1, \psi_2, \psi_3) &= \left(e^{i\theta_1} \frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{12}\|_{M_1}} y, e^{i\theta_2} \frac{\|\phi_{12}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}} x, \right. \\
&\quad \left. \pm e^{i\frac{\theta_1+\theta_2}{2}} \frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}} z \right) \\
3. \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} &= \begin{pmatrix} \alpha r e^{-i\theta} a_{31} & \frac{1}{r} e^{-i\theta} a_{32} & \pm a_{13} \\ \frac{1}{\alpha r} e^{i\theta} a_{31} & r e^{i\theta} a_{32} & \pm \frac{1}{\alpha} e^{2i\theta} a_{13} \\ a_{31} & a_{32} & \pm \frac{\alpha r^2 + 1}{2\alpha r} e^{i\theta} a_{13} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}
\end{aligned}$$

where

$$\alpha = \frac{-\|\phi_{10}\|_{M_2}}{\|\phi_{12}\|_{M_2}} \quad (2.2.32)$$

$$a_{31} = \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}} \frac{r\|\phi_{10}\|_{M_2}\|\phi_{12}\|_{M_2}}{\|\phi_{00}\|_{M_2}(r^2\|\phi_{10}\|_{M_2} + \|\phi_{12}\|_{M_2})} e^{i\theta_{31}} \quad (2.2.33)$$

$$a_{32} = \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{12}\|_{M_1}} \frac{r\|\phi_{10}\|_{M_2}\|\phi_{12}\|_{M_2}}{\|\phi_{00}\|_{M_2}(\|\phi_{12}\|_{M_2} + r^2\|\phi_{10}\|_{M_2})} e^{i\theta_{32}} \quad (2.2.34)$$

$$a_{13} = \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}} \frac{2r\|\phi_{10}\|_{M_2}\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}(r^2\|\phi_{10}\|_{M_2} + \|\phi_{12}\|_{M_2})} e^{i\left(\frac{\pi}{2} + \frac{\theta_{31}}{2} + \frac{\theta_{32}}{2} - \theta\right)} \quad (2.2.35)$$

$$0 < r < \infty, 0 \leq \theta_{31}, \theta_{32}, \theta < 2\pi.$$

Proof: By going through the proof of Theorem 3.7 in (22), we see that statements (1) and (2) in the theorem follow directly from Theorem 2.2.2.6. The remaining case that we need to deal with is the case 1 (b) in the proof of Theorem 3.7 in (22). Let us summarize the situation.

$$\nu_{V_2}^{(1,0)} = \frac{\|\phi_{11}\|_{M_2}^2}{\|\phi_{10}\|_{M_2}\|\phi_{12}\|_{M_2}} = \frac{1}{2} = \nu_{V_1}^{(1,0)} = \frac{\|\phi_{11}\|_{M_1}^2}{\|\phi_{10}\|_{M_1}\|\phi_{12}\|_{M_1}} \quad (2.2.36)$$

$$a_{11} = r_1 a_{31}, \quad a_{21} = \frac{1}{r_1} a_{31} \quad (2.2.37)$$

$$a_{22} = r_2 a_{32}, \quad a_{12} = \frac{1}{r_2} a_{32} \quad (2.2.38)$$

$$a_{23} = \frac{r_2}{r_1} a_{13}, \quad a_{33} = \left(\frac{r_2}{2} + \frac{1}{2r_1} \right) a_{13} \quad (2.2.39)$$

$$\alpha = \frac{r_1}{\bar{r}_2} = \frac{-2\|\phi_{11}\|_{M_2}^2}{\|\phi_{12}\|_{M_2}^2} = \frac{-\|\phi_{10}\|_{M_2}}{\|\phi_{12}\|_{M_2}} \quad (2.2.40)$$

$$r_2 = r e^{i\theta} \quad (2.2.41)$$

$$r_1 = \alpha r e^{-i\theta} = -\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{12}\|_{M_2}} r e^{-i\theta} \quad (2.2.42)$$

Putting Equation 2.2.37 into Equation 2.2.18, we get

$$|a_{31}|^2 \|\phi_{00}\|_{M_2}^2 \left[\frac{|r_1|^2}{\|\phi_{10}\|_{M_2}^2} + \frac{1}{\|\phi_{11}\|_{M_2}^2} + \frac{1}{|r_1|^2 \|\phi_{12}\|_{M_2}^2} \right] = \frac{\|\phi_{00}\|_{M_1}^2}{\|\phi_{10}\|_{M_1}^2}$$

By applying Equation 2.2.36 in the above equation we get

$$|a_{31}| = \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}} \frac{|r_1| \|\phi_{10}\|_{M_2} \|\phi_{12}\|_{M_2}}{\|\phi_{00}\|_{M_2} (|r_1|^2 \|\phi_{12}\|_{M_2} + \|\phi_{10}\|_{M_2})}$$

In view of Equation 2.2.42, we have

$$|a_{31}| = \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}} \frac{r \|\phi_{10}\|_{M_2} \|\phi_{12}\|_{M_2}}{\|\phi_{00}\|_{M_2} (\|\phi_{10}\|_{M_2} r^2 + \|\phi_{12}\|_{M_2})} \quad (2.2.43)$$

Putting Equation 2.2.38 into Equation 2.2.19, we get

$$|a_{32}|^2 \left[\frac{\|\phi_{00}\|_{M_2}^2}{|r_2|^2 \|\phi_{10}\|_{M_2}^2} + \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2} + \frac{|r_2|^2 \|\phi_{00}\|_{M_2}^2}{\|\phi_{12}\|_{M_2}^2} \right] = \frac{\|\phi_{00}\|_{M_1}^2}{\|\phi_{12}\|_{M_1}^2}$$

By applying Equation 2.2.36 in the above equation, we get

$$|a_{32}|^2 \|\phi_{00}\|_{M_2}^2 \left[\frac{\|\phi_{12}\|_{M_2}^2 + 2|r_2|^2 \|\phi_{10}\|_{M_2} \|\phi_{12}\|_{M_2} + |r_2|^4 \|\phi_{10}\|_{M_2}^2}{|r_2|^2 \|\phi_{10}\|_{M_2}^2 \|\phi_{12}\|_{M_2}^2} \right] = \frac{\|\phi_{00}\|_{M_1}^2}{\|\phi_{12}\|_{M_1}^2}$$

In view of Equation 2.2.41, we have

$$|a_{32}| = \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{12}\|_{M_1}} \frac{r \|\phi_{10}\|_{M_2} \|\phi_{12}\|_{M_2}}{\|\phi_{00}\|_{M_2} (\|\phi_{12}\|_{M_2} + r^2 \|\phi_{10}\|_{M_2})} \quad (2.2.44)$$

Putting Equation 2.2.39 into Equation 2.2.20, we get

$$|a_{13}|^2 \left[\frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{10}\|_{M_2}^2} + \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{11}\|_{M_2}^2} + \frac{(r_1 r_2 + 1)^2}{4|r_1|^2} + \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{12}\|_{M_2}^2} \left| \frac{r_2}{r_1} \right|^2 \right] = \frac{\|\phi_{00}\|_{M_1}^2}{\|\phi_{11}\|_{M_1}^2}$$

By applying Equation 2.2.36 in the above equation, we get

$$|a_{13}|^2 = \frac{\|\phi_{00}\|_{M_1}^2}{\|\phi_{11}\|_{M_1}^2} \cdot \frac{|r_1|^2 \|\phi_{10}\|_{M_2}^2 \|\phi_{12}\|_{M_2}^2}{\|\phi_{00}\|_{M_2}^2 \left[|r_1|^2 \|\phi_{12}\|_{M_2}^2 + \frac{(1+r_1 r_2)^2}{2} \|\phi_{10}\|_{M_2} \|\phi_{12}\|_{M_2} + |r_2|^2 \|\phi_{10}\|_{M_2}^2 \right]}$$

In view of Equation 2.2.41 and Equation 2.2.42, we have

$$|a_{13}| = \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}} \frac{2r \|\phi_{10}\|_{M_2} \|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2} (r^2 \|\phi_{10}\|_{M_2} + \|\phi_{12}\|_{M_2})} \quad (2.2.45)$$

Putting Equation 2.2.37, Equation 2.2.38, Equation 2.2.39 into Equation 2.2.7, we get

$$a_{13}^2 = a_{31} a_{32} \frac{4r_1}{r_2} \quad (2.2.46)$$

Putting Equation 2.2.43, Equation 2.2.44 and Equation 2.2.45 into Equation 2.2.46, we get

$$\theta_{13} = \frac{\pi}{2} + \frac{\theta_{31}}{2} + \frac{\theta_{31}}{2} - \theta \quad (2.2.47)$$

We have shown that

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} \alpha r e^{-i\theta} a_{31} & \frac{1}{r} e^{-i\theta} a_{32} & \pm a_{13} \\ \frac{1}{\alpha r} e^{i\theta} a_{31} & r e^{i\theta} a_{32} & \pm \frac{1}{\alpha} e^{2i\theta} a_{13} \\ a_{31} & a_{32} & \pm \frac{\alpha r^2 + 1}{2\alpha r} e^{i\theta} a_{13} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ + \text{higher order terms} \quad (2.2.48)$$

where α , a_{31} , a_{32} , a_{13} , are described in Equation 2.2.32–Equation 2.2.35. It remains to prove that all the higher order terms vanish. In view of Theorem 2.1.0.14, we have

$$B_{V_1}(x, y, z) = B_{V_2}(\psi(x, y, z)). \quad (2.2.49)$$

Putting Equation 2.2.48 into Equation 2.2.49 and comparing the 3rd order terms in Equation 2.2.49, we see easily that the 2nd order terms of (ψ_1, ψ_2, ψ_3) are zero. By repeating this argument, we see that (ψ_1, ψ_2, ψ_3) has only linear terms. Q. E. D.

2.2.3 Continuous numerical invariants of bounded complete Reinhardt domains in A_n -variety

In (22), we have succeeded in constructing the continuous numerical invariant of domains in A_1 -variety $\nu_V^{(1,0)} = \frac{\|\phi_{11}\|_M^2}{\|\phi_{10}\|_M \|\phi_{12}\|_M}$ from 1st order Bergman function. In this section, we shall construct infinitely many continuous numerical invariants of domains in A_n -variety from Bergman function.

Lemma 2.2.3.1. *Let V_i , $i = 1, 2$, be two bounded complete Reinhardt domains in A_n -variety $\widetilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ and M_i is a resolution of V_i . If $(\psi_1, \psi_2, \psi_3) = (a_{11}x, a_{22}y, a_{33}z)$ and $a_{33}^{n+1} = a_{11}a_{22}$,*

where

$$|a_{11}| = \frac{\|\phi_{10}\|_{M_2} \|\phi_{00}\|_{M_1}}{\|\phi_{00}\|_{M_2} \|\phi_{10}\|_{M_1}}, \quad |a_{22}| = \frac{\|\phi_{n,n+1}\|_{M_2} \|\phi_{00}\|_{M_1}}{\|\phi_{00}\|_{M_2} \|\phi_{n,n+1}\|_{M_1}},$$

$$|a_{33}| = \frac{\|\phi_{11}\|_{M_2} \|\phi_{00}\|_{M_1}}{\|\phi_{00}\|_{M_2} \|\phi_{11}\|_{M_1}},$$

is a biholomorphism from V_1 to V_2 , then the following equations hold:

$$\frac{\|\phi_{10}\|_{M_1}^{\alpha - \frac{n}{n+1}\beta} \|\phi_{n,n+1}\|_{M_1}^{\frac{\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_{M_1} \|\phi_{00}\|_{M_1}^{\alpha - \frac{n-1}{n+1}\beta - 1}} = \frac{\|\phi_{10}\|_{M_2}^{\alpha - \frac{n}{n+1}\beta} \|\phi_{n,n+1}\|_{M_2}^{\frac{\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_{M_2} \|\phi_{00}\|_{M_2}^{\alpha - \frac{n-1}{n+1}\beta - 1}}, \quad \alpha \geq \frac{n}{n+1}\beta.$$

Proof: From Equation 2.2.31 $\|\phi_{00}\|_{M_1}^2 \Theta_{V_1}^{(1)} = \|\phi_{00}\|_{M_2}^2 \Theta_{V_2}^{(1)}$, we have

$$\|\phi_{00}\|_{M_1}^2 \sum_{\substack{\alpha \geq \frac{n}{n+1}\beta \\ \alpha \geq 1}} \frac{|x|^{2\alpha - \frac{2n\beta}{n+1}} |y|^{\frac{2\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_{M_1}^2} = \|\phi_{00}\|_{M_2}^2 \sum_{\substack{\alpha \geq \frac{n}{n+1}\beta \\ \alpha \geq 1}} \frac{|\psi_1|^{2\alpha - \frac{2n\beta}{n+1}} |\psi_2|^{\frac{2\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_{M_2}^2}.$$

Comparing the coefficients of $|x|^{2\alpha - \frac{2n\beta}{n+1}} |y|^{\frac{2\beta}{n+1}}$ each side, we get

$$\frac{\|\phi_{00}\|_{M_1}^2}{\|\phi_{\alpha\beta}\|_{M_1}^2} = \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{\alpha\beta}\|_{M_2}^2} \cdot \frac{\|\phi_{00}\|_{M_1}^{2\alpha - \frac{2n\beta}{n+1}}}{\|\phi_{10}\|_{M_1}^{2\alpha - \frac{2n\beta}{n+1}}} \cdot \frac{\|\phi_{10}\|_{M_2}^{2\alpha - \frac{2n\beta}{n+1}}}{\|\phi_{00}\|_{M_2}^{2\alpha - \frac{2n\beta}{n+1}}}.$$

$$\frac{\|\phi_{00}\|_{M_1}^{\frac{2\beta}{n+1}}}{\|\phi_{n,n+1}\|_{M_1}^{\frac{2\beta}{n+1}}} \cdot \frac{\|\phi_{n,n+1}\|_{M_2}^{\frac{2\beta}{n+1}}}{\|\phi_{00}\|_{M_2}^{\frac{2\beta}{n+1}}}.$$

Simplifying this equation, we get

$$\frac{\|\phi_{10}\|_{M_1}^{\alpha - \frac{n}{n+1}\beta} \|\phi_{n,n+1}\|_{M_1}^{\frac{\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_{M_1} \|\phi_{00}\|_{M_1}^{\alpha - \frac{n-1}{n+1}\beta - 1}} = \frac{\|\phi_{10}\|_{M_2}^{\alpha - \frac{n}{n+1}\beta} \|\phi_{n,n+1}\|_{M_2}^{\frac{\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_{M_2} \|\phi_{00}\|_{M_2}^{\alpha - \frac{n-1}{n+1}\beta - 1}}, \quad \alpha \geq \frac{n}{n+1}\beta.$$

Q. E. D.

Lemma 2.2.3.2. *Let V_i , $i = 1, 2$, be two bounded complete Reinhardt domains in A_n -variety $\widetilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ and M_i is a resolution of V_i . If $(\psi_1, \psi_2, \psi_3) = (a_{12}y, a_{21}x, a_{33}z)$ and $a_{33}^{n+1} = a_{12}a_{21}$,*

where

$$|a_{12}| = \frac{\|\phi_{10}\|_{M_2} \|\phi_{00}\|_{M_1}}{\|\phi_{00}\|_{M_2} \|\phi_{n,n+1}\|_{M_1}}, \quad |a_{21}| = \frac{\|\phi_{n,n+1}\|_{M_2} \|\phi_{00}\|_{M_1}}{\|\phi_{00}\|_{M_2} \|\phi_{10}\|_{M_1}},$$

$$|a_{33}| = \frac{\|\phi_{11}\|_{M_2} \|\phi_{00}\|_{M_1}}{\|\phi_{00}\|_{M_2} \|\phi_{11}\|_{M_1}},$$

is a biholomorphism from V_1 to V_2 , then the following equations hold:

$$\frac{\|\phi_{10}\|_{M_1}^{\alpha - \frac{n}{n+1}\beta} \|\phi_{n,n+1}\|_{M_1}^{\frac{\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_{M_1} \|\phi_{00}\|_{M_1}^{\alpha - \frac{n-1}{n+1}\beta - 1}} = \frac{\|\phi_{10}\|_{M_2}^{\frac{\beta}{n+1}} \|\phi_{n,n+1}\|_{M_2}^{\alpha - \frac{n}{n+1}\beta}}{\|\phi_{n\alpha - (n-1)\beta, (n+1)\alpha - n\beta}\|_{M_2} \|\phi_{00}\|_{M_2}^{\alpha - \frac{n-1}{n+1}\beta - 1}},$$

where

$$\alpha \geq \frac{n}{n+1}\beta.$$

Proof: From Equation 2.2.31, $\|\phi_{00}\|_{M_1}^2 \Theta_{V_1}^{(1)} = \|\phi_{00}\|_{M_2}^2 \Theta_{V_2}^{(1)}$, we have

$$\|\phi_{00}\|_{M_1}^2 \sum_{\substack{\alpha \geq \frac{n}{n+1}\beta \\ \alpha \geq 1}} \frac{|x|^{2\alpha - \frac{2n\beta}{n+1}} |y|^{\frac{2\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_{M_1}^2} = \|\phi_{00}\|_{M_2}^2 \sum_{\substack{\alpha \geq \frac{n}{n+1}\beta \\ \alpha \geq 1}} \frac{|\psi_1|^{2\alpha - \frac{2n\beta}{n+1}} |\psi_2|^{\frac{2\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_{M_2}^2}.$$

Comparing the coefficients of $|x|^{2\alpha - \frac{2n\beta}{n+1}} |y|^{\frac{2\beta}{n+1}}$ each side, we get

$$\begin{aligned} \frac{\|\phi_{00}\|_{M_1}^2}{\|\phi_{\alpha\beta}\|_{M_1}^2} &= \frac{\|\phi_{00}\|_{M_2}^2}{\|\phi_{n\alpha - (n-1)\beta, (n+1)\alpha - n\beta}\|_{M_2}^2} \cdot \frac{\|\phi_{00}\|_{M_1}^{2\alpha - \frac{2n\beta}{n+1}}}{\|\phi_{10}\|_{M_1}^{2\alpha - \frac{2n\beta}{n+1}}} \cdot \frac{\|\phi_{n,n+1}\|_{M_2}^{2\alpha - \frac{2n\beta}{n+1}}}{\|\phi_{00}\|_{M_2}^{2\alpha - \frac{2n\beta}{n+1}}} \\ &= \frac{\|\phi_{00}\|_{M_1}^{\frac{2\beta}{n+1}}}{\|\phi_{n,n+1}\|_{M_1}^{\frac{2\beta}{n+1}}} \cdot \frac{\|\phi_{10}\|_{M_2}^{\frac{2\beta}{n+1}}}{\|\phi_{00}\|_{M_2}^{\frac{2\beta}{n+1}}}. \end{aligned}$$

Simplifying this equation, we get

$$\frac{\|\phi_{10}\|_{M_1}^{\alpha - \frac{n}{n+1}\beta} \|\phi_{n,n+1}\|_{M_1}^{\frac{\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_{M_1} \|\phi_{00}\|_{M_1}^{\alpha - \frac{n-1}{n+1}\beta - 1}} = \frac{\|\phi_{10}\|_{M_2}^{\frac{\beta}{n+1}} \|\phi_{n,n+1}\|_{M_2}^{\alpha - \frac{n}{n+1}\beta}}{\|\phi_{n\alpha - (n-1)\beta, (n+1)\alpha - n\beta}\|_{M_2} \|\phi_{00}\|_{M_2}^{\alpha - \frac{n-1}{n+1}\beta - 1}},$$

where $\alpha \geq \frac{n}{n+1}\beta$.

Q. E. D.

Theorem 2.2.3.3. (Theorem D)

Proof of Theorem D:

(\Leftarrow) In view of Theorem 2.1.0.16, there exists a biholomorphic map $\Phi(z_1, z_2) = (az_1, bz_2)$ or (az_2, bz_1) . Observe that the fiber of the quotient map π is of the form

$$\{(z_1, z_2), (\delta z_1, \delta^n z_2), (\delta^2 z_1, \delta^{2n} z_2), \dots, (\delta^n z_1, \delta^{n^2} z_2)\},$$

where δ is a primitive $(n + 1)$ -th root of unit. And Φ sends one fiber to another fiber. Hence Φ descends to a biholomorphic map $\Psi : V_1 \rightarrow V_2$ given by

$$\Psi(x, y, z) = (a^{n+1}x, b^{n+1}y, abz)$$

or

$$\Psi(x, y, z) = (a^{n+1}y, b^{n+1}x, abz).$$

(\Rightarrow) Suppose Ψ is a biholomorphic map from V_1 to V_2 . Observe that W_1 is a simply connected domain in \mathbb{C}^2 . It follows that $W_1 \setminus \{0\}$ is also simply connected. Observe also that $\pi_i : W_i \setminus \{0\} \rightarrow V_i \setminus \{0\}$, $i = 1, 2$, are $n + 1$ -fold covering maps. The holomorphic map $\Psi \circ \pi_1 : W_1 \setminus \{0\} \rightarrow V_2 \setminus \{0\}$ can be uniquely lifted to a holomorphic map $\Phi : W_1 \setminus \{0\} \rightarrow W_2 \setminus \{0\}$.

Similarly the holomorphic map $\Psi^{-1} \circ \pi_2 : W_2 \setminus \{0\} \rightarrow V_1 \setminus \{0\}$ can be uniquely lifted to a holomorphic map $\tilde{\Phi} : W_2 \setminus \{0\} \rightarrow W_1 \setminus \{0\}$. Thus we have the following commutative diagram.

$$\begin{array}{ccccc}
W_1 \setminus \{0\} & \xrightarrow{\Phi} & W_2 \setminus \{0\} & \xrightarrow{\tilde{\Phi}} & W_1 \setminus \{0\} \\
\downarrow \pi_1 & & \downarrow \pi_2 & & \downarrow \pi_1 \\
V_1 \setminus \{0\} & \xrightarrow{\Psi} & V_2 \setminus \{0\} & \xrightarrow{\Psi^{-1}} & V_1 \setminus \{0\}
\end{array}$$

By the unique lifting property, we have $\tilde{\Phi} \circ \Phi = 1_{W_1 \setminus \{0\}}$ and $\Phi \circ \tilde{\Phi} = 1_{W_2 \setminus \{0\}}$. By Hartog theorem, Φ extends to a biholomorphism from W_1 to W_2 . Q. E. D.

Corollary 2.2.3.4. *Let $\pi : \mathbb{C}^2 \rightarrow \tilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ with $\pi(z_1, z_2) = (z_1^{n+1}, z_2^{n+1}, z_1 z_2)$. Let $\mathcal{V} = \{V : V \text{ a bounded complete Reinhardt domain in } A_n\text{-variety}\}$ and $\mathcal{W} = \{W = \pi^{-1}(V) : V \in \mathcal{V}\}$. Then the moduli space of \mathcal{V} is equal to the moduli space of \mathcal{W} .*

Proof: It is a direct consequence of the Theorem 2.2.3.3. Q. E. D.

Using Lemma 2.2.3.1, 2.2.3.2 and Theorem 2.2.3.3, we can get a lot of biholomorphic invariants. In order to simplify the notation, we let

$$g^{(\alpha, \beta)} = \frac{\|\phi_{10}\|^{\alpha - \frac{n}{n+1}\beta} \|\phi_{n, n+1}\|^{\frac{\beta}{n+1}}}{\|\phi_{\alpha\beta}\| \|\phi_{00}\|^{\alpha - \frac{n-1}{n+1}\beta - 1}}.$$

Theorem 2.2.3.5. (Theorem A)

Proof of Theorem A:

According to the Theorem 2.2.3.3, there is a biholomorphism Ψ either of the form in the Lemma 2.2.3.1 or in the Lemma 2.2.3.2. So either

case 1)

$$g_{M_1}^{(\alpha, \beta)} = g_{M_2}^{(\alpha, \beta)}$$

for any α and β satisfying $\alpha \geq \frac{n}{n+1}\beta, \alpha \geq 1, \beta \geq 0$, or

case 2)

$$g_{M_1}^{(\alpha, \beta)} = g_{M_2}^{(n\alpha - (n-1)\beta, (n+1)\alpha - n\beta)}$$

for any α and β satisfying $\alpha \geq \frac{n}{n+1}\beta, \alpha \geq 1, \beta \geq 0$.

For the case 1), we have $g_{M_1}^{(\alpha, \beta)} = g_{M_2}^{(\alpha, \beta)}$ and $g_{M_1}^{(n\alpha - (n-1)\beta, (n+1)\alpha - n\beta)} = g_{M_2}^{(n\alpha - (n-1)\beta, (n+1)\alpha - n\beta)}$

for any α and β satisfying

$$\alpha \geq \frac{n}{n+1}\beta, \alpha \geq 1, \beta \geq 0.$$

So it is easy to see that

$$\xi_{V_1}^{(\alpha, \beta)} = \xi_{V_2}^{(\alpha, \beta)}, \zeta_{V_1}^{(\alpha, \beta)} = \zeta_{V_2}^{(\alpha, \beta)}, \eta_{V_1}^{(\alpha, p, q)} = \eta_{V_2}^{(\alpha, p, q)},$$

$$\omega_{V_1}^{(\alpha_1, \alpha_2, p_1, p_2)} = \omega_{V_2}^{(\alpha_1, \alpha_2, p_1, p_2)}.$$

For the case 2), we have

$$g_{M_1}^{(\alpha, \beta)} = g_{M_2}^{(n\alpha - (n-1)\beta, (n+1)\alpha - n\beta)}, g_{M_1}^{(n\alpha - (n-1)\beta, (n+1)\alpha - n\beta)} = g_{M_2}^{(\alpha, \beta)}$$

for any α and β satisfying

$$\alpha \geq \frac{n}{n+1}\beta, \alpha \geq 1, \beta \geq 0.$$

So it is easy to see

$$\xi_{V_1}^{(\alpha, \beta)} = \xi_{V_2}^{(\alpha, \beta)}, \zeta_{V_1}^{(\alpha, \beta)} = \zeta_{V_2}^{(\alpha, \beta)}.$$

Moreover, we notice that

$$\begin{aligned} g_{M_1}^{(\alpha, p)} &= g_{M_2}^{(n\alpha - (n-1)p, (n+1)\alpha - np)}, & g_{M_1}^{(n\alpha - (n-1)p, (n+1)\alpha - np)} &= g_{M_2}^{(\alpha, p)}, \\ g_{M_1}^{(\alpha, q)} &= g_{M_2}^{(n\alpha - (n-1)q, (n+1)\alpha - nq)}, & g_{M_1}^{(n\alpha - (n-1)q, (n+1)\alpha - nq)} &= g_{M_2}^{(\alpha, q)}, \end{aligned}$$

for

$$\alpha \geq 1, 0 \leq p, q \leq \left\lceil \frac{n+1}{n}\alpha \right\rceil, p \neq q,$$

So

$$\eta_{V_1}^{(\alpha, p, q)} = \eta_{V_2}^{(\alpha, p, q)}.$$

For the same reason,

$$g_{M_1}^{(\alpha_1, p_1)} = g_{M_2}^{(n\alpha_1 - (n-1)p_1, (n+1)\alpha_1 - np_1)},$$

$$g_{M_1}^{(n\alpha_1 - (n-1)p_1, (n+1)\alpha_1 - np_1)} = g_{M_2}^{(\alpha_1, p_1)},$$

$$g_{M_1}^{(\alpha_2, p_2)} = g_{M_2}^{(n\alpha_2 - (n-1)p_2, (n+1)\alpha_2 - np_2)},$$

$$g_{M_1}^{(n\alpha_2 - (n-1)p_2, (n+1)\alpha_2 - np_2)} = g_{M_2}^{(\alpha_2, p_2)},$$

where

$$0 \leq p_i \leq \left[\frac{n+1}{n} \alpha_i \right], \alpha_i \geq 1, \alpha_1 \neq \alpha_2, i = 1, 2,$$

imply

$$\omega_{V_1}^{(\alpha_1, \alpha_2, p_1, p_2)} = \omega_{V_2}^{(\alpha_1, \alpha_2, p_1, p_2)}.$$

Therefore $\xi^{(\alpha, \beta)}$, $\zeta^{(\alpha, \beta)}$, $\eta^{(\alpha, p, q)}$ and $\omega^{(\alpha_1, \alpha_2, p_1, p_2)}$ are all invariants.

Q. E. D.

Corollary 2.2.3.6. *Let V_i , $i = 1, 2$, be two bounded complete Reinhardt domains in A_n -variety*

$\widetilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$. *If V_1 is a biholomorphic to V_2 , then*

$$\nu^{(\alpha, \beta)} := \frac{\|\phi_{\alpha\alpha}\|^2}{\|\phi_{\alpha\beta}\| \cdot \|\phi_{n\alpha - (n-1)\beta, (n+1)\alpha - n\beta}\|} \quad \alpha \geq 1, \alpha \geq \frac{n}{n+1}\beta$$

are all invariants, i.e.

$$\nu_{V_1}^{(\alpha, \beta)} = \nu_{V_2}^{(\alpha, \beta)}$$

Proof: Since

$$\nu^{(\alpha, \beta)} = \frac{\sqrt{\xi^{(\alpha, \beta)} \cdot \xi^{(n\alpha - (n-1)\beta, (n+1)\alpha - n\beta)}}}{\sqrt{\xi^{(\alpha, \alpha)}}},$$

$\nu^{(\alpha, \beta)}$'s are all invariants.

Q. E. D.

Remark: The fundamental invariant ν_X mentioned in (22) is $\nu^{(1, 0)}$.

The following Corollary is an immediate consequence of Theorem 2.2.3.3 and Theorem 2.2.3.5.

Corollary 2.2.3.7. *Let $\pi : \mathbb{C}^2 \rightarrow \widetilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ with $\pi(z_1, z_2) = (z_1^{n+1}, z_2^{n+1}, z_1 z_2)$. Let $\mathcal{V} = \{V : V \text{ a bounded complete Reinhardt domain in } A_n\text{-variety } \widetilde{V}\}$ and $\mathcal{W} = \{W : W = \pi^{-1}(V) : V \in \mathcal{V}\}$. Then*

$$\xi^{(\alpha, \beta)}, \zeta^{(\alpha, \beta)}, \eta^{(\alpha, p, q)} \omega^{(\alpha_1, \alpha_2, p_1, p_2)},$$

$$\alpha \geq 1, \alpha \geq \frac{n}{n+1}\beta, 0 \leq p, q \leq \left\lfloor \frac{n+1}{n}\alpha \right\rfloor, p \neq q,$$

$$0 \leq p_i \leq \left\lfloor \frac{n+1}{n}\alpha_i \right\rfloor, \alpha_i \geq 1, \alpha_1 \neq \alpha_2, i = 1, 2,$$

are holomorphic invariants of \mathcal{W} .

Q. E. D.

We have seen that by the Theorem 2.2.3.5, we can get a lot of invariants from Bergman functions. However it is natural to ask whether these invariants are sufficient to recover the Bergman function up to automorphism of A_n -variety. The answer is positive. For proving this, we need the following lemma.

Lemma 2.2.3.8. *If $x_i, x'_i \in \mathbb{R}$, where $1 \leq i \leq n, i, 2 \leq n \in \mathbb{N}$ satisfying*

$$x_j x_{n-j+1} = x'_j x'_{n-j+1}, \quad (2.2.50)$$

$$x_j + x_{n-j+1} = x'_j + x'_{n-j+1}, \quad (2.2.51)$$

$$(x_k - x_{n-k+1})(x_l - x_{n-l+1}) = (x'_k - x'_{n-k+1})(x'_l - x'_{n-l+1}), \quad (2.2.52)$$

for all $1 \leq j, k, l \leq n$, $j, k, l \in \mathbb{N}$, then $x_i = x'_i$ for all $1 \leq i \leq n$ or $x_i = x'_{n-i+1}$ for all $1 \leq i \leq n$. Q. E. D.

Theorem 2.2.3.9. *Let V_i , $i = 1, 2$, be two bounded complete Reinhardt domains in A_n -variety $\widetilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ and M_i is a resolution of V_i . If*

$$\xi_{V_1}^{(\alpha, \beta)} = \xi_{V_2}^{(\alpha, \beta)}, \zeta_{V_1}^{(\alpha, \beta)} = \zeta_{V_2}^{(\alpha, \beta)}, \eta_{V_1}^{(\alpha, p, q)} = \eta_{V_2}^{(\alpha, p, q)},$$

$$\omega_{V_1}^{(\alpha_1, \alpha_2, p_1, p_2)} = \omega_{V_2}^{(\alpha_1, \alpha_2, p_1, p_2)},$$

where

$$\alpha \geq 1, \alpha \geq \frac{n}{n+1}\beta, 0 \leq p, q \leq \left[\frac{n+1}{n}\alpha \right], p \neq q,$$

$$0 \leq p_i \leq \left[\frac{n+1}{n}\alpha_i \right], \alpha_i \geq 1, \alpha_1 \neq \alpha_2, i = 1, 2,$$

then there exists an automorphism $\Psi = (\psi_1, \psi_2, \psi_3)$ of A_n -variety $\widetilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ given by

$$(\psi_1, \psi_2, \psi_3) =$$

$$\left(\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}} x, \frac{\|\phi_{n, n+1}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{n, n+1}\|_{M_1}} y, \frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}} z \right),$$

or

$$(\psi_1, \psi_2, \psi_3) =$$

$$\left(\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{n, n+1}\|_{M_1}} y, \frac{\|\phi_{n, n+1}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}} x, \frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}} z \right).$$

such that

$$B_{V_1}(x, y, z) = B_{V_2}(\Psi(x, y, z)).$$

Proof: $\xi_{V_1}^{(\alpha, \beta)} = \xi_{V_2}^{(\alpha, \beta)}$, $\zeta_{V_1}^{(\alpha, \beta)} = \zeta_{V_2}^{(\alpha, \beta)}$ and $\eta_{V_1}^{(\alpha, p, q)} = \eta_{V_2}^{(\alpha, p, q)}$, means

$$g_{M_1}^{(\alpha, \beta)} \cdot g_{M_1}^{(n\alpha - (n-1)\beta, (n+1)\alpha - n\beta)} = g_{M_2}^{(\alpha, \beta)} \cdot g_{M_2}^{(n\alpha - (n-1)\beta, (n+1)\alpha - n\beta)},$$

$$g_{M_1}^{(\alpha, \beta)} + g_{M_1}^{(n\alpha - (n-1)\beta, (n+1)\alpha - n\beta)} = g_{M_2}^{(\alpha, \beta)} + g_{M_2}^{(n\alpha - (n-1)\beta, (n+1)\alpha - n\beta)}$$

and

$$\begin{aligned} & (g_{M_1}^{(\alpha, p)} - g_{M_1}^{(n\alpha - (n-1)p, (n+1)\alpha - np)}) \cdot (g_{M_1}^{(\alpha, q)} - g_{M_1}^{(n\alpha - (n-1)q, (n+1)\alpha - nq)}) \\ &= (g_{M_2}^{(\alpha, p)} - g_{M_2}^{(n\alpha - (n-1)p, (n+1)\alpha - np)}) \cdot (g_{M_2}^{(\alpha, q)} - g_{M_2}^{(n\alpha - (n-1)q, (n+1)\alpha - nq)}) \end{aligned}$$

where

$$\alpha \geq 1, \alpha \geq \frac{n}{n+1}\beta, 0 \leq p, q \leq \left\lceil \frac{n+1}{n}\alpha \right\rceil, p \neq q.$$

For fixed $\alpha \geq 1$ by Lemma 2.2.3.8, we can get

$$g_{M_1}^{(\alpha, \beta)} = g_{M_2}^{(\alpha, \beta)},$$

for any $\beta \geq 0$ satisfying $\alpha \geq \frac{n}{n+1}\beta$ or

$$g_{M_1}^{(\alpha, \beta)} = g_{M_2}^{(n\alpha - (n-1)\beta, (n+1)\alpha - n\beta)},$$

for any $\beta \geq 0$ satisfying $\alpha \geq \frac{n}{n+1}\beta$.

If for some different $\alpha_1, \alpha_2 \geq 1$, and p_1, p_2 satisfying $0 \leq p_i \leq \left\lceil \frac{n+1}{n}\alpha_i \right\rceil$, $i = 1, 2$, such that

$$g_{M_1}^{(\alpha_1, p_1)} = g_{M_2}^{(\alpha_1, p_1)},$$

$$g_{M_1}^{(n\alpha_1 - (n-1)p_1, (n+1)\alpha_1 - np_1)} = g_{M_2}^{(n\alpha_1 - (n-1)p_1, (n+1)\alpha_2 - np_1)},$$

$$g_{M_1}^{(\alpha_2, p_2)} = g_{M_2}^{(n\alpha_2 - (n-1)p_2, (n+1)\alpha_2 - np_2)},$$

$$g_{M_1}^{(n\alpha_2 - (n-1)p_2, (n+1)\alpha_2 - np_2)} = g_{M_2}^{(\alpha_2, p_2)},$$

then

$$\omega_{V_1}^{(\alpha_1, \alpha_2, p_1, p_2)} = \omega_{V_2}^{(\alpha_1, \alpha_2, p_1, p_2)}$$

forces that

$$g_{M_1}^{(\alpha_1, p_1)} = g_{M_1}^{(n\alpha_1 - (n-1)p_1, (n+1)\alpha_1 - np_1)},$$

$$g_{M_2}^{(\alpha_1, p_1)} = g_{M_2}^{(n\alpha_1 - (n-1)p_1, (n+1)\alpha_2 - np_1)},$$

or

$$g_{M_1}^{(\alpha_2, p_2)} = g_{M_1}^{(n\alpha_2 - (n-1)p_2, (n+1)\alpha_2 - np_2)},$$

$$g_{M_2}^{(n\alpha_2 - (n-1)p_2, (n+1)\alpha_2 - np_2)} = g_{M_2}^{(\alpha_2, p_2)}.$$

So we have

$$\begin{aligned} g_{M_1}^{(\alpha_1, p_1)} &= g_{M_1}^{(n\alpha_1 - (n-1)p_1, (n+1)\alpha_1 - np_1)} \\ &= g_{M_2}^{(\alpha_1, p_1)} = g_{M_2}^{(n\alpha_1 - (n-1)p_1, (n+1)\alpha_2 - np_1)} \end{aligned}$$

or

$$\begin{aligned} g_{M_2}^{(\alpha_2, p_2)} &= g_{M_2}^{(n\alpha_2 - (n-1)p_2, (n+1)\alpha_2 - np_2)} \\ &= g_{M_2}^{(\alpha_2, p_2)} = g_{M_2}^{(n\alpha_2 - (n-1)p_2, (n+1)\alpha_2 - np_2)} \end{aligned}$$

So we only have following two cases:

1.

$$g_{M_1}^{(\alpha, \beta)} = g_{M_2}^{(\alpha, \beta)},$$

for any

$$\alpha \geq 1, \beta \geq 0, \alpha \geq \frac{n}{n+1}\beta$$

or

2.

$$g_{M_1}^{(\alpha, \beta)} = g_{M_2}^{(n\alpha - (n-1)\beta, (n+1)\alpha - n\beta)},$$

for any

$$\alpha \geq 1, \beta \geq 0, \alpha \geq \frac{n}{n+1}\beta.$$

Therefore if for the first case, we take

$$(\psi_1, \psi_2, \psi_3) = \left(\frac{\|\phi_{10}\|_{M_2} \|\phi_{00}\|_{M_1}}{\|\phi_{00}\|_{M_2} \|\phi_{10}\|_{M_1}} x, \frac{\|\phi_{n,n+1}\|_{M_2} \|\phi_{00}\|_{M_1}}{\|\phi_{00}\|_{M_2} \|\phi_{n,n+1}\|_{M_1}} y, \frac{\|\phi_{11}\|_{M_2} \|\phi_{00}\|_{M_1}}{\|\phi_{00}\|_{M_2} \|\phi_{11}\|_{M_1}} z \right),$$

and if for the second case, we take

$$(\psi_1, \psi_2, \psi_3) = \left(\frac{\|\phi_{10}\|_{M_2} \|\phi_{00}\|_{M_1}}{\|\phi_{00}\|_{M_2} \|\phi_{n,n+1}\|_{M_1}} y, \frac{\|\phi_{n,n+1}\|_{M_2} \|\phi_{00}\|_{M_1}}{\|\phi_{00}\|_{M_2} \|\phi_{10}\|_{M_1}} x, \frac{\|\phi_{11}\|_{M_2} \|\phi_{00}\|_{M_1}}{\|\phi_{00}\|_{M_2} \|\phi_{11}\|_{M_1}} z \right).$$

Then we can always get

$$\begin{aligned} & \|\phi_{00}\|_{M_1}^2 \sum_{\substack{\alpha \geq \frac{n}{n+1}\beta \\ \alpha \geq 1}} \frac{|x|^{2\alpha - \frac{2n\beta}{n+1}} |y|^{\frac{2\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_{M_1}^2} \\ &= \|\phi_{00}\|_{M_2}^2 \sum_{\substack{\alpha \geq \frac{n}{n+1}\beta \\ \alpha \geq 1}} \frac{|\psi_1|^{2\alpha - \frac{2n\beta}{n+1}} |\psi_2|^{\frac{2\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_{M_2}^2} \end{aligned}$$

by comparing the coefficients of $|x|^{2\alpha - \frac{2n\beta}{n+1}} |y|^{\frac{2\beta}{n+1}}$, i.e. $\|\phi_{00}\|_{M_1}^2 \Theta_{V_1}^{(1)} = \|\phi_{00}\|_{M_2}^2 \Theta_{V_2}^{(1)}$. So $B_{V_1}^{(1)}(x, y, z) = B_{V_2}^{(1)}(\Psi(x, y, z)$, by Equation 2.2.3. Q. E. D.

Theorem 2.2.3.10. (Theorem B)

Proof of Theorem B:

Proof: By the Fornaess Lemma (See Lemma 2.2.3.12 below), there exists a dense set in the boundary of M_i such that the Bergman kernel blows up at the points in this dense set. It follows that the Bergman function B_{V_i} is equal to 1 in a dense subset of ∂V_i . Recall that B_{V_i} is zero at the origin and $0 < B_V < 1$ on $V \setminus \{(0, 0, 0)\}$. In view of Theorem 2.2.3.9, $B_{V_1}(x, y, z) = B_{V_2}(\Psi(x, y, z))$, we see immediately that Ψ preserves the level sets of Bergman functions and hence sends a dense subset of ∂V_1 to a dense subset of ∂V_2 . By continuity, Ψ sends ∂V_1 to ∂V_2 . Q. E. D.

Lemma 2.2.3.11. (Henkin (32), Ramirez (33)) *Let D be a bounded strictly pseudoconvex domain in \mathbb{C}^2 . Let p be a point in the boundary of D . Then there exists an L^2 holomorphic function on D which blows up only at p .*

Proof: There exists, (32) (33), a holomorphic function f defined on a neighborhood of \bar{D} such that $f(p) = 0$, $\Re f \leq 0$ on \bar{D} and moreover $|f(q) - f(p)| \geq |q - p|^2$ on \bar{D} . We can then set $F(z) = \frac{1}{f^{\frac{3}{4}}}$. Then

$$\int_D |F|^2 \leq \int_D \frac{1}{|f|^{\frac{3}{2}}} \leq \int_D \frac{1}{|z - p|^3} < \infty.$$

Q. E. D.

Remark. To find a function f as in the proof of the Lemma all we need is that p is a strictly pseudoconvex boundary point and that \bar{D} has a Stein neighborhood basis.

Lemma 2.2.3.12. (Fornaess) *Let D be a bounded complete Reinhardt pseudoconvex domain with real analytic boundary in \mathbb{C}^2 . Let $E = \{p \in \partial D; \exists g \in H^2(D) \text{ which blows up only at } p\}$. Then E is dense in the boundary of D , and the Bergman kernel of D blows up at points in E .*

Proof: We note that since ∂D has a real analytic boundary it follows that strictly pseudoconvex boundary points are dense. Moreover \bar{D} has a Stein neighborhood basis (34). Therefore the lemma follows from the remark and the previous lemma. Q. E. D.

From Theorem 2.2.3.10 and Corollary 2.2.3.4 we can get the following two corollaries easily.

Corollary 2.2.3.13. *The moduli space of bounded complete Reinhardt strictly pseudoconvex (respectively C^ω -smooth pseudoconvex) domains in A_n -variety $\widetilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ is given by the image of the map $\Phi : \{V : V \text{ a bounded complete Reinhardt strictly pseudoconvex (respectively } C^\omega\text{-smooth pseudoconvex) domain in } \widetilde{V}_n\} \rightarrow \mathbb{R}^\infty$, where the component function of Φ are the invariant functions*

$$\xi^{(\alpha, \beta)}, \zeta^{(\alpha, \beta)}, \eta^{(\alpha, p, q)}, \omega^{(\alpha_1, \alpha_2, p_1, p_2)},$$

$$\alpha \geq 1, \alpha \geq \frac{n}{n+1}\beta, 0 \leq p, q \leq \left[\frac{n+1}{n}\alpha \right], p \neq q,$$

$$0 \leq p_i \leq \left[\frac{n+1}{n}\alpha_i \right], \alpha_i \geq 1, \alpha_1 \neq \alpha_2, i = 1, 2.$$

Q. E. D.

Corollary 2.2.3.14. *Let $\mathcal{W}_P = \{W : W = \pi^{-1}(V) \text{ where } V \text{ is a complete Reinhardt pseudoconvex } C^\omega\text{-smooth domain in } A_n\text{-variety}\}$ and $\mathcal{W}_{SP} = \{W : W = \pi^{-1}(V) \text{ where } V \text{ is a complete Reinhardt strictly pseudoconvex domain in } A_n\text{-variety}\}$. Then the moduli space of \mathcal{W}_P (respectively \mathcal{W}_{SP}) is given by the image of the map $\widetilde{\Phi}_P : \mathcal{W}_P \rightarrow \mathbb{R}^\infty$ (respectively $\widetilde{\Phi}_{SP} : \mathcal{W}_{SP} \rightarrow \mathbb{R}^\infty$), where the component functions of $\widetilde{\Phi}_P$ (respectively $\widetilde{\Phi}_{SP}$) are the invariant functions*

$$\xi^{(\alpha,\beta)}, \zeta^{(\alpha,\beta)}, \eta^{(\alpha,p,q)}, \omega^{(\alpha_1, \alpha_2, p_1, p_2)},$$

$$\alpha \geq 1, \alpha \geq \frac{n}{n+1}\beta, 0 \leq p, q \leq \left[\frac{n+1}{n}\alpha \right], p \neq q,$$

$$0 \leq p_i \leq \left[\frac{n+1}{n}\alpha_i \right], \alpha_i \geq 1, \alpha_1 \neq \alpha_2, i = 1, 2.$$

In particular, the moduli space of \mathcal{W}_P (respectively \mathcal{W}_{SP}) is as same as the moduli space of bounded complete Reinhardt pseudoconvex C^ω -smooth domains (respectively bounded complete Reinhardt strictly pseudoconvex domains) in A_n -variety $\widetilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$.

Q. E. D.

Remark. In fact in the view of Proposition 2.1.0.17, Theorem B, Theorem C, Theorem E and their corollaries are all true for weaker condition “ C^1 -bounding domains”. However, we noticed Proposition 2.1.0.17 after (1) was accepted. So we still keep the results as those in (1).

2.2.4 Explicit computation of new invariants

Let a, b, c be positive real number and d be an integer greater than or equal to 1. We shall follow the notations in our previous section. Let $V_{(a,b,c)}^{(d)} = \{(x, y, z) \in \mathbb{C}^3 : xy = z^2, a|x|^{2d} + b|y|^{2d} + c|z|^{2d} < \varepsilon_0\}$.

Recall that $(x, y, z) = (u_0, u_0 v_0^2, u_0 v_0)$. Then $M_{(a,b,c)}^{(d)}$ be the resolution of $V_{(a,b,c)}^{(d)}$ with local coordinate chart $W_0 = \{(u_0, v_0) : a|u_0|^{2d} + b|u_0|^{2d}|v_0|^{4d} + c|u_0|^{2d}|v_0|^{2d} < \varepsilon_0\}$. Next write $u_0 = r e^{i\theta}$, and $v_0 = \rho e^{i\phi}$.

In the following paragraphs, we denote

$$V_1 = V_{(a_1, b_1, c_1)}^{(d)}, V_2 = V_{(a_2, b_2, c_2)}^{(d)}, M_1 = M_{(a_1, b_1, c_1)}^{(d)}, M_2 = M_{(a_2, b_2, c_2)}^{(d)}.$$

First let us consider the case $d = 1$ and fix ε_0 .

$$\begin{aligned} \|\phi_{\alpha\beta}\|_{M_{(a,b,c)}}^2 &= \int_{M_{(a,b,c)}} \phi_{\alpha\beta} \wedge \overline{\phi_{\alpha\beta}} = \int_{W_0} |u_0^\alpha|^2 |v_0^\beta|^2 du_0 \wedge dv_0 \wedge \overline{du_0} \wedge \overline{dv_0} \\ &= 16\pi^2 \iint_D r^{2\alpha+1} \rho^{2\beta+1} dr d\rho \end{aligned}$$

where $D = \{(r, \rho) : r \geq 0, \rho \geq 0, ar^2 + br^2\rho^4 + cr^2\rho^2 < \varepsilon_0\}$. Then

$$\begin{aligned} \|\phi_{00}\|^2 &= 16\pi^2 \int_0^\infty \int_0^{\frac{\sqrt{\varepsilon_0}}{\sqrt{a+c\rho^2+b\rho^4}}} \sqrt{a+c\rho^2+b\rho^4} r \rho dr d\rho \\ &= 8\pi^2 \int_0^\infty \frac{\varepsilon_0 \rho}{a+c\rho^2+b\rho^4} d\rho = 4\varepsilon_0 \pi^2 \int_0^\infty \frac{1}{a+c\rho+b\rho^2} d\rho, \end{aligned}$$

$$\begin{aligned}
\|\phi_{1\beta}\|^2 &= 16\pi^2 \int_0^\infty \int \frac{\sqrt{\varepsilon_0}}{\sqrt{a+c\rho^2+b\rho^4}} r^3 \rho^{2\beta+1} dr d\rho \\
&= 4\varepsilon_0^2 \pi^2 \int_0^\infty \frac{\rho^{2\beta+1}}{(a+c\rho^2+b\rho^4)^2} d\rho = 2\varepsilon_0^2 \pi^2 \int_0^\infty \frac{\rho^\beta}{(a+c\rho+b\rho^2)^2} d\rho.
\end{aligned}$$

By calculation, we can get the following results:

If $c^2 - 4ab < 0$,

$$\|\phi_{00}\|^2 = 4\varepsilon_0 \pi^2 \left(\frac{\pi}{\sqrt{4ab-c^2}} - \frac{2}{\sqrt{4ab-c^2}} \arctan \frac{c}{\sqrt{4ab-c^2}} \right), \quad (2.2.53)$$

$$\|\phi_{10}\|^2 = 2\varepsilon_0 \pi^2 \left(-\frac{c}{(4ab-c^2)a} + \frac{2\pi b}{(4ab-c^2)^{\frac{3}{2}}} - \frac{4b}{(4ab-c^2)^{\frac{3}{2}}} \arctan \frac{c}{\sqrt{4ab-c^2}} \right), \quad (2.2.54)$$

$$\|\phi_{11}\|^2 = 2\varepsilon_0 \pi^2 \left(\frac{2}{4ab-c^2} - \frac{\pi c}{(4ab-c^2)^{\frac{3}{2}}} + \frac{2c}{(4ab-c^2)^{\frac{3}{2}}} \arctan \frac{c}{\sqrt{4ab-c^2}} \right), \quad (2.2.55)$$

$$\|\phi_{12}\|^2 = 2\varepsilon_0 \pi^2 \left(-\frac{c}{(4ab-c^2)b} + \frac{2\pi a}{(4ab-c^2)^{\frac{3}{2}}} - \frac{4a}{(4ab-c^2)^{\frac{3}{2}}} \arctan \frac{c}{\sqrt{4ab-c^2}} \right). \quad (2.2.56)$$

If $c^2 - 4ab > 0$,

$$\|\phi_{00}\|^2 = 4\varepsilon_0 \pi^2 \frac{1}{\sqrt{c^2-4ab}} \cdot \ln \frac{c + \sqrt{c^2-4ab}}{c - \sqrt{c^2-4ab}}, \quad (2.2.57)$$

$$\|\phi_{10}\|^2 = 2\varepsilon_0^2 \pi^2 \frac{c\sqrt{c^2-4ab} - 2ab \ln \frac{c + \sqrt{c^2-4ab}}{c - \sqrt{c^2-4ab}}}{(c^2-4ab)^{\frac{3}{2}} \cdot a}, \quad (2.2.58)$$

$$\|\phi_{11}\|^2 = 2\varepsilon_0^2 \pi^2 \frac{-2\sqrt{c^2-4ab} + c \ln \frac{c + \sqrt{c^2-4ab}}{c - \sqrt{c^2-4ab}}}{(c^2-4ab)^{\frac{3}{2}}}, \quad (2.2.59)$$

$$\|\phi_{12}\|^2 = 2\varepsilon_0^2 \pi^2 \frac{c\sqrt{c^2-4ab} - 2ab \ln \frac{c + \sqrt{c^2-4ab}}{c - \sqrt{c^2-4ab}}}{(c^2-4ab)^{\frac{3}{2}} b}. \quad (2.2.60)$$

If $c^2 - 4ab = 0$,

$$\|\phi_{00}\|^2 = 4\varepsilon_0\pi^2 \cdot \frac{2}{c}, \quad (2.2.61)$$

$$\|\phi_{10}\|^2 = 2\varepsilon_0^2\pi^2 \cdot \frac{8b}{3c^3}, \quad (2.2.62)$$

$$\|\phi_{11}\|^2 = 2\varepsilon_0^2\pi^2 \cdot \frac{2}{3c^2}, \quad (2.2.63)$$

$$\|\phi_{12}\|^2 = 2\varepsilon_0^2\pi^2 \cdot \frac{2}{3bc}. \quad (2.2.64)$$

Remark: $\|\phi_{12}\|^2 = \frac{a}{b}\|\phi_{10}\|^2$ for all the three cases above.

Lemma 2.2.4.1. *If $x > 0$, then $\arctan x > \frac{-x + x\sqrt{9 + 8x^2}}{2(1 + x^2)}$.*

Proof: Let

$$f(x) = \arctan x - \frac{-x + x\sqrt{9 + 8x^2}}{2(1 + x^2)}.$$

Then

$$f'(x) = \frac{(x^2 + 3)\sqrt{9 + 8x^2} - 7x^2 - 9}{2(1 + x^2)^2\sqrt{9 + 8x^2}}.$$

$$(x^2 + 3)\sqrt{9 + 8x^2} > 7x^2 + 9 \iff x^6 + x^4 > 0$$

So $f'(x) > 0$, $f(x) > f(0) = 0$, i.e., $\arctan x > \frac{-x + x\sqrt{9 + 8x^2}}{2(1 + x^2)}$

Q. E. D.

Lemma 2.2.4.2. *If $0 < x < 1$, $\ln \frac{1+x}{1-x} > \frac{-x + x\sqrt{9 - 8x^2}}{1 - x^2}$*

Proof: Let

$$f(x) = \ln \frac{1+x}{1-x} - \frac{-x + x\sqrt{9-8x^2}}{1-x^2}.$$

Then

$$f'(x) = \frac{(3-x^2)\sqrt{9-8x^2} - (9-7x^2)}{\sqrt{9-8x^2}(1-x^2)^2}.$$

$$(3-x^2)\sqrt{9-8x^2} > 9-7x^2 \iff x^4 + x^6 > 0.$$

So $f'(x) > 0$, $f(x) > f(0) = 0$, i.e., $\ln \frac{1+x}{1-x} > \frac{-x + x\sqrt{9-8x^2}}{1-x^2}$.

Q. E. D.

Proposition 2.2.4.3. *If $c^2 - 4ab < 0$, let $\frac{\sqrt{4ab-c^2}}{c} = x$, then*

$$f(x) = \frac{\|\phi_{11}\|^2}{\|\phi_{10}\|\|\phi_{12}\|} = \frac{x - \arctan x}{\sqrt{1+x^2} \left(-\frac{x}{1+x^2} + \arctan x \right)}$$

which is a strict increasing function in terms of x . In particular $\frac{1}{2} < \nu_V^{(1,0)} < \frac{2}{\pi}$.

Proof:

$$\frac{\|\phi_{11}\|^2}{\|\phi_{10}\|\|\phi_{12}\|} = \frac{\frac{2}{4ab-c^2} - \frac{\pi c}{(4ab-c^2)^{\frac{3}{2}}} + \frac{2c}{(4ab-c^2)^{\frac{3}{2}}} \arctan \frac{c}{\sqrt{4ab-c^2}}}{\sqrt{\frac{a}{b}} \left(-\frac{c}{(4ab-c^2)a} + \frac{2\pi \cdot b}{(4ab-c^2)^{\frac{3}{2}}} - \frac{4b}{(4ab-c^2)^{\frac{3}{2}}} \arctan \frac{c}{\sqrt{4ab-c^2}} \right)}.$$

Notice that $\frac{ab}{c^2} = \frac{x^2 + 1}{4}$, so by calculation,

$$f(x) = \frac{x - \arctan x}{\sqrt{1+x^2} \left(-\frac{x}{1+x^2} + \arctan x \right)}$$

and

$$f'(x) = \frac{x(-2x^2 + x \arctan x + (x^2 + 1) \arctan^2 x)}{\sqrt{1-x^2} (x - (1+x^2) \cdot \arctan x)^2}.$$

By Lemma 2.2.4.1,

$$\begin{aligned} -2x^2 + x \arctan x + (x^2 + 1) \arctan^2 x &> \\ -2x^2 + \frac{-x^2 + x^2 \sqrt{9+8x^2}}{2(1+x^2)} + \frac{(-x + x \sqrt{9+8x^2})^2}{4(1+x^2)} &= 0. \end{aligned}$$

So $f'(x) > 0$.

In particular,

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{x - \arctan x}{\sqrt{1+x^2} \left(-\frac{x}{1+x^2} + \arctan x \right)} = \frac{1}{2}.$$

So $\nu^{(1,0)} > \frac{1}{2}$. Observe that $\lim_{x \rightarrow \infty} f(x) = \frac{2}{\pi}$, so $\nu^{(1,0)} < \frac{2}{\pi}$.

Q. E. D.

Proposition 2.2.4.4. *If $c^2 - 4ab > 0$, let $\frac{\sqrt{c^2 - 4ab}}{c} = x$, then*

$$f(x) = \frac{\|\phi_{11}\|^2}{\|\phi_{10}\|\|\phi_{12}\|} = \frac{-2x + \ln \frac{1+x}{1-x}}{\sqrt{1-x^2} \left(\frac{2x}{1-x^2} - \ln \frac{1+x}{1-x} \right)}$$

which is a strictly decreasing function in term of x . In particular, $0 < \nu^{(1,0)} < \frac{1}{2}$.

Proof:

$$\frac{\|\phi_{11}\|^2}{\|\phi_{10}\|\|\phi_{12}\|} = \frac{\left(-2\sqrt{c^2 - 4ab} + c \ln \frac{c + \sqrt{c^2 - 4ab}}{c - \sqrt{c^2 - 4ab}} \right) / (c^2 - 4ab)^{\frac{3}{2}}}{\sqrt{\frac{a}{b}} \left(c\sqrt{c^2 - 4ab} - 2ab \ln \frac{c + \sqrt{c^2 - 4ab}}{c - \sqrt{c^2 - 4ab}} \right) / a(c^2 - 4ab)^{\frac{3}{2}}}.$$

Notice that $\frac{ab}{c^2} = \frac{1-x^2}{4}$, so by calculation

$$f(x) = \frac{-2x + \ln \frac{1+x}{1-x}}{\sqrt{1-x^2} \left(\frac{2x}{1-x^2} - \ln \frac{1+x}{1-x} \right)}$$

and

$$f'(x) = \frac{x \left(8x^2 - 2x \ln \frac{1+x}{1-x} + (x^2 - 1) \ln^2 \frac{1+x}{1-x} \right)}{\sqrt{1-x^2} \left(2x + (x^2 - 1) \ln \frac{1+x}{1-x} \right)^2}.$$

By Lemma 2.2.4.2,

$$(1-x^2) \ln^2 \frac{1+x}{1-x} + 2x \ln \frac{1+x}{1-x} - 8x^2 > \frac{(-x + x\sqrt{9-8x^2})^2}{1-x^2} + 2x \frac{(-x + x\sqrt{9-8x^2})}{1-x^2} - 8x^2 = 0.$$

Thus $f'(x) < 0$.

In particular,

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{-x + \ln \frac{1+x}{1-x}}{\frac{2x}{\sqrt{1-x^2}} - \sqrt{1-x^2} \ln \frac{1+x}{1-x}} = \frac{1}{2}.$$

So $\nu^{(1,0)} < \frac{1}{2}$. Observe that $\lim_{x \rightarrow 1^-} f(x) = 0$. So $\nu^{(1,0)} > 0$.

Q. E. D.

Proposition 2.2.4.5. *Let*

$$V_i = \{(x, y, z) \in \mathbb{C}^3 : xy = z^2, a_i|x|^2 + b_i|y|^2 + c_i|z|^2 < \varepsilon_0\},$$

$i = 1, 2$. Then V_1 is biholomorphic to V_2 if and only if $\frac{a_1 b_1}{c_1^2} = \frac{a_2 b_2}{c_2^2}$.

Proof: (\Rightarrow) Since $\nu^{(1,0)}$ is a biholomorphic invariant, we can get the result from the Proposition 2.2.4.3 and 2.2.4.4.

$$(\Leftarrow) \text{ Let } \Psi = \left(\sqrt{\frac{b_2}{b_1}} \cdot \frac{c_1}{c_2} x, \sqrt{\frac{a_2}{a_1}} \cdot \frac{c_1}{c_2} y, \sqrt{\frac{c_1}{c_2}} z \right).$$

Then

$$\begin{aligned} a_2 |\psi_1|^2 + b_2 |\psi_2|^2 + c_2 |\psi_3|^2 &= a_2 \cdot \frac{b_2 c_1^2}{b_1 c_2^2} |x|^2 + b_2 \frac{a_2 c_1^2}{a_1 c_2^2} |y|^2 + c_2 \cdot \frac{c_1}{c_2} |z|^2 \\ &= a_1 |x|^2 + b_1 |y|^2 + c_1 |z|^2, \end{aligned}$$

i.e. Ψ maps the boundary of V_1 to the boundary of V_2 . So ψ is a biholomorphic map from V_1 to V_2 .

Q. E. D.

Theorem 2.2.4.6. *Let*

$$V_{(a,b,c)}^{(1)} = \{(x, y, z) : xy = z^2, a|x|^2 + b|y|^2 + c|z|^2 < \varepsilon_0\}.$$

Let \sim denote the biholomorphic equivalence. Then the map

$$\varphi: \{V_{(a,b,c)}^{(1)}\} \rightarrow \mathbb{R}_+, V_{(a,b,c)}^{(1)} \mapsto \nu^{(1,0)}$$

is injective up to a biholomorphism. More precisely the induced map $\tilde{\varphi}: \{V_{(a,b,c)}^{(1)}\}/\sim \rightarrow \mathbb{R}_+$ is one-to-one map from $\{V_{(a,b,c)}^{(1)}\}/\sim$ onto $\left(0, \frac{2}{\pi}\right)$.

Proof: The theorem follows from Proposition 2.2.4.3, 2.2.4.4 and 2.2.4.5 directly.

Q. E. D.

In view of Proposition 2.2.4.5, Theorem 2.2.2.6 and Theorem 2.2.2.8, we can simplify the forms of biholomorphic map from V_1 to V_2 .

Theorem 2.2.4.7. *Let*

$$V_i = \{(x, y, z) \in \mathbb{C}^3 : xy = z^2, a_i|x|^2 + b_i|y|^2 + c_i|z|^2 < \varepsilon_0\},$$

$i = 1, 2$.

If $\nu_{V_1}^{(1,0)} \neq \frac{1}{2}$ or $\nu_{V_2}^{(1,0)} \neq \frac{1}{2}$, then the biholomorphism Ψ from V_1 to V_2 must be one of the following forms:

form (1):

$$\Psi = \left(e^{i\theta_1} \sqrt{\frac{b_2}{b_1}} \cdot \frac{c_1}{c_2} x, e^{i\theta_2} \sqrt{\frac{a_2}{a_1}} \cdot \frac{c_1}{c_2} y, \pm e^{i\frac{\theta_1+\theta_2}{2}} \sqrt{\frac{c_1}{c_2}} z \right),$$

form (2):

$$\Psi = \left(e^{i\theta_1} \sqrt{\frac{b_1}{a_2}} y, e^{i\theta_2} \sqrt{\frac{a_1}{b_2}} x, \pm e^{i\frac{\theta_1+\theta_2}{2}} \sqrt{\frac{c_1}{c_2}} z \right).$$

If $\nu_{V_1}^{(1,0)} = \nu_{V_2}^{(1,0)} = \frac{1}{2}$, then the biholomorphism Ψ from V_1 to V_2 must be one of the following

forms:

form (1):

$$\Psi = \left(e^{i\theta_1} \sqrt{\frac{b_2}{b_1}} \cdot \frac{c_1}{c_2} x, e^{i\theta_2} \sqrt{\frac{b_1}{b_2}} y, \pm e^{i\frac{\theta_1+\theta_2}{2}} \sqrt{\frac{c_1}{c_2}} z \right),$$

form (2):

$$\Psi = \left(e^{i\theta_1} \frac{2\sqrt{b_1 b_2}}{c_2} y, e^{i\theta_2} \frac{c_1}{2\sqrt{b_1 b_2}} x, \pm e^{i\frac{\theta_1+\theta_2}{2}} \sqrt{\frac{c_1}{c_2}} z \right),$$

and form (3):

$\Psi =$

$$\begin{pmatrix} \frac{-2c_1 b_2 \sqrt{b_2} r^2}{\sqrt{b_1} c_2 (2b_2 r^2 + c_2)} e^{i(\theta_{31}-\theta)}, & \frac{2\sqrt{b_1 b_2}}{2b_2 r^2 + c_2} e^{i(\theta_{32}-\theta)}, & \pm \frac{4\sqrt{c_1} b_2 r}{\sqrt{c_2} (2b_2 r^2 + c_2)} e^{i\theta_{13}} \\ -\frac{c_1 c_2}{2\sqrt{b_1 b_2} (2b_2 r^2 + c_2)} e^{i(\theta_{31}+\theta)}, & \frac{2\sqrt{b_1 b_2} r^2}{2b_2 r^2 + c_2} e^{i(\theta_{32}+\theta)}, & \pm \frac{-2\sqrt{c_1 c_2} r}{2b_2 r^2 + c_2} e^{i(\theta_{13}+2\theta)} \\ \frac{c_1 \sqrt{b_2} r}{\sqrt{b_1} (2b_2 r^2 + c_2)} e^{i\theta_{31}}, & \frac{2\sqrt{b_1 b_2} r}{2b_2 r^2 + c_2} e^{i\theta_{32}}, & \pm \frac{\sqrt{c_1} (2b_2 r^2 - c_2)}{\sqrt{c_2} (2b_2 r^2 + c_2)} e^{i(\theta_{13}+\theta)} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Proof: Since $\frac{a_1 b_1}{c_1^2} = \frac{a_2 b_2}{c_2^2}$, $\frac{\sqrt{4a_1 b_1 - c_1^2}}{c_1^2} = \frac{\sqrt{4a_2 b_2 - c_2^2}}{c_2^2}$. If $c_1^2 - 4a_1 b_1 < 0$, then $c_2^2 - 4a_2 b_2 <$

0.

Using Equation 2.2.53, Equation 2.2.54, Equation 2.2.55 and Equation 2.2.56, we can get

$$\begin{aligned}
\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \cdot \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}} &= \frac{\|\phi_{10}\|_{M_2} \cdot \sqrt{\frac{c_2^3}{b_2}}}{\|\phi_{00}\|_{M_2} \cdot \sqrt{c_2}} \cdot \frac{\|\phi_{00}\|_{M_1} \cdot \sqrt{c_1}}{\|\phi_{10}\|_{M_1} \cdot \sqrt{\frac{c_1^3}{b_1}}} \cdot \sqrt{\frac{b_2}{b_1}} \cdot \frac{c_1}{c_2} \\
&= \sqrt{\frac{b_2}{b_1}} \cdot \frac{c_1}{c_2} \\
\frac{\|\phi_{12}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \cdot \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{12}\|_{M_1}} &= \frac{\|\phi_{12}\|_{M_2} \cdot \sqrt{\frac{c_2^3}{a_2}}}{\|\phi_{00}\|_{M_2} \cdot \sqrt{c_2}} \cdot \frac{\|\phi_{00}\|_{M_1} \cdot \sqrt{c_1}}{\|\phi_{12}\|_{M_1} \cdot \sqrt{\frac{c_1^3}{a_1}}} \cdot \sqrt{\frac{a_2}{a_1}} \cdot \frac{c_1}{c_2} \\
&= \sqrt{\frac{a_2}{a_1}} \cdot \frac{c_1}{c_2} \\
\frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \cdot \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}} &= \frac{\|\phi_{11}\|_{M_2} \cdot c_2}{\|\phi_{00}\|_{M_2} \cdot \sqrt{c_2}} \cdot \frac{\|\phi_{00}\|_{M_1} \cdot \sqrt{c_1}}{\|\phi_{11}\|_{M_1} \cdot c_1} \cdot \sqrt{\frac{c_1}{c_2}} \\
&= \sqrt{\frac{c_1}{c_2}} \\
\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \cdot \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{12}\|_{M_1}} &= \frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \cdot \frac{\|\phi_{00}\|_{M_1}}{\sqrt{\frac{a_1}{b_1}} \|\phi_{10}\|_{M_1}} \\
&= \sqrt{\frac{b_2}{b_1}} \cdot \frac{c_1}{c_2} \cdot \sqrt{\frac{b_1}{a_1}} = \sqrt{\frac{b_2}{a_1}} \cdot \frac{c_1}{c_2} = \sqrt{\frac{b_1}{a_2}} \\
\frac{\|\phi_{12}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \cdot \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}} &= \frac{\sqrt{\frac{a_2}{b_2}} \|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}} \cdot \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}} \\
&= \sqrt{\frac{a_2}{b_2}} \cdot \sqrt{\frac{b_2}{b_1}} \cdot \frac{c_1}{c_2} = \sqrt{\frac{a_2}{b_1}} \cdot \frac{c_1}{c_2} = \sqrt{\frac{a_1}{b_2}}
\end{aligned}$$

If $c_1^2 - 4a_1b_1 > 0$, then $c_2^2 - 4a_2b_2 > 0$.

Using Equation 2.2.57, Equation 2.2.58, Equation 2.2.59 and Equation 2.2.60, we can get the same results.

So form (1):

$$\Psi = \left(e^{i\theta_1} \sqrt{\frac{b_2}{b_1}} \cdot \frac{c_1}{c_2} x, e^{i\theta_2} \sqrt{\frac{a_2}{a_1}} \cdot \frac{c_1}{c_2} y, \pm e^{i\frac{\theta_1+\theta_2}{2}} \sqrt{\frac{c_1}{c_2}} z \right), \quad (2.2.65)$$

and form (2):

$$\Psi = \left(e^{i\theta_1} \sqrt{\frac{b_1}{a_2}} y, e^{i\theta_2} \sqrt{\frac{a_1}{b_2}} x, \pm e^{i\frac{\theta_1+\theta_2}{2}} \sqrt{\frac{c_1}{c_2}} z \right), \quad (2.2.66)$$

If $c_1^2 - 4a_1b_1 = 0$, then $c_2^2 - 4a_2b_2 = 0$. Using Equation 2.2.61, Equation 2.2.62, Equation 2.2.63 and Equation 2.2.64, we can get the same result. But in this case $\nu_{V_1}^{(1,0)} = \nu_{V_2}^{(1,0)} = \frac{1}{2}$, the form (3) will appear.

Form (3)

$$\Psi = \begin{pmatrix} \alpha r e^{-i\theta} a_{31} & \frac{1}{r} e^{-i\theta} a_{32} & \pm a_{13} \\ \frac{1}{\alpha r} e^{i\theta} a_{31} & r e^{i\theta} a_{32} & \pm \frac{1}{\alpha} e^{2i\theta} a_{13} \\ a_{31} & a_{32} & \pm \frac{\alpha r^2 + 1}{2\alpha r} e^{i\theta} a_{13} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where $\alpha = \frac{-\|\phi_{10}\|_{M_2}}{\|\phi_{12}\|_{M_2}} = -\frac{2b_2}{c_2}$.

$$\begin{aligned} a_{31} &= \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}} \cdot \frac{r\|\phi_{10}\|_{M_2} \cdot \|\phi_{12}\|_{M_2}}{\|\phi_{00}\|_{M_2} \cdot (r^2\|\phi_{10}\|_{M_2} + \|\phi_{12}\|_{M_2})} e^{i\theta_{31}} \\ &= \frac{\sqrt{\frac{2}{c_1}}}{\sqrt{\frac{8b_1}{3c_1^3}}} \cdot \frac{r\sqrt{\frac{8b_2}{3c_2^3}} \cdot \sqrt{\frac{2}{3b_2c_2}}}{\sqrt{\frac{2}{c_2}} \cdot \left(r^2\sqrt{\frac{8b_2}{3c_2^3}} + \sqrt{\frac{2}{3b_2c_2}} \right)} e^{i\theta_{31}} = \frac{c_1\sqrt{b_2}r}{\sqrt{b_1}(2b_2r^2 + c_2)} e^{i\theta_{31}} \\ &= \frac{2\sqrt{a_1b_1} \cdot \sqrt{b_2}r}{\sqrt{b_1}(2b_2r^2 + 2\sqrt{a_2b_2})} e^{i\theta_{31}} = \frac{\sqrt{a_1} \cdot r}{\sqrt{b_2r^2 + \sqrt{a_2}}} e^{i\theta_{31}}, \end{aligned}$$

$$\begin{aligned}
a_{32} &= \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{12}\|_{M_1}} \cdot \frac{r\|\phi_{10}\|_{M_2} \cdot \|\phi_{12}\|_{M_2}}{\|\phi_{00}\|_{M_2} (\|\phi_{12}\|_{M_2} + r^2\|\phi_{00}\|_{M_2})} e^{i\theta_{32}} \\
&= \frac{\sqrt{\frac{2}{c_1}}}{\sqrt{\frac{2}{3b_1c_1}}} \cdot \frac{r\sqrt{\frac{8b_2}{3c_2^3}} \cdot \sqrt{\frac{2}{3b_2c_2}}}{\sqrt{\frac{2}{c_2}} \left(r^2\sqrt{\frac{8b_2}{3c_2^3}} + \sqrt{\frac{2}{3b_2c_2}} \right)} e^{i\theta_{32}} \\
&= \frac{2\sqrt{b_1b_2}r}{2b_2r^2 + c_2} e^{i\theta_{32}} = \frac{2\sqrt{b_1b_2}r}{2b_2r^2 + 2\sqrt{a_2b_2}} e^{i\theta_{32}} = \frac{\sqrt{b_1}r}{\sqrt{b_2}r^2 + \sqrt{a_2}} e^{i\theta_{32}},
\end{aligned}$$

$$\begin{aligned}
a_{13} &= \frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}} \cdot \frac{2r\|\phi_{10}\|_{M_2} \cdot \|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2} (r^2\|\phi_{10}\|_{M_2} + \|\phi_{12}\|_{M_2})} e^{i\theta_{13}} \\
&= \frac{\sqrt{\frac{2}{c_1}}}{\sqrt{\frac{2}{3c_1^2}}} \cdot \frac{2r\sqrt{\frac{8b_2}{3c_2^3}} \cdot \sqrt{\frac{2}{3c_2^3}}}{\sqrt{\frac{2}{c_2}} \left(r^2\sqrt{\frac{8b_2}{3c_2^3}} + \sqrt{\frac{2}{3b_2c_2}} \right)} e^{i\theta_{13}} = \frac{4\sqrt{c_1}b_2r}{\sqrt{c_2} (2b_2r^2 + c_2)} e^{i\theta_{13}} \\
&= \frac{4\sqrt{2}\sqrt[4]{a_1b_1b_2}r}{\sqrt{2}\sqrt[4]{a_2b_2} (2b_2r^2 + 2\sqrt{a_2b_2})} e^{i\theta_{13}} = \frac{2\sqrt[4]{a_1b_1}\sqrt{b_2}r}{\sqrt[4]{a_2b_2} (\sqrt{b_2}r^2 + \sqrt{a_2})} e^{i\theta_{13}}.
\end{aligned}$$

So, form (1):

$$\Psi = \left(e^{i\theta_1} \sqrt{\frac{b_2}{b_1}} \cdot \frac{c_1}{c_2} x, e^{i\theta_2} \sqrt{\frac{b_1}{b_2}} y, \pm e^{i\frac{\theta_1+\theta_2}{2}} \sqrt{\frac{c_1}{c_2}} z \right), \quad (2.2.67)$$

form (2):

$$\Psi = \left(e^{i\theta_1} \frac{2\sqrt{b_1b_2}}{c_2} y, e^{i\theta_2} \frac{c_1}{2\sqrt{b_1b_2}} x, \pm e^{i\frac{\theta_1+\theta_2}{2}} \sqrt{\frac{c_1}{c_2}} z \right), \quad (2.2.68)$$

and form (3):

$$\Psi =$$

$$\begin{pmatrix} \frac{-2c_1b_2\sqrt{b_2}r^2}{\sqrt{b_1}c_2(2b_2r^2+c_2)}e^{i(\theta_{31}-\theta)}, \frac{2\sqrt{b_1b_2}}{2b_2r^2+c_2}e^{i(\theta_{32}-\theta)}, \pm\frac{4\sqrt{c_1}b_2r}{\sqrt{c_2}(2b_2r^2+c_2)}e^{i\theta_{13}} \\ -\frac{c_1c_2}{2\sqrt{b_1b_2}(2b_2r^2+c_2)}e^{i(\theta_{31}+\theta)}, \frac{2\sqrt{b_1b_2}r^2}{2b_2r^2+c_2}e^{i(\theta_{32}+\theta)}, \pm\frac{-2\sqrt{c_1c_2}r}{2b_2r^2+c_2}e^{i(\theta_{13}+2\theta)} \\ \frac{c_1\sqrt{b_2}r}{\sqrt{b_1}(2b_2r^2+c_2)}e^{i\theta_{31}}, \frac{2\sqrt{b_1b_2}r}{2b_2r^2+c_2}e^{i\theta_{32}}, \pm\frac{\sqrt{c_1}(2b_2r^2-c_2)}{\sqrt{c_2}(2b_2r^2+c_2)}e^{i(\theta_{13}+\theta)} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \quad (2.2.69)$$

Q. E. D.

Next, let's consider the case $d \geq 2$ for $V_{(a,b,c)}^{(d)} = \{(x, y, z) \in \mathbb{C}^3 : xy = z^2, a|x|^{2d} + b|y|^{2d} + c|z|^{2d} < \varepsilon_0\}$

From the Corollary 2.2.3.6, we know

$$\nu^{(2d-1,d-1)} = \frac{\|\phi_{2d-1,2d-1}\|^2}{\|\phi_{2d-1,d-1}\| \cdot \|\phi_{2d-1,3d-1}\|}$$

is an invariant.

Recall that

$$\|\phi_{\alpha\beta}\|^2 = 16\pi^2 \iint_D r^{2\alpha+1} \rho^{2\beta+1} dr d\rho$$

where $D = \{(r, \rho) : r \geq 0, \rho \geq 0, ar^{2d} + br^{2d}\rho^{4d} + cr^{2d}\rho^{2d} < \varepsilon_0\}$. So

$$\begin{aligned} \|\phi_{2d-1,2d-1}\|^2 &= 16\pi^2 \int_0^\infty \int_0^{\sqrt[2d]{\frac{\varepsilon_0}{a + c\rho^{2d} + b\rho^{4d}}}} r^{4d-1} \rho^{4d-1} dr d\rho \\ &= \frac{4\varepsilon_0^2\pi^2}{d} \int_0^\infty \frac{\rho^{4d-1}}{(a + c\rho^{2d} + b\rho^{4d})^2} d\rho \\ &= \frac{2\varepsilon_0^2\pi^2}{d^2} \int_0^\infty \frac{\rho}{(a + c\rho + b\rho^2)^2} d\rho \end{aligned}$$

$$\begin{aligned}
\|\phi_{2d-1,d-1}\|^2 &= 16\pi^2 \int_0^\infty \int_0^\infty \frac{\sqrt[2d]{\varepsilon_0}}{\sqrt[2d]{a + c\rho^{2d} + b\rho^{4d}}} r^{4d-1} \rho^{2d-1} dr d\rho \\
&= \frac{4\varepsilon_0^2 \pi^2}{d} \int_0^\infty \frac{\rho^{2d-1}}{(a + c\rho^{2d} + b\rho^{4d})^2} d\rho \\
&= \frac{2\varepsilon_0^2 \pi^2}{d^2} \int_0^\infty \frac{1}{(a + c\rho + b\rho^2)^2} d\rho \\
\|\phi_{2d-1,3d-1}\|^2 &= 16\pi^2 \int_0^\infty \int_0^\infty \frac{\sqrt[2d]{\varepsilon_0}}{\sqrt[2d]{a + c\rho^{2d} + b\rho^{4d}}} r^{4d-1} \rho^{6d-1} dr d\rho \\
&= \frac{4\varepsilon_0^2 \pi^2}{d} \int_0^\infty \frac{\rho^{6d-1}}{(a + c\rho^{2d} + b\rho^{4d})^2} d\rho \\
&= \frac{2\varepsilon_0^2 \pi^2}{d^2} \int_0^\infty \frac{\rho^2}{(a + c\rho + b\rho^2)^2} d\rho
\end{aligned}$$

Notice that the value of the invariant

$$\frac{\|\phi_{2d-1,2d-1}\|^2}{\|\phi_{2d-1,d-1}\| \cdot \|\phi_{2d-1,3d-1}\|}$$

is equal to

$$\frac{\|\phi_{11}\|^2}{\|\phi_{10}\| \cdot \|\phi_{12}\|}.$$

as the case $d = 1$.

Then we can get the following proposition.

Proposition 2.2.4.8. *let*

$$V_1 = \{(x, y, z) \in \mathbb{C}^3 : xy = z^2, a_1|x|^{2d} + b_1|y|^{2d} + c_1|z|^{2d} < \varepsilon_0\},$$

and

$$V_2 = \{(x, y, z) \in \mathbb{C}^3 : xy = z^2, a_2|x|^{2d} + b_2|y|^{2d} + c_2|z|^{2d} < \varepsilon_0\}.$$

Then V_1 is biholomorphic to V_2 if and only if $\frac{a_1b_1}{c_1^2} = \frac{a_2b_2}{c_2^2}$.

Proof: (\Rightarrow) From the discussion above, we get

$$\frac{\|\phi_{2d-1,2d-1}\|_{M_1}^2}{\|\phi_{2d-1,d-1}\|_{M_1} \cdot \|\phi_{2d-1,3d-1}\|_{M_1}} = \frac{\|\phi_{2d-1,2d-1}\|_{M_2}^2}{\|\phi_{2d-1,d-1}\|_{M_2} \cdot \|\phi_{2d-1,3d-1}\|_{M_2}}.$$

And using the Proposition 2.2.4.3 and the Proposition 2.2.4.4, we can get $\frac{a_1b_1}{c_1^2} = \frac{a_2b_2}{c_2^2}$.

(\Leftarrow) Just take a biholomorphic map $\Psi = \left(\sqrt[2d]{\frac{a_1}{a_2}}x, \sqrt[2d]{\frac{b_1}{b_2}}y, \sqrt[d]{\frac{c_1}{c_2}}z \right)$.

Q. E. D.

Combining the Theorem 2.2.4.6 and the Proposition 2.2.4.8, we can get the following theorem.

Theorem 2.2.4.9. *Let d be a fixed positive integer and*

$$V_{(a,b,c)}^{(d)} = \{(x, y, z) \in \mathbb{C}^3 : xy = z^2, a|x|^{2d} + b|y|^{2d} + c|z|^{2d} < \varepsilon_0\}.$$

Let \sim denote the biholomorphic equivalence. Then the map

$$\varphi: \{V_{(a,b,c)}^{(d)}\} \rightarrow \mathbb{R}_+, V_{(a,b,c)}^{(d)} \mapsto \nu^{(2d-1,d-1)}$$

is injective up to a biholomorphism. More precisely the induced map $\tilde{\varphi}: \{V_{(a,b,c)}^{(d)}\}/\sim \rightarrow \mathbb{R}_+$ is one-to-one map from $\{V_{(a,b,c)}^{(d)}\}/\sim$ onto $\left(0, \frac{2}{\pi}\right)$. So the moduli space of $\{V_{(a,b,c)}^{(d)}\}$ is of dimension 1 and independent on d .

Let

$$W_{(a,b,c)}^{(d)} = \{(x, y) \in \mathbb{C}^2: a|x|^{4d} + b|y|^{4d} + c|xy|^{2d} < \varepsilon_0\},$$

then this is a special Reinhardt domain in \mathbb{C}^2 . Using the Corollary 2.2.3.4, we can get the moduli space of $W_{(a,b,c)}^{(d)}$ which coincides with the moduli space of $V_{(a,b,c)}^{(d)}$.

Last, as an application to our theory, we compute explicitly the invariant $\nu^{(3,1)}$ for two domains $V_{(1,1,1)}^{(1)}$ and $V_{(1,1,1)}^{(2)}$ in A_1 -variety.

We have known that for the domain $V_{(1,1,1)}^{(2)}$,

$$\begin{aligned} \nu^{(3,1)} &= \frac{\|\phi_{3,3}\|^2}{\|\phi_{3,1}\| \cdot \|\phi_{3,5}\|} = \frac{\int_0^\infty \frac{\rho}{(1+\rho+\rho^2)^2} d\rho}{\left(\int_0^\infty \frac{1}{1+\rho+\rho^2} d\rho \cdot \int_0^\infty \frac{\rho^2}{(1+\rho+\rho^2)^2} d\rho\right)^{\frac{1}{2}}} \\ &= \frac{2(9 - \sqrt{3}\pi)}{4\sqrt{3}\pi - 9}. \end{aligned}$$

However, for the domain $V_{(1,1,1)}^{(1)}$,

$$\begin{aligned} \|\phi_{3,3}\|^2 &= 16\pi^2 \int_0^\infty \int_0^{\sqrt{\varepsilon_0}} \sqrt{1 + \rho^2 + \rho^4} r^7 \rho^7 dr d\rho \\ &= 2\varepsilon_0^4 \pi^2 \int_0^\infty \frac{\rho^7}{(1 + \rho^2 + \rho^4)^4} d\rho = \varepsilon_0^4 \pi^2 \cdot \left(\frac{1}{3} - \frac{14\pi}{81\sqrt{3}}\right), \end{aligned}$$

$$\begin{aligned}
\|\phi_{3,1}\|^2 &= 16\pi^2 \int_0^\infty \int_0^{\sqrt{\varepsilon_0}} \frac{\sqrt{\varepsilon_0}}{\sqrt{1+\rho^2+\rho^4}} r^7 \rho^3 dr d\rho \\
&= 2\varepsilon_0^4 \pi^2 \int_0^\infty \frac{\rho^3}{(1+\rho^2+\rho^4)^4} d\rho = \varepsilon_0^4 \pi^2 \cdot \left(\frac{1}{2} - \frac{20\pi}{81\sqrt{3}}\right), \\
\|\phi_{3,5}\|^2 &= 16\pi^2 \int_0^\infty \int_0^{\sqrt{\varepsilon_0}} \frac{\sqrt{\varepsilon_0}}{\sqrt{1+\rho^2+\rho^4}} r^7 \rho^{11} dr d\rho \\
&= 2\varepsilon_0^4 \pi^2 \int_0^\infty \frac{\rho^{11}}{(1+\rho^2+\rho^4)^4} d\rho = \varepsilon_0^4 \pi^2 \cdot \left(\frac{1}{2} - \frac{20\pi}{81\sqrt{3}}\right).
\end{aligned}$$

Then

$$\nu^{(3,1)} = \frac{\|\phi_{3,3}\|^2}{\|\phi_{3,1}\| \cdot \|\phi_{3,5}\|} = \frac{\frac{1}{3} - \frac{14\pi}{81\sqrt{3}}}{\frac{1}{2} - \frac{20\pi}{81\sqrt{3}}} = \frac{162 - 28\sqrt{3}\pi}{243 - 40\sqrt{3}\pi}.$$

So the invariant $\nu^{(3,1)}$ is different for the domains $V_{(1,1,1)}^{(1)}$ and $V_{(1,1,1)}^{(2)}$, i.e. $V_{(1,1,1)}^{(1)}$ is not biholomorphic to $V_{(1,1,1)}^{(2)}$. And by the Corollary 2.2.3.4, the domain $W_{(1,1,1)}^{(1)}$ in \mathbb{C}^2 is not biholomorphic to the domain $W_{(1,1,1)}^{(2)}$ in \mathbb{C}^2 .

2.3 Moduli space of bounded complete Reinhardt domains in \mathbb{C}^n

In this section, we will consider bounded complete Reinhardt domains in \mathbb{C}^n .

2.3.1 Continuous invariants of bounded complete Reinhardt domains in \mathbb{C}^n

In what follows, we shall use the following notations $\vec{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the i th position.

\mathbb{Z}_+ = set of nonnegative integers

$$\vec{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n.$$

Proposition 2.3.1.1. Let D be a bounded complete Reinhardt domain in \mathbb{C}^n . Let $\phi_{\vec{\alpha}} = \prod_{i=1}^n z_i^{\alpha_i} dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$, $\alpha_i \in \mathbb{Z}_+$. Then $\left\{ \frac{\phi_{\vec{\alpha}}}{\|\phi_{\vec{\alpha}}\|_D} \right\}$ is a complete orthonormal base of F_D , and $\left\{ \frac{\phi_{\vec{\alpha}}}{\|\phi_{\vec{\alpha}}\|_D} : \vec{\alpha} \neq 0 \right\}$ is a complete orthonormal basis of $F_{D,0}$. The Bergman kernel $K_{D,0}$ vanish at the origin and the Bergman kernel K_D are given by

$$K_{D,0} = \Theta_D dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \quad (2.3.1)$$

and

$$K_D = \left(\frac{1}{\|\phi_{\vec{0}}\|_D^2} + \Theta_D \right) dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \quad (2.3.2)$$

where

$$\Theta_D = \sum_{\vec{\alpha} \neq 0} \frac{\prod_{i=1}^n |z_i|^{2\alpha_i}}{\|\phi_{\vec{\alpha}}\|_D^2}. \quad (2.3.3)$$

Proof. This is a consequence of the proof of Proposition 3.2 of (22). Q.E.D.

Proposition 2.3.1.2. Let D be a bounded complete Reinhardt domain in \mathbb{C}^n . With the notations in the above proposition, $\|\phi_{\vec{0}}\|_D^2 \Theta_D$ is invariant under biholomorphic maps which send the origin to the origin.

Proof. Let $\Psi: D_1 \rightarrow D_2$ be a biholomorphic map between two bounded complete Reinhardt domains such that $\Psi(0) = 0$. By Theorem 2.1.0.6 and Proposition 2.3.1.1, we have

$$B_{D_1}(z) = B_{D_2}(\Psi(z))$$

$$\begin{aligned} \Rightarrow \frac{\Theta_{D_1}(z)}{\frac{1}{\|\phi_{\bar{0}}\|_{D_1}^2 + \Theta_{D_1}(z)}} &= \frac{\Theta_{D_2}(\Psi(z))}{\frac{1}{\|\phi_{\bar{0}}\|_{D_2}^2} + \Theta_{D_2}(\Psi(z))} \\ \Rightarrow \|\phi_{\bar{0}}\|_{D_1}^2 \Theta_{D_1}(z) &= \|\phi_{\bar{0}}\|_{D_2}^2 \Theta_{D_2}(\Psi(z)). \end{aligned}$$

Q.E.D.

Theorem 2.3.1.3. Let D_i , $i = 1, 2$, be two bounded complete Reinhardt domains in \mathbb{C}^n . If D_1 is biholomorphically equivalent to D_2 , then there exists a biholomorphic map Ψ_σ of the following form

$$\Psi_\sigma(z) = (a_1 z_{\sigma(1)}, \dots, a_n z_{\sigma(n)}) \quad (2.3.4)$$

where σ is in S_n , a symmetric group of order n , and

$$a_i = \frac{\|\phi_{\bar{0}}\|_{D_1} \|\phi_{\bar{e}_i}\|_{D_2}}{\|\phi_{\bar{e}_{\sigma(i)}}\|_{D_1} \|\phi_{\bar{0}}\|_{D_2}}. \quad (2.3.5)$$

Proof. In view of Theorem 2.1.0.16, there exists a biholomorphism $\Psi_\sigma: D_1 \rightarrow D_2$ of the form Equation 2.3.4. By Proposition 2.3.1.2

$$\begin{aligned} \|\phi_{\bar{0}}\|_{D_1}^2 \Theta_{D_1}(z) &= \|\phi_{\bar{0}}\|_{D_2}^2 \Theta_{D_2}(\Psi_\sigma(z)) \\ \Rightarrow \|\phi_{\bar{0}}\|_{D_1}^2 \sum_{\bar{\alpha} \neq 0} \frac{\prod_{i=1}^n |z_n|^{2\alpha_i}}{\|\phi_{\bar{\alpha}}\|_{D_1}^2} &= \|\phi_{\bar{0}}\|_{D_2}^2 \sum_{\bar{\alpha} \neq 0} \frac{\prod_{i=1}^n |a_i z_{\sigma(i)}|^{2\alpha_i}}{\|\phi_{\bar{\alpha}}\|_{D_2}^2}. \end{aligned} \quad (2.3.6)$$

Comparing the coefficient of $|z_i|^2$ in Equation 2.3.6, we can get

$$\begin{aligned} \|\phi_{\bar{0}}\|_{D_1}^2 \frac{1}{\|\phi_{\bar{e}_i}\|_{D_1}^2} &= \frac{\|\phi_{\bar{0}}\|_{D_2}^2 |a_{\sigma^{-1}(i)}|^2}{\|\phi_{\bar{e}_{\sigma^{-1}(i)}}\|_{D_2}^2} \\ \Rightarrow a_{\sigma^{-1}(i)} &= \frac{\|\phi_{\bar{0}}\|_{D_1} \|\phi_{\bar{e}_{\sigma^{-1}(i)}}\|_{D_2}}{\|\phi_{\bar{e}_i}\|_{D_1} \|\phi_{\bar{0}}\|_{D_2}}, \text{ i.e., } a_i = \frac{\|\phi_{\bar{0}}\|_{D_1} \|\phi_{\bar{e}_i}\|_{D_2}}{\|\phi_{\bar{e}_{\sigma(i)}}\|_{D_1} \|\phi_{\bar{0}}\|_{D_2}}. \end{aligned}$$

Q.E.D.

Now we are ready to prove Theorem F.

Proof of Theorem F. Let D_1 and D_2 be biholomorphically equivalent bounded complete Reinhardt domains. By Theorem 2.3.1.3, there exists a biholomorphic map

$$\begin{aligned} \Psi_{\sigma}: D_1 &\longrightarrow D_2 \\ (z_1, \dots, z_n) &\longrightarrow (a_1 z_{\sigma(1)}, \dots, a_n z_{\sigma(n)}) \end{aligned}$$

$$\text{where } a_i = \frac{\|\phi_{\bar{0}}\|_{D_1} \|\phi_{\bar{e}_i}\|_{D_2}}{\|\phi_{\bar{e}_{\sigma(i)}}\|_{D_1} \|\phi_{\bar{0}}\|_{D_2}}$$

.

By Proposition 2.3.1.2, we have Equation 2.3.6. Comparing the coefficient of $|z_1|^{2\alpha_1} |z_2|^{2\alpha_2} \dots |z_n|^{2\alpha_n}$ in Equation 2.3.6 both sides, we get

$$\frac{\|\phi_{\bar{0}}\|_{D_1}^2}{\|\phi_{\bar{\alpha}}\|_{D_1}^2} = \frac{\|\phi_{\bar{0}}\|_{D_2}^2 \cdot \prod_{i=1}^n |a_{\sigma^{-1}(i)}|^{2\alpha_i}}{\|\phi_{\sigma(\bar{\alpha})}\|_{D_2}^2}$$

$$\begin{aligned}
&\Rightarrow \frac{\|\phi_{\vec{0}}\|_{D_1}^2}{\|\phi_{\vec{\alpha}}\|_{D_1}^2} = \frac{\|\phi_{\vec{0}}\|_{D_2}^2}{\|\phi_{\sigma(\vec{\alpha})}\|_{D_2}^2} \prod_{i=1}^n \left(\frac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e}_{\sigma^{-1}(i)}}\|_{D_2}}{\|\phi_{\vec{e}_i}\|_{D_1} \|\phi_{\vec{0}}\|_{D_2}} \right)^{2\alpha_i} \\
&\Rightarrow \frac{\|\phi_{\vec{0}}\|_{D_1}^{\sum \alpha_i - 1} \cdot \|\phi_{\vec{\alpha}}\|_{D_1}}{\prod_{i=1}^n \|\phi_{\vec{e}_i}\|_{D_1}^{\alpha_i}} = \frac{\|\phi_{\vec{0}}\|_{D_2}^{\sum \alpha_i - 1} \cdot \|\phi_{\sigma(\vec{\alpha})}\|_{D_2}}{\prod_{i=1}^n \|\phi_{\vec{e}_{\sigma^{-1}(i)}}\|_{D_2}^{\alpha(i)}} = \frac{\|\phi_{\vec{0}}\|_{D_2}^{\sum \alpha_i - 1} \|\phi_{\sigma(\vec{\alpha})}\|_{D_2}}{\prod_{i=1}^n \|\phi_{\vec{e}_i}\|_{D_2}^{\alpha_{\sigma(i)}}} \quad (2.3.7)
\end{aligned}$$

i.e., $g_{D_1}^{\text{Id}}(\vec{\alpha}) = g_{D_2}^{\sigma}(\vec{\alpha})$.

Similarly, by comparing the coefficient of $|z_1|^{2\alpha_{\tau(1)}} |z_2|^{2\alpha_{\tau(2)}} \dots |z_n|^{2\alpha_{\tau(n)}}$ in Equation 2.3.6 both sides, we know

$$g_{D_1}^{\tau}(\vec{\alpha}) = g_{D_2}^{\tau \cdot \sigma}(\vec{\alpha}),$$

which implies

$$\sum_{\tau \in S_n} g_{D_1}^{\tau}(\vec{\alpha}) \tau = \sum_{\tau \in S_n} g_{D_2}^{\tau \cdot \sigma}(\vec{\alpha}) \cdot \tau = \sum_{\tau \in S_n} g_{D_2}^{\tau \cdot \sigma}(\vec{\alpha}) \tau \cdot \sigma \cdot \sigma^{-1} = \sigma^{-1} \left(\sum_{\tau \in S_n} g_{D_2}^{\tau}(\vec{\alpha}) \tau \right).$$

For the same reason, we have

$$\sum_{\tau \in S_n} g_{D_1}^{\tau}(\vec{\alpha}_i) \tau = \sigma^{-1} \left(\sum_{\tau \in S_n} g_{D_2}^{\tau}(\vec{\alpha}_i) \tau \right), \quad i = 1, \dots, n!. \quad (2.3.8)$$

From Equation 2.3.8, we have $\xi_{D_1}^{\vec{\alpha}, \dots, \vec{\alpha}_n!} \sim \xi_{D_2}^{\vec{\alpha}, \dots, \vec{\alpha}_n!}$.

Q.E.D.

In order to prove Theorem G, we need to establish the following theorem.

Theorem 2.3.1.4. Let D_i , $i = 1, 2$, be two bounded complete Reinhardt domains in \mathbb{C}^n . Suppose for all $\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}$ n -tuples of nonnegative integers, $\xi_{D_1}^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}} = \xi_{D_2}^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}}$ in $(\mathbb{R}[S_n] \times \dots \times \mathbb{R}[S_n]) / \sim$, i.e., $\exists \sigma^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}} \in S_n$ such that

$$\xi_{D_1}^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}} = \sigma^{\alpha_1, \dots, \alpha_{n!}} \left(\xi_{D_2}^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}} \right) \quad (2.3.9)$$

Then, there exists $\sigma \in S_n$ and a biholomorphic map

$$\begin{aligned} \Psi_\sigma: \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ (z_1, \dots, z_n) &\longrightarrow (a_1 z_{\sigma(i)}, \dots, a_n z_{\sigma(n)}), \end{aligned}$$

where $a_i = \frac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e}_i}\|_{D_2}}{\|\phi_{e_{\sigma(i)}}\|_{D_2} \|\phi_{\vec{0}}\|_{D_1}}$, such that $B_{D_1}(z) = B_{D_2}(\Psi_\sigma(z))$.

Proof. We claim that $\exists \sigma \in S_n$ such that $g_{D_1}^\tau(\vec{\beta}) = g_{D_2}^{\tau \circ \sigma}(\vec{\beta}) \forall \tau \in S_n$ and $\forall \vec{\beta}$ n -tuple of nonnegative integers. For any $\vec{\alpha}$ n -tuple of nonnegative integers, let

$$I_{\vec{\alpha}} = \{\sigma \in S_n : g_{D_1}^\tau(\vec{\alpha}) = g_{D_2}^{\tau \circ \sigma}(\vec{\alpha}), \quad \forall \tau \in S_n\}.$$

If our claim is not true, then $\forall \sigma_i \in S_n = \{\sigma_1, \dots, \sigma_{n!}\}$, $\exists \vec{\alpha}_i$, n -tuple of nonnegative integers such that $\sigma_i \notin I_{\vec{\alpha}_i}$. It follows that

$$\bigcup_{i=1}^{n!} (S_n \setminus I_{\vec{\alpha}_i}) = S_n,$$

which implies $\bigcap_{i=1}^{n!} I_{\vec{\alpha}_i} = \emptyset$. On the other hand for these n -tuples of integers $\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}$, we have $\xi_{D_1}^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}} = \xi_{D_2}^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}}$, i.e., $\exists \sigma^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}} \in S_n$ such that Equation 2.3.9 hold. Let $\tau_{(\vec{\alpha}_1, \dots, \vec{\alpha}_{n!})} = (\sigma^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}})^{-1}$. In view of Equation 2.3.9, one easily sees that $\tau_{(\vec{\alpha}_1, \dots, \vec{\alpha}_{n!})} \in \bigcap_{i=1}^{n!} I_{\alpha_i}$. This leads to a contradiction and our claim is proven.

We now take σ in the claim and let

$$\begin{aligned} \Psi_\sigma: \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ (z_1, \dots, z_n) &\longrightarrow (a_1 z_{\sigma(1)}, \dots, a_n z_{\sigma(n)}), \end{aligned}$$

where $a_i = \frac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e}_i}\|_{D_2}}{\|\phi_{e_{\sigma(i)}}\|_{D_2} \|\phi_{\vec{0}}\|_{D_2}}$. After computation as in the proof of Theorem F, we get

$$\|\phi_{\vec{0}}\|_{D_1}^2 \sum \frac{|z_1|^{2\alpha_1} \dots |z_n|^{2\alpha_n}}{\|\phi_{\vec{\alpha}}\|_{D_1}} = \|\phi_{\vec{0}}\|_{D_2}^2 \sum \frac{\prod_{i=1}^n |a_i z_{\sigma(i)}|^{2\alpha_i}}{\|\phi_{\sigma(\vec{\alpha})}\|_{D_2}^2}.$$

It follows that $B_{D_1}(z) = B_{D_2}(\Psi_\sigma(z))$.

Q.E.D.

Now we are ready to prove Theorem G.

Proof of Theorem G. It is easy to see that the Bergman function B_{D_i} is zero at the origin and $0 < B_{D_i} < 1$ on $D_i \setminus \{(0, 0, \dots, 0)\}$. In view of Proposition 2.1.0.17, B_{D_i} is identically equal to 1 on ∂D_i . By Theorem 2.3.1.4 there exists a biholomorphic map Ψ_σ from \mathbb{C}^n to \mathbb{C}^n such that $B_{D_1}(z) = B_{D_2}(\Psi_\sigma(z))$. In particular Ψ_σ preserves the level sets of the Bergman functions.

It follows that Ψ_σ sends ∂D_1 to ∂D_2 .

Q.E.D.

2.3.2 Complete continuous numerical invariants and Hilbert 14th problem

We shall now construct the complete continuous numerical invariants for complete Reinhardt domains with C^1 boundaries. For $n = 2$, we have Theorem H.

Proof of Theorem H. Since Equation 1.2.3, Equation 1.2.4 and Equation 1.2.5 hold for all non-negative integers α_i, β_i , we have either

$$(1) \quad g_{D_1}(\alpha_1, \alpha_2) = g_{D_2}(\alpha_1, \alpha_2), \quad g_{D_1}(\alpha_2, \alpha_1) = g_{D_2}(\alpha_2, \alpha_1),$$

$$g_{D_1}(\beta_1, \beta_2) = g_{D_2}(\beta_1, \beta_2), \quad g_{D_1}(\beta_2, \beta_1) = g_{D_2}(\beta_2, \beta_1),$$

or

$$(2) \quad g_{D_1}(\alpha_1, \alpha_2) = g_{D_2}(\alpha_2, \alpha_1), \quad g_{D_1}(\alpha_2, \alpha_1) = g_{D_2}(\alpha_1, \alpha_2),$$

$$g_{D_1}(\beta_1, \beta_2) = g_{D_2}(\beta_2, \beta_1), \quad g_{D_1}(\beta_2, \beta_1) = g_{D_2}(\beta_1, \beta_2),$$

for all nonnegative integers α_i, β_i . It is easy to see that in both cases we get

$$\begin{aligned} \xi_{D_1}^{\vec{\alpha}, \vec{\beta}} &:= \left(g_{D_1}(\alpha_1, \alpha_2) \cdot \text{Id} + g_{D_1}(\alpha_2, \alpha_1) \cdot \sigma, \right. \\ &\quad \left. g_{D_1}(\beta_1, \beta_2) \cdot \text{Id} + g_{D_2}(\beta_2, \beta_1) \cdot \sigma \right) \\ &\sim \xi_{D_2}^{\vec{\alpha}, \vec{\beta}} := \left(g_{D_2}(\alpha_1, \alpha_2) \cdot \text{Id} + g_{D_2}(\alpha_2, \alpha_1) \cdot \sigma, \right. \\ &\quad \left. g_{D_2}(\beta_1, \beta_2) \cdot \text{Id} + g_{D_2}(\beta_2, \beta_1) \cdot \sigma \right), \end{aligned} \tag{2.3.10}$$

where $S_2 = \{\text{Id}, \sigma\}$. In view of Theorem B, we know that D_1 is biholomorphic to D_2 .

Conversely, if D_1 is biholomorphic to D_2 then by Theorem F Equation 2.3.10 holds. Then it is easy to check that Equation 1.2.3, Equation 1.2.4 and Equation 1.2.5 hold. Q.E.D.

Theorem H says that

$$\begin{aligned} g_D(\alpha_1, \alpha_2) + g_D(\alpha_2, \alpha_1), \\ g_D(\alpha_1, \alpha_2) \cdot g_D(\alpha_2, \alpha_1), \end{aligned}$$

and

$$(g_D(\alpha_1, \alpha_2) - g_D(\alpha_2, \alpha_1))(g_D(\beta_1, \beta_2) - g_D(\beta_2, \beta_1)),$$

$\forall \alpha_i, \beta_i \geq 0, \alpha_i, \beta_i \in \mathbb{Z}$, are complete numerical biholomorphic invariants of bounded complete Reinhardt pseudoconvex domains in \mathbb{C}^n with C^1 boundaries. If we want to find the complete numerical biholomorphic invariants for bounded complete Reinhardt pseudoconvex domains in \mathbb{C}^n , $n \geq 3$, we need to consider the following problem.

Let S_n be the symmetric group of order n . S_n acts on $\mathbb{C}[x_\sigma, \dots, y_\sigma]$ in the following manner.

$$\tau \in S_n, \quad \tau(x_\sigma) = x_{\sigma \cdot \tau}, \dots, \tau(y_\sigma) = y_{\sigma \cdot \tau}$$

Let R be subring of invariants in $\mathbb{C}[x_\sigma, \dots, y_\sigma]$. In other words,

$$R := \mathbb{C}[x_\sigma, \dots, y_\sigma]^{S_n} = \{f \in \mathbb{C}[x_\sigma, \dots, y_\sigma] : \forall \tau \in S_n, \tau(f) = f\}.$$

We want to know whether R is finitely generated and what are the generators of R . Actually the first problem is a special case of Hilbert's 14th problem. At the International Congress of Mathematicians at Paris in 1900, David Hilbert asked the following question.

Hilbert 14th Problem. If an algebraic group acts linearly on a polynomial ring in finitely many variables, is the ring of invariants always finitely generated?

Recall that an algebraic group G is said to be linearly reductive if, for every epimorphism $\phi: V \rightarrow W$ of G representations, the induced map on G -invariants $\phi^G: V^G \rightarrow W^G$ is surjective. The answer to the Hilbert 14th problem is positive if the group G is linearly reductive (30). In fact, the answer is also positive if G is additive (non-reductive) group \mathbb{C} (or, more generally, the field k . See (35) and (36)). In view of the following lemma, we know that finding the complete numerical biholomorphic invariants is equivalent to finding the generators of R .

Lemma 2.3.2.1. Let the symmetric group $S_n = \{\sigma_1, \dots, \sigma_n\}$ act on $\mathbb{C}^{n!n!}$ via the following formula

$$\begin{aligned} \tau \in S_n, \quad p &= (x_{\sigma_1}, \dots, x_{\sigma_n}, \dots, y_{\sigma_1}, \dots, y_{\sigma_n}) \in \mathbb{C}^{n!n!} \\ \tau(p) &= (x_{\sigma_{1\tau}}, \dots, x_{\sigma_{n\tau}}, \dots, y_{\sigma_{1\tau}}, \dots, y_{\sigma_{n\tau}}). \end{aligned}$$

Let $\pi: \mathbb{C}^{n!n!} \rightarrow \mathbb{C}^{n!n!}/S_n$ be the quotient map. For any p and p' in $\mathbb{C}^{n!n!}$, the following two statements are equivalent

$$(1) \quad \pi(p) = \pi(p')$$

(2) For any $f \in R = \mathbb{C}[x_{\sigma_1}, \dots, x_{\sigma_{n!}}, y_{\sigma_1}, \dots, y_{\sigma_{n!}}]^{S_n}$, $f(p) = f(p')$.

Proof. See Theorem 5.16 on p. 167 of (37).

In view of Lemma 2.3.2.1, we want to show that $R = \mathbb{C}[x_1, \dots, x_{n!}, \dots, y_1, \dots, y_{n!}]^{S_n}$ is finitely generated.

Proposition 2.3.2.2. Every finite group is linearly reductive.

Theorem 2.3.2.3 (Hilbert (30)). For linearly reductive group, the ring of invariant polynomials is finitely generated.

From Proposition 2.3.2.2 and Theorem 2.3.2.3, we know that

$$R = \mathbb{C}[x_{\sigma_1}, \dots, x_{\sigma_{n!}}, \dots, y_{\sigma_1}, \dots, y_{\sigma_{n!}}]^{S_n}$$

is finitely generated. Now we assume that the generators of R are $f_1(x_\sigma, \dots, y_\sigma), \dots, f_N(x_\sigma, \dots, y_\sigma)$ where σ runs over all the elements in S_n . We want to show that

$$f_1(g_D^\sigma(\vec{\alpha}_1), \dots, g_D^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n}, \dots, f_N(g_D^\sigma(\vec{\alpha}_1), \dots, g_D^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n},$$

for all n -tuples of nonnegative integers $\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}$, are complete numerical invariants of bounded complete Reinhardt pseudoconvex domains in \mathbb{C}^n .

Proof of Theorem J. Let D_1, D_1 be two bounded complete Reinhardt domains in \mathbb{C}^n . If D_1 is biholomorphic to D_2 , then there exists $\tau \in S_n$ such that $g_{D_1}^\sigma(\vec{\alpha}) = g_{D_2}^{\sigma\tau}(\vec{\alpha}), \forall \sigma \in S_n, \forall \vec{\alpha}$ n -tuple of nonnegative integers. We take

$$P_i = \left(g_{D_i}^{\sigma_1}(\vec{\alpha}_1), \dots, g_{D_i}^{\sigma_{n!}}(\vec{\alpha}_1), \dots, g_{D_i}^{\sigma_1}(\vec{\alpha}_{n!}), \dots, g_{D_i}^{\sigma_{n!}}(\vec{\alpha}_{n!}) \right) \in \mathbb{C}^{n!n!}$$

where $i = 1, 2$ and $S_n = \{\sigma_1, \sigma_2, \dots, \sigma_{n!}\}$. Then $\pi(P_1) = \pi(P_2)$ where $\pi: \mathbb{C}^{n!n!} \rightarrow \mathbb{C}^{n!n!}/S_n$ is the quotient map in Lemma 2.3.2.1. It follows that

$$f_i \left(g_{D_1}^\sigma(\vec{\alpha}_1), \dots, g_{D_1}^\sigma(\vec{\alpha}_{n!}) \right)_{\sigma \in S_n} = f_i \left(g_{D_2}^\sigma(\vec{\alpha}_1), \dots, g_{D_2}^\sigma(\vec{\alpha}_{n!}) \right)_{\sigma \in S_n},$$

$$1 \leq i \leq N.$$

Q.E.D.

Proof of Theorem K. Since

$$f_i \left(g_{D_1}^\sigma(\vec{\alpha}_1), \dots, g_{D_1}^\sigma(\vec{\alpha}_{n!}) \right)_{\sigma \in S_n} = f_i \left(g_{D_2}^\sigma(\vec{\alpha}_1), \dots, g_{D_2}^\sigma(\vec{\alpha}_{n!}) \right)_{\sigma \in S_n},$$

$$i = 1, 2, \dots, N$$

and $\{f_1, \dots, f_N\}$ generates $\mathbb{C}[x_{\sigma_1}, \dots, x_{\sigma_{n!}}, \dots, y_{\sigma_1}, \dots, y_{\sigma_{n!}}]^{S_n}$, we know by Lemma 2.3.2.1 that

$$P_1 = \left(g_{D_1}^{\sigma_1}(\vec{\alpha}_1), \dots, g_{D_1}^{\sigma_{n!}}(\vec{\alpha}_1), \dots, g_{D_1}^{\sigma_1}(\vec{\alpha}_{n!}), \dots, g_{D_1}^{\sigma_{n!}}(\vec{\alpha}_{n!}) \right)$$

and $P_2 = \left(g_{D_2}^{\sigma_1}(\vec{\alpha}_1), \dots, g_{D_2}^{\sigma_{n!}}(\vec{\alpha}_{n!}), \dots, g_{D_2}^{\sigma_1}(\vec{\alpha}_{n!}), \dots, g_{D_2}^{\sigma_{n!}}(\vec{\alpha}_{n!}) \right)$ are in the same S_n orbit. Hence there exists $\sigma^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}} \in S_n$ such that

$$g_{D_1}^{\sigma}(\vec{\alpha}_1) = g_{D_2}^{\sigma \cdot \sigma^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}}}(\vec{\alpha}_1), \dots, g_{D_1}^{\sigma}(\vec{\alpha}_{n!}) = g_{D_2}^{\sigma \cdot \sigma^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}}}(\vec{\alpha}_{n!}) \quad (2.3.11)$$

Equation 2.3.11 implies that $\xi_{D_1}^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}} = \xi_{D_2}^{\vec{\alpha}_1, \dots, \vec{\alpha}_{n!}}$ in $(\mathbb{R}[S_n] \times \dots \times \mathbb{R}[S_n]) / \sim$. By Theorem 2.3.1.4, there exists $\sigma \in S_n$ and a biholomorphic map $\Psi_\sigma: \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that $B_{D_1}(z) = B_{D_2}(\Psi_\sigma(z))$. The proof of Theorem B shows that Ψ_σ sends D_1 onto D_2 . Q.E.D.

For complete Reinhardt pseudoconvex domains with real analytic boundaries, we can use fewer numerical invariants to characterize these domains.

Proof of Theorem G'.

“ \Rightarrow ” This follows from the proof of Theorem F.

“ \Leftarrow ” Suppose that for all $\vec{\alpha}$ n -tuple nonnegative integers,

$$\xi_{D_1}^{\vec{\alpha}} = \xi_{D_2}^{\vec{\alpha}} \text{ in } R[S_n] / \sim .$$

Then $\exists \sigma_{\vec{\alpha}} \in S_n$ such that

$$\sum_{\tau \in S_n} g_{D_1}^{\tau}(\vec{\alpha})\tau = \sigma_{\vec{\alpha}} \left(\sum_{\tau \in S_n} g_{D_2}^{\tau}(\vec{\alpha})\tau \right) = \sum_{\tau \in S_n} g_{D_2}^{\tau}(\vec{\alpha})\tau \cdot \sigma_{\vec{\alpha}} = \sum_{\tau \in S_n} g_{D_2}^{\tau \cdot \sigma_{\vec{\alpha}}^{-1}}(\vec{\alpha})\tau \quad (2.3.12)$$

$$\Rightarrow g_{D_1}^{\tau}(\vec{\alpha}) = g_{D_2}^{\tau \cdot \sigma_{\vec{\alpha}}^{-1}}(\vec{\alpha})$$

Let $S_n = \{\tau_1, \dots, \tau_n\}$. Denote $I_j = \{\vec{\alpha} \in (\mathbb{Z}_+)^n : \sigma_{\vec{\alpha}}^{-1} = \tau_j\}$, $j = 1, 2, \dots, n!$. We shall introduce the concept of partial Bergman function in the following manner. Let

$$\Theta_D^{(j)}(z) = \sum_{\vec{\alpha} \in I_j} \frac{\prod_{k=1}^n |z_k|^{2\alpha_k}}{\|\phi_{\vec{\alpha}}\|_D^2}, \quad B_D^{(j)}(z) = \frac{\Theta_D^{(j)}(z)}{\frac{1}{\|\phi_{\vec{\alpha}}\|_D^2} + \Theta_D^{(j)}(z)}.$$

Clearly we have

$$\Theta_D(z) = \sum_{\vec{\alpha} \neq 0} \frac{\prod_{k=1}^n |z_k|^{2\alpha_k}}{\|\phi_{\vec{\alpha}}\|_D^2} = \sum_{j=1}^{n!} \Theta_D^{(j)}(z) \quad (2.3.13)$$

In view of Equation 2.3.12, by routine computation we get

$$\|\phi_{\vec{0}}\|_{D_1}^2 \Theta_{D_1}^{(j)}(z) = \|\phi_{\vec{0}}\|_{D_2}^2 \Theta_{D_2}^{(j)}(\Psi_{\tau_j}(z)) \quad (2.3.14)$$

and

$$B_{D_1}^{(j)}(z) = B_{D_2}^{(j)}(\Psi_{\tau_j}(z)), \quad (2.3.15)$$

where

$$\begin{aligned} \Psi_{\tau_j} : \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ (z_1, \dots, z_n) &\longrightarrow (a_1 z_{\tau_j(1)}, \dots, a_n z_{\tau_j(n)}) \\ a_i &= \frac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e}_i}\|_{D_2}}{\|\phi_{e_{\tau_j(i)}}\|_{D_1} \|\phi_{\vec{0}}\|_{D_2}}. \end{aligned}$$

By Proposition 2.1.0.17, the Bergman kernel and hence Θ_{D_1} blows up at the boundary points. It follows from Equation 2.3.13 that there exists some m such that $\Theta_{D_1}^{(m)}$ is infinite on a nonempty open set \mathcal{U}_1 of ∂D_1 . It follows that $B_{D_1}^{(m)}(z)$ is equal to 1 in the nonempty open set \mathcal{U}_1 of ∂D_1 . Recall that by the definition, $B_{D_i}^{(m)}$ is zero at the origin and is $0 < B_{D_i}^{(m)} < 1$ on $D_i \setminus \{(0, \dots, 0)\}$. Since $B_{D_1}^{(m)}(z) = B_{D_2}^{(m)}(\Psi_{\tau_m}(z))$, we see immediately that Ψ_{τ_m} sends $\mathcal{U}_1 \subseteq \partial D_1$ to an open subset of ∂D_2 . In view of the fact that ∂D_i , $i = 1, 2$, are real analytic and compact, it follows easily that Ψ_{τ_m} sends ∂D_1 to ∂D_2 . Q.E.D.

Proof of Theorem H'. This follows immediately from Theorem G'. Q.E.D.

Proof of Theorem K'.

“ \Rightarrow ” This follows from the proof of Theorem J.

“ \Leftarrow ” Suppose that for all $\vec{\alpha}$ n -tuple nonnegative integers, we have

$$f_i(g_{D_1}^\sigma(\vec{\alpha}))_{\sigma \in S_n} = f_i(g_{D_2}^\sigma(\vec{\alpha}))_{\sigma \in S_n}, \quad i = 1, \dots, N.$$

Then by the analogy of Lemma 2.3.2.1, we have

$$\pi(g_{D_1}^{\tau_1}(\vec{\alpha}), \dots, g_{D_1}^{\tau_{n!}}(\vec{\alpha})) = \pi(g_{D_2}^{\tau_1}(\vec{\alpha}), \dots, g_{D_2}^{\tau_{n!}}(\vec{\alpha}))$$

when $\pi: \mathbb{C}^{n!} \longrightarrow \mathbb{C}^{n!}/S_n$. Therefore there exist $\sigma_{\vec{\alpha}}$ such that $g_{D_1}^\sigma(\vec{\alpha}) = g_{D_2}^{\sigma \cdot \sigma_{\vec{\alpha}}}(\vec{\alpha})$ for all $\sigma \in S_n$. By the proof similar to the proof of Theorem G', we get a biholomorphic map Ψ_τ which sends D_1 onto D_2 . Q.E.D.

2.3.3 Application in concrete examples

In 1907 Poincaré discovered the following theorem ((38), p. 24).

Theorem 2.3.3.1 (Poincaré). Let $B_n = \{z \in \mathbb{C}^n \mid \sum_{i=1}^n |z_i|^2 < 1\}$ be the unit ball and $\Delta_n = \{z \in \mathbb{C}^n \mid |z_i| < 1, i = 1, \dots, n\}$ be the unit polydisc in \mathbb{C}^n , then there exists no biholomorphic map between Δ_n and B_n .

Using our method, we can easily get this result.

Notation: $\vec{0} = (0, \dots, 0)$, $\vec{e}_i = (0, \dots, 1, \dots, 0)$, $\vec{\alpha} = (2, 0, \dots, 0)$.

Proof of Theorem 2.3.3.1.

For any $\vec{\beta} \in \mathbb{Z}^n$, $D = \{z \in \mathbb{C}^n \mid f(z_1, \dots, z_n) < 0\}$ is the bounded Reinhardt domain.

$$\begin{aligned} \|\phi_{\vec{\beta}}\|_D &= \int_D \phi_{\vec{\beta}} \wedge \bar{\phi}_{\vec{\beta}} \\ &= \int_D |z_1|^{2\beta_1} \dots |z_n|^{2\beta_n} dz_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_1 \wedge \dots \wedge d\bar{z}_n \\ &= (4\pi)^n \int_{\{\vec{r}: f(r_1, \dots, r_n) < 0\}} \prod_{j=1}^n r_j^{2\beta_j+1} dr_1 \wedge \dots \wedge dr_n \end{aligned}$$

After computation, we get

$$\begin{aligned} \|\phi_{\vec{0}}\|_{\Delta_n}^2 &= (4\pi)^n \int_{\Delta_n} r_1 r_2 \dots r_n dr_1 \dots dr_n = (2\pi)^n, \\ \|\phi_{\vec{e}_1}\|_{\Delta_n}^2 &= (4\pi)^n \int_{\Delta_n} r_1^3 r_2 \dots r_n dr_1 \dots dr_n = \frac{1}{2} \cdot (2\pi)^n, \\ \|\phi_{\vec{\alpha}}\|_{\Delta_n}^2 &= (4\pi)^n \int_{\Delta_n} r_1^5 r_2 \dots r_n dr_1 \dots dr_n = \frac{1}{3} \cdot (2\pi)^n, \end{aligned}$$

$$\begin{aligned}\|\phi_{\vec{0}}\|_{B_n}^2 &= (4\pi^n) \int_{B_n} \cdots \int r_1 r_2 \cdots r_n dr_1 \cdots dr_n = \frac{(4\pi)^n}{\prod_{i=1}^n 2i}, \\ \|\phi_{\vec{e}_1}\|_{B_n}^2 &= (4\pi^n) \int_{B_n} \cdots \int r_1^3 r_2 \cdots r_n dr_1 \cdots dr_n = \frac{(4\pi)^n}{\prod_{i=1}^{n+1} 2i}, \\ \|\phi_{\vec{\alpha}}\|_{B_n}^2 &= (4\pi^n) \int_{B_n} \cdots \int r_1^5 r_2 \cdots r_n dr_1 \cdots dr_n = \frac{(4\pi)^n}{\prod_{i=3}^{n+2} 2i}.\end{aligned}$$

For Δ_n and B_n are symmetric domains

$$\begin{aligned}\|\phi_{\vec{e}_i}\|_{\Delta_n} &= \|\phi_{\vec{e}_j}\|_{\Delta_n}, & \|\phi_{\vec{e}_i}\|_{B_n} &= \|\phi_{\vec{e}_j}\|_{B_n}, & \forall i, j \in \{1, \dots, n\}, \\ \|\phi_{\vec{\alpha}}\|_{\Delta_n} &= \|\phi_{\tau(\vec{\alpha})}\|_{\Delta_n}, & \|\phi_{\vec{\alpha}}\|_{B_n} &= \|\phi_{\tau(\vec{\alpha})}\|_{B_n}, & \forall \tau \in S_n,\end{aligned}$$

$$\begin{aligned}g_{\Delta_n}^{\vec{\tau}}(\vec{\alpha}) &= g_{\Delta_n}^{\text{Id}}(\vec{\alpha}) = \frac{\|\phi_{\vec{0}}\|_{\Delta_n}^{\sum \alpha_i - 1} \|\phi_{\vec{\alpha}}\|_{\Delta_n}}{\prod_{i=1}^n \|\phi_{\vec{e}_i}\|_{\Delta_n}^{\alpha_i}} = \frac{2\sqrt{3}}{3}, \\ g_{B_n}^{\vec{\tau}}(\vec{\alpha}) &= g_{B_n}^{\text{Id}}(\vec{\alpha}) = \frac{\|\phi_{\vec{0}}\|_{B_n}^{\sum \alpha_i - 1} \|\phi_{\vec{\alpha}}\|_{B_n}}{\prod_{i=1}^n \|\phi_{\vec{e}_i}\|_{B_n}^{\alpha_i}} = \frac{\sqrt{2(n+1)(n+2)}}{n+2}.\end{aligned}$$

So $\sum_{\tau \in S_n} g_{\Delta_n}^{\vec{\tau}}(\vec{\alpha}) \cdot \tau \neq \sum_{\tau \in S_n} g_{B_n}^{\vec{\tau}}(\vec{\alpha}) \cdot \tau$ in $\mathbb{R}[S_n]/\sim$.

Then Δ_n is not biholomorphic to B_n by Theorem F.

Q.E.D.

Example 2.3.3.2. In addition, we can show $A_n = \{z \in \mathbb{C}^n \mid \sum_{i=1}^n |z_i| < 1\}$ is not biholomorphic to B_n

$$\begin{aligned}\|\phi_{\vec{0}}\|_{A_n} &= (4\pi^n) \int_{A_n} \cdots \int r_1 r_2 \cdots r_n dr_1 \cdots dr_n = \frac{(4\pi)^n}{\prod_{i=1}^n i}, \\ \|\phi_{\vec{e}_1}\|_{A_n} &= (4\pi^n) \int_{A_n} \cdots \int r_1^3 r_2 \cdots r_n dr_1 \cdots dr_n = \frac{(4\pi)^n}{2(n+1) \prod_{i=4}^n i}, \\ \|\phi_{\vec{\alpha}}\|_{A_n} &= (4\pi^n) \int_{A_n} \cdots \int r_1^5 r_2 \cdots r_n dr_1 \cdots dr_n = \frac{(4\pi)^n}{2(n+2) \prod_{i=6}^n i}.\end{aligned}$$

Using the same method, we get

$$g_{A_n}^\tau(\vec{\alpha}) = g_{A_n}^{\text{Id}}(\vec{\alpha}) = \frac{\|\phi_{\vec{0}}\|_{A_n}^{\sum \alpha_i - 1} \|\phi_{\vec{\alpha}}\|_{A_n}}{\prod_{i=1}^n \|\phi_{\vec{e}_i}\|_{A_n}^{\alpha_i}} = \frac{\sqrt{5(2n+1)(2n+2)}}{\sqrt{3(2n+3)(2n+4)}}.$$

So $\sum_{\tau \in S_n} g_{B_n}^\tau(\vec{\alpha}) \cdot \tau \neq \sum_{\tau \in S_n} g_{A_n}^\tau(\vec{\alpha}) \cdot \tau$ in $\mathbb{R}[S_n]/\sim$.

Then B_n is not biholomorphic to A_n by Theorem F.

Example 2.3.3.3. Δ_n is not biholomorphic to A_n .

According to the computation above,

$$\sum_{\tau \in S_n} g_{\Delta_n}^\tau(\vec{\alpha}) \cdot \tau \neq \sum_{\tau \in S_n} g_{A_n}^\tau(\vec{\alpha}) \cdot \tau \text{ in } \mathbb{R}[S_n]/\sim.$$

So Δ_n is not biholomorphic to A_n .

From the examples above, it is reasonable to suspect the problem lies with the fact that B_n has a C^∞ smooth boundary, whereas Δ_n and A_n have “corners” in its boundaries. However Δ_n is not biholomorphic to A_n though their boundaries both have “corners”. So the biholomorphic equivalence problem of domains in \mathbb{C}^n is very complicated even in the Reinhardt domain case.

CHAPTER 3

COMPLEX PLATEAU PROBLEM

3.1 Preliminaries

Kohn-Rossi cohomology was first introduced by Kohn-Rossi. Following Tanaka (28), we reformulate the definition in a way independent of the interior manifold.

Definition 3.1.0.4. *Let X be a connected orientable manifold of real dimension $2n - 1$. A CR structure on X is a rank $n - 1$ subbundle S of $\mathbb{C}T(X)$ (complexified tangent bundle) such that*

1. $S \cap \bar{S} = \{0\}$,
2. *If L, L' are local sections of S , then so is $[L, L']$.*

Such a manifold with a CR structure is called a CR manifold. There is a unique subbundle \mathcal{H} of $T(X)$ such that $\mathbb{C}\mathcal{H} = S \oplus \bar{S}$. Furthermore, there is a unique homomorphism $J : \mathcal{H} \rightarrow \mathcal{H}$ such that $J^2 = -1$ and $S = \{v - iJv : v \in \mathcal{H}\}$. The pair (\mathcal{H}, J) is called the real expression of the CR structure.

Let X be a CR manifold with structure S . For a complex valued C^∞ function u defined on X , the section $\bar{\partial}_b u \in \Gamma(S)$ is defined by

$$\bar{\partial}_b u(\bar{L}) = \bar{L}(u), L \in S.$$

The differential operator $\bar{\partial}_b$ is called the (tangential) Cauchy-Riemann operator, and a solution u of the equation $\bar{\partial}_b u = 0$ is called a holomorphic function.

Definition 3.1.0.5. *A complex vector bundle E over X is said to be holomorphic if there is a differential operator*

$$\bar{\partial}_E : \Gamma(E) \longrightarrow \Gamma(E) \otimes \bar{S}^*$$

satisfying the following conditions:

1. $\bar{\partial}_E(fu)(\bar{L}_1) = (\bar{\partial}_b f)(\bar{L}_1)u + f(\bar{\partial}_E u)(\bar{L}_1) = (\bar{L}_1 f)u + f(\bar{\partial}_E u)(\bar{L}_1),$
2. $(\bar{\partial}_E u)[\bar{L}_1, \bar{L}_2] = \bar{\partial}_E(\bar{\partial}_E u(\bar{L}_2))(\bar{L}_1) - \bar{\partial}_E(\bar{\partial}_E u(\bar{L}_1))(\bar{L}_2),$

where $u \in \Gamma(E)$, $f \in C^\infty(X)$ and $L_1, L_2 \in \Gamma(S)$.

The operator $\bar{\partial}_E$ is called the Cauchy-Riemann operator and a solution u of the equation $\bar{\partial}_E u = 0$ is called a holomorphic cross section.

A basic holomorphic vector bundle over a CR manifold X is the vector bundle $\hat{T}(X) = \mathbb{C}T(X)/\bar{S}$. The corresponding operator $\bar{\partial} = \bar{\partial}_{\hat{T}(X)}$ is defined as follows. Let p be the projection from $\mathbb{C}T(X)$ to $\hat{T}(X)$. Take any $u \in \Gamma(\hat{T}(X))$ and express it as $u = p(Z)$, $Z \in \Gamma(\mathbb{C}T(X))$. For any $L \in \Gamma(S)$, define a cross section $(\bar{\partial}u)(\bar{L})$ of $\hat{T}(X)$ by $(\bar{\partial}u)(\bar{L}) = p([\bar{L}, Z])$. One can show that $(\bar{\partial}u)(\bar{L})$ does not depend on the choice of Z and that $\bar{\partial}u$ gives a cross section of $\hat{T}(X) \otimes \bar{S}^*$. Furthermore one can show that the operator $u \mapsto \bar{\partial}u$ satisfies (1) and (2) of Definition 3.1.0.5, using the Jacobi identity in the Lie algebra $\Gamma(\mathbb{C}T(X))$. The resulting holomorphic vector bundle $\hat{T}(X)$ is called the holomorphic tangent bundle of X .

If X is a real hypersurface in a complex manifold M , we may identify $\widehat{T}(X)$ with the holomorphic vector bundle of all $(1,0)$ tangent vectors to M and $\widehat{T}(X)$ with the restriction of $\widehat{T}(M)$ to X . In fact, since the structure S of X is the bundle of all $(1,0)$ tangent vectors to X , the inclusion map $\mathbb{C}T(X) \longrightarrow \mathbb{C}T(X)$ induces a natural map $\widehat{T}(X) \xrightarrow{\phi} \widehat{T}(X)|_X$ which is a bundle isomorphism satisfying $\bar{\partial}(\phi(u))(\bar{L}) = \phi(\bar{\partial}u(\bar{L}))$, $u \in \Gamma(\widehat{T}(X))$, $L \in S$.

For a holomorphic vector bundle E over X , set

$$C^q(X, E) = E \otimes \wedge^q \bar{S}^*, \mathcal{C}^q(X, E) = \Gamma(C^q(X, E))$$

and define a differential operator

$$\bar{\partial}_E^q : \mathcal{C}^q(X, E) \longrightarrow \mathcal{C}^{q+1}(X, E)$$

by

$$\begin{aligned} (\bar{\partial}_E^q \phi)(\bar{L}_1, \dots, \bar{L}_{q+1}) &= \sum_i (-1)^{i+1} \bar{\partial}_E(\phi(\bar{L}_1, \dots, \widehat{\bar{L}}_i, \dots, \bar{L}_{q+1}))(\bar{L}_i) \\ &+ \sum_{i < j} (-1)^{i+j} \phi([\bar{L}_i, \bar{L}_j], \bar{L}_1, \dots, \widehat{\bar{L}}_i, \dots, \bar{L}_{q+1}) \end{aligned}$$

for all $\phi \in \mathcal{C}^q(X, E)$ and $L_1, \dots, L_{q+1} \in \Gamma(S)$. One shows by standard arguments that $\bar{\partial}_E^q \phi$ gives an element of $\mathcal{C}^{q+1}(X, E)$ and that $\bar{\partial}_E^{q+1} \bar{\partial}_E^q = 0$. The cohomology groups of the resulting complex $\{\mathcal{C}^q(X, E), \bar{\partial}_E^q\}$ is denoted by $H^q(X, E)$.

Let $\{\mathcal{A}^k(X), d\}$ be the De Rham complex of X with complex coefficients, and let $H^k(X)$ be the De Rham cohomology groups. There is a natural filtration of the De Rham complex,

as follows. For any integer p and k , put $A^k(X) = \wedge^k(\mathbb{C}T(X))^*$ and denote by $F^p(A^k(X))$ the subbundle of $A^k(X)$ consisting of all $\phi \in A^k(X)$ which satisfy the equality

$$\phi(Y_1, \dots, Y_{p-1}, \bar{Z}_1, \dots, \bar{Z}_{k-p+1}) = 0$$

for all $Y_1, \dots, Y_{p-1} \in \mathbb{C}T(X)_0$ and $Z_1, \dots, Z_{k-p+1} \in S_0$, 0 being the origin of ϕ . Then

$$\begin{aligned} A^k(X) &= F^0(A^k(X)) \supset F^1(A^k(X)) \supset \dots \\ &\supset F^k(A^k(X)) \supset F^{k+1}(A^k(X)) = 0 \end{aligned}$$

Setting $F^p(\mathcal{A}^k(X)) = \Gamma(F^p(A^k(X)))$, we have

$$\begin{aligned} \mathcal{A}^k(X) &= F^0(\mathcal{A}^k(X)) \supset F^1(\mathcal{A}^k(X)) \supset \dots \\ &\supset F^k(\mathcal{A}^k(X)) \supset F^{k+1}(\mathcal{A}^k(X)) = 0. \end{aligned}$$

Since clearly $dF^p(\mathcal{A}^k(X)) \subseteq F^p(\mathcal{A}^{k+1}(X))$, the collection $\{F^p(\mathcal{A}^k(X))\}$ gives a filtration of the De Rham complex.

We denote by $H_{KR}^{p,q}(X)$ the groups $E_1^{p,q}(X)$ of the spectral sequence $\{E_r^{p,q}(X)\}$ associated with the filtration $\{F^p(\mathcal{A}^k(X))\}$. We call $H_{KR}^{p,q}(X)$ the Kohn-Rossi cohomology group of type (p, q) . More explicitly, let

$$A^{p,q}(X) = F^p(A^{p+q}(X)), \mathcal{A}^{p,q}(X) = \Gamma(A^{p,q}(X)),$$

$$C^{p,q}(X) = A^{p,q}(X)/A^{p+1,q-1}(X), \mathcal{C}^{p,q}(X) = \Gamma(C^{p,q}(X)).$$

Since $d : \mathcal{A}^{p,q}(X) \longrightarrow \mathcal{A}^{p,q+1}(X)$ maps $\mathcal{A}^{p+1,q-1}(X)$ into $\mathcal{A}^{p+1,q}(X)$, it induces an operator $d'' : \mathcal{C}^{p,q}(X) \longrightarrow \mathcal{C}^{p,q+1}(X)$. $H_{KR}^{p,q}(X)$ are then the cohomology groups of the complex $\{\mathcal{C}^{p,q}(X), d''\}$.

Alternatively $H_{KR}^{p,q}(X)$ may be described in terms of the vector bundle $E^p = \wedge^p(\widehat{T}(X)^*)$. If for $\phi \in \Gamma(E^p)$, $u_1, \dots, u_p \in \Gamma(\widehat{T}(X))$, $Y \in S$, we define $(\bar{\partial}_{E^p}\phi)(\bar{Y}) = \bar{Y}\phi$ by

$$\bar{Y}\phi(u_1, \dots, u_p) = \bar{Y}(\phi(u_1, \dots, u_p)) + \sum_i (-1)^i \phi(\bar{Y}u_i, u_1, \dots, \widehat{u}_i, \dots, u_p)$$

where $\bar{Y}u_i = (\bar{\partial}_{\widehat{T}(X)}u_i)(\bar{Y})$, then we easily verify that E^p with $\bar{\partial}_{E^p}$ is a holomorphic vector bundle. Tanaka [Ta] proves that $C^{p,q}(X)$ may be identified with $C^q(X, E^p)$ in a natural manner such that

$$d''\phi = (-1)^p \bar{\partial}_{E^p}\phi, \phi \in \mathcal{C}^{p,q}(X).$$

Thus, $H_{KR}^{p,q}(X)$ may be identified with $H^q(X, E^p)$.

We denote by $H_h^k(X)$ the groups $E_2^{k,0}(X)$ of the spectral sequence $\{E_r^{p,q}(X)\}$ associated with the filtration $\{F^p(\mathcal{A}^k(X))\}$. We call $H_h^k(X)$ the holomorphic De Rham cohomology groups. The groups $H_h^k(X)$ are the cohomology groups of the complex $\{\mathcal{S}^k(X), d\}$, where we put $\mathcal{S}^k(X) = E_1^{k,0}(X)$ and $d = d_1 : E_1^{k,0} \longrightarrow E_1^{k+1,0}$. Recall that $\mathcal{S}^k(X)$ is the kernel of the following mapping:

$$\begin{aligned} d_0 : E_0^{k,0} = F^k \mathcal{A}^k &= \mathcal{A}^{k,0}(X) \\ &\rightarrow E_0^{k,1} = F^k \mathcal{A}^{k+1} / F^{k+1} \mathcal{A}^{k+1} = \mathcal{A}^{k,1}(X) / \mathcal{A}^{k+1,0}. \end{aligned}$$

Note that \mathcal{S} may be characterized as the space of holomorphic k -forms, namely holomorphic cross sections of E^k . Thus the complex $\{\mathcal{S}^k(X), d\}$ (respectively, the groups $H_h^k(X)$) will be called the holomorphic De Rham complex (respectively, the holomorphic De Rham cohomology groups).

Definition 3.1.0.6. *Let L_1, \dots, L_{n-1} be a local frame of the CR structure S on X so that $\bar{L}_1, \dots, \bar{L}_{n-1}$ is a local frame of \bar{S} . Since $S \oplus \bar{S}$ has complex codimension one in $\mathbb{C}T(X)$, we may choose a local section N of $\mathbb{C}T(X)$ such that $L_1, \dots, L_{n-1}, \bar{L}_1, \dots, \bar{L}_{n-1}, N$ span $\mathbb{C}T(X)$.*

We may assume that N is purely imaginary. Then the matrix (c_{ij}) defined by

$$[L_i, \bar{L}_j] = \sum_k a_{i,j}^k L_k + \sum_k b_{i,j}^k \bar{L}_k + c_{i,j} N$$

is Hermitian, and is called the Levi form of X .

Proposition 3.1.0.7. *The number of non-zero eigenvalues and the absolute value of the signature of (c_{ij}) at each point are independent of the choice of L_1, \dots, L_{n-1}, N .*

Definition 3.1.0.8. *X is said to be strictly pseudoconvex if the Levi form is positive definite at each point of X .*

Definition 3.1.0.9. *Let X be a CR manifold of real dimension $2n - 1$. X is said to be Calabi-Yau if there exists a nowhere vanishing holomorphic section in $\Gamma(\wedge^n \widehat{T}(X)^*)$, where $\widehat{T}(X)$ is the holomorphic tangent bundle of X .*

Remark. :

1. Let X be a CR manifold of real dimension $2n - 1$ in \mathbb{C}^n . Then X is a Calabi-Yau CR manifold.
2. Let X be a strictly pseudoconvex CR manifold of real dimension $2n - 1$ contained in the boundary of bounded strictly pseudoconvex domain in \mathbb{C}^{n+1} . Then X is a Calabi-Yau CR manifold.

3.2 Invariants of singularities and CR-invariants

3.2.1 Sheaves of germs of holomorphic p -forms

Let V be a n -dimensional complex analytic subvariety in \mathbb{C}^N with only isolated singularities.

In (39), Yau considered four kinds of sheaves of germs of holomorphic p -forms

1. $\bar{\Omega}_V^p := \pi_* \Omega_M^p$, where $\pi : M \longrightarrow V$ is a resolution of singularities of V ,
2. $\bar{\bar{\Omega}}_V^p := \theta_* \Omega_{V \setminus V_{sing}}^p$ where $\theta : V \setminus V_{sing} \longrightarrow V$ is the inclusion map and V_{sing} is the singular set of V ,

3. $\Omega_V^p := \Omega_{\mathbb{C}^N}^p / \mathcal{K}^p$, where $\mathcal{K}^p = \{f\alpha + dg \wedge \beta : \alpha \in \Omega_{\mathbb{C}^N}^p; \beta \in \Omega_{\mathbb{C}^N}^{p-1}; f, g \in \mathcal{I}\}$ and \mathcal{I} is the ideal sheaf of V in \mathbb{C}^N ,
4. $\tilde{\Omega}_V^p := \Omega_{\mathbb{C}^N}^p / \mathcal{H}^p$, where $\mathcal{H}^p = \{\omega \in \Omega_{\mathbb{C}^N}^p : \omega|_{V \setminus V_{sing}} = 0\}$.

Clearly $\Omega_V^p, \tilde{\Omega}_V^p$ are coherent. $\bar{\Omega}_V^p$ is a coherent sheaf because π is a proper map. $\bar{\bar{\Omega}}_V^p$ is also a coherent sheaf by a theorem of Siu (cf Theorem A of (40)). In case V is a normal variety, the dualizing sheaf ω_V of Grothendieck is actually the sheaf $\bar{\bar{\Omega}}_V^n$.

Definition 3.2.1.1. *The Siu complex is a complex of coherent sheaves J^\bullet supported on the singular points of V which is defined by the following exact sequence*

$$0 \longrightarrow \bar{\Omega}^\bullet \longrightarrow \bar{\bar{\Omega}}^\bullet \longrightarrow J^\bullet \longrightarrow 0. \quad (3.2.1)$$

Definition 3.2.1.2. *Let V be a n -dimensional Stein space with 0 as its only singular point. Let $\pi : (M, A) \rightarrow (V, 0)$ be a resolution of the singularity with A as exceptional set. The geometric genus p_g , the irregularity q and $g^{(p)}$ -invariant of the singularity are defined as follows (cf. (39), (41)):*

$$p_g := \dim \Gamma(M \setminus A, \Omega^n) / \Gamma(M, \Omega^n), \quad (3.2.2)$$

$$q := \dim \Gamma(M \setminus A, \Omega^{n-1}) / \Gamma(M, \Omega^{n-1}), \quad (3.2.3)$$

$$g^{(p)} := \dim \Gamma(M, \Omega_M^p) / \pi^* \Gamma(V, \Omega_V^p). \quad (3.2.4)$$

The s -invariant of the singularity is defined as follows

$$s := \dim \Gamma(M \setminus A, \Omega^n) / [\Gamma(M, \Omega^n) + d\Gamma(M \setminus A, \Omega^{n-1})]. \quad (3.2.5)$$

Lemma 3.2.1.3. (27) *Let V be a n -dimensional Stein space with 0 as its only singular point.*

Let $\pi : (M, A) \rightarrow (V, 0)$ be a resolution of the singularity with A as exceptional set. Let J^\bullet be the Siu complex of coherent sheaves supported on 0 . Then

1. $\dim J^n = p_g$,
2. $\dim J^{n-1} = q$,
3. $\dim J^i = 0$, for $1 \leq i \leq n - 2$

Proposition 3.2.1.4. (27) *Let V be a n -dimensional Stein space with 0 as its only singular point. Let $\pi : (M, A) \rightarrow (V, 0)$ be a resolution of the singularity with A as exceptional set. Let J^\bullet be the Siu complex of coherent sheaves supported on 0 . Then the s -invariant is given by*

$$s := \dim H^n(J^\bullet) = p_g - q \quad (3.2.6)$$

and

$$\dim H^{n-1}(J^\bullet) = 0 \quad (3.2.7)$$

3.2.2 New invariants

Definition 3.2.2.1. Let $(V, 0)$ be a Stein germ of a 2-dimensional analytic space with an isolated singularity at 0. Let $\pi : (M, A) \rightarrow (V, 0)$ be a resolution of the singularity with A as exceptional set. Define a sheaf of germs $\Omega_M^{1,1}$ by the sheaf associated to the presheaf

$$U \mapsto \langle \Gamma(U, \Omega_M^1) \wedge \Gamma(U, \Omega_M^1) \rangle,$$

where U is an open set of M . Let $\bar{\Omega}_V^{1,1} := \pi_* \Omega_M^{1,1}$.

Lemma 3.2.2.2. Let $(V, 0)$ be a Stein germ of a 2-dimensional analytic space with an isolated singularity at 0. Let $\pi : (M, A) \rightarrow (V, 0)$ be a resolution of the singularity with A as exceptional set. Then $\bar{\Omega}_V^{1,1}$ is coherent and there is a short exact sequence

$$0 \longrightarrow \bar{\Omega}_V^{1,1} \longrightarrow \bar{\Omega}_V^2 \longrightarrow \mathcal{F}^{(1,1)} \longrightarrow 0 \quad (3.2.8)$$

where $\mathcal{F}^{(1,1)}$ is a sheaf supported on the singular point of V . Let

$$F^{(1,1)}(M) := \Gamma(M, \Omega_M^2) / \langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle, \quad (3.2.9)$$

then $\dim \mathcal{F}_0^{(1,1)} = \dim F^{(1,1)}(M)$.

Proof. Since the sheaf of germ Ω_M^1 is coherent, for any point $w \in M$ there exists an open neighborhood U of w in M such that $\Gamma(U, \Omega_M^1)$ is finitely generated over $\Gamma(U, \mathcal{O}_M)$. So $\Gamma(U, \Omega_M^1) \wedge \Gamma(U, \Omega_M^1)$ is finitely generated over $\Gamma(U, \mathcal{O}_M)$, which means $\Gamma(U, \Omega_M^{1,1})$ is finitely

generated over $\Gamma(U, \mathcal{O}_M)$, i.e. $\Omega_M^{1,1}$ is a sheaf of finite type. It is obvious that $\Omega_M^{1,1}$ is a subsheaf of Ω_M^2 which is also coherent. So $\Omega_M^{1,1}$ is coherent. And π is a proper map, therefore $\bar{\Omega}_V^{1,1}$ is also coherent.

Notice that the stalk of $\bar{\Omega}_V^{1,1}$ and $\bar{\Omega}_V^2$ coincide at each point different from the singular point 0, $\mathcal{F}^{(1,1)}$ is supported at 0. And from Cartan Theorem B

$$\dim \mathcal{F}_0^{(1,1)} = \dim \Gamma(V, \bar{\Omega}_V^2) / \Gamma(V, \bar{\Omega}_V^{1,1}) = \dim F^{(1,1)}(M).$$

Q.E.D.

Observe that $\Gamma(M, \Omega_M^2)$ and $\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle$ are birational invariants. Thus, from Lemma 3.2.2.2, we can define a local invariant of a singularity which is independent of resolution.

Definition 3.2.2.3. *Let V be a 2-dimensional Stein space with 0 as its only singular point.*

Let $\pi : (M, A) \rightarrow (V, 0)$ be a resolution of the singularity with A as exceptional set. Let

$$f^{(1,1)}(0) := \dim \mathcal{F}_0^{(1,1)} = \dim F^{(1,1)}(M). \quad (3.2.10)$$

We will omit 0 in $f^{(1,1)}(0)$ if there is no confusion from the context.

Let X be a compact connected strictly pseudoconvex CR manifold of real dimension 3, in the boundary of a bounded strictly pseudoconvex domain D in \mathbb{C}^N . By Harvey and Lawson [Ha-La], there is a unique complex variety V in \mathbb{C}^N such that the boundary of V is X . Let

$\pi : (M, A_1, \dots, A_k) \rightarrow (V, 0_1, \dots, 0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i)$, $1 \leq i \leq k$, as exceptional sets. Then the dimension of the following vector space

$$F^{(1,1)}(M) := \Gamma(M, \Omega_M^2) / \langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle$$

is an invariant of strictly pseudoconvex CR manifold X . Moreover s -invariant defined in Definition 3.2.1.2 is also CR invariant which was also called $s(X)$.

Definition 3.2.2.4. *Let X be a compact connected strictly pseudoconvex CR manifold of real dimension 3, in the boundary of a bounded strictly pseudoconvex domain D in \mathbb{C}^N . Suppose V in \mathbb{C}^N such that the boundary of V is X . Let $\pi : (M, A_1, \dots, A_k) \rightarrow (V, 0_1, \dots, 0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i)$, $1 \leq i \leq k$, as exceptional sets. Then we set CR -invariant*

$$f^{(1,1)}(X) := \dim F^{(1,1)}(M). \quad (3.2.11)$$

$f^{(1,1)}(X)$ is independent of the resolution of V in view of the following Lemma 3.2.2.5, which is an easy consequence of Lemma 3.2.2.2.

Lemma 3.2.2.5. *If X is a compact connected strictly pseudoconvex CR manifold of real dimension 3, in the boundary of a bounded strictly pseudoconvex domain D in \mathbb{C}^N . Suppose V in \mathbb{C}^N such that the boundary of V is X . Let $\pi : (M, A_1, \dots, A_k) \rightarrow (V, 0_1, \dots, 0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i)$, $1 \leq i \leq k$, as exceptional sets. Then*

$$f^{(1,1)}(X) = \sum_i f^{(1,1)}(0_i) = \sum_i \dim \mathcal{F}_{0_i}^{(1,1)}.$$

The following proposition is to show that $f^{(1,1)}$ is bounded above by $g^{(2)}$.

Proposition 3.2.2.6. *Let V be a 2-dimensional Stein space with 0 as its only singular point. Then $f^{(1,1)} \leq g^{(2)}$.*

Proof. Since

$$f^{(1,1)} = \dim \Gamma(M, \Omega_M^2) / \langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle,$$

$$g^{(2)} := \dim \Gamma(M, \Omega^2) / \pi^* \Gamma(V, \Omega_V^2).$$

and

$$\pi^* \Gamma(V, \Omega_V^2) = \langle \pi^* \Gamma(V, \Omega_V^1) \wedge \pi^* \Gamma(V, \Omega_V^1) \rangle \subseteq \langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle,$$

the result follows. Q.E.D.

We want to use the CR -invariant $f^{(1,1)}(X)$ to solve the classical Plateau problem. However $f^{(1,1)}(X)$ is in terms of M although it is independent of M . It will be nice to find some CR -invariant which can be calculated directly from the boundary X .

For this purpose, we define another sheaf of germs $\bar{\Omega}_V^{1,1}$.

Definition 3.2.2.7. *Let $(V, 0)$ be a Stein germ of a 2-dimensional analytic space with an isolated singularity at 0. Let $\theta : V \setminus \{0\} \hookrightarrow V$ be the embedding map. Define a sheaf of germs $\Omega_{V \setminus 0}^{1,1}$ by the sheaf associated to the presheaf*

$$U \mapsto \langle \Gamma(U, \Omega_V^1) \wedge \Gamma(U, \Omega_V^1) \rangle,$$

where U is an open set of $V \setminus 0$. Let $\bar{\bar{\Omega}}_V^{1,1} := \theta_*(\Omega_{V \setminus 0}^{1,1})$.

Lemma 3.2.2.8. *Let V be a 2-dimensional Stein space with 0 as its only singular point in \mathbb{C}^N .*

Let $\pi : (M, A) \rightarrow (V, 0)$ be a resolution of the singularity with A as exceptional set. Then $\bar{\bar{\Omega}}_V^{1,1}$ is coherent and there is a short exact sequence

$$0 \longrightarrow \bar{\bar{\Omega}}_V^{1,1} \longrightarrow \bar{\bar{\Omega}}_V^2 \longrightarrow \mathcal{G}^{(1,1)} \longrightarrow 0 \quad (3.2.12)$$

where $\mathcal{G}^{(1,1)}$ is a sheaf supported on the singular point of V . Let

$$G^{(1,1)}(M \setminus A) := \Gamma(M \setminus A, \Omega_M^2) / \langle \Gamma(M \setminus A, \Omega_M^1) \wedge \Gamma(M \setminus A, \Omega_M^1) \rangle, \quad (3.2.13)$$

then $\dim \mathcal{G}_0^{(1,1)} = \dim G^{(1,1)}(M \setminus A)$.

Proof. Since the sheaf of germ $\bar{\bar{\Omega}}_V^1$ is coherent by a theorem of Siu (cf Theorem A of [Si]), for any point $w \in V$ there exists an open neighborhood U of w in V such that $\Gamma(U, \bar{\bar{\Omega}}_V^1)$ is finitely generated over $\Gamma(U, \mathcal{O}_V)$. So $\Gamma(U, \bar{\bar{\Omega}}_V^1) \wedge \Gamma(U, \bar{\bar{\Omega}}_V^1)$ is finitely generated over $\Gamma(U, \mathcal{O}_V)$, which means $\Gamma(U, \bar{\bar{\Omega}}_V^{1,1})$ is finitely generated over $\Gamma(U, \mathcal{O}_V)$, i.e. $\bar{\bar{\Omega}}_V^{1,1}$ is a sheaf of finite type. It is obvious that $\bar{\bar{\Omega}}_V^{1,1}$ is a subsheaf of $\bar{\bar{\Omega}}_V^2$ which is also coherent. So $\bar{\bar{\Omega}}_V^{1,1}$ is coherent.

Notice that the stalk of $\bar{\bar{\Omega}}_V^{1,1}$ and $\bar{\bar{\Omega}}_V^2$ coincide at each point different from the singular point 0 , $\mathcal{G}^{(1,1)}$ is supported at 0 . And from Cartan Theorem B

$$\dim \mathcal{G}_0^{(1,1)} = \dim \Gamma(V, \bar{\bar{\Omega}}_V^2) / \Gamma(V, \bar{\bar{\Omega}}_V^{1,1}) = \dim G^{(1,1)}(M \setminus A).$$

Q.E.D.

Thus, from Lemma 3.2.2.8, we can define another local invariant of a singularity which is independent on resolution.

Definition 3.2.2.9. *Let V be a 2-dimensional Stein space with 0 as its only singular point.*

Let $\pi : (M, A) \rightarrow (V, 0)$ be a resolution of the singularity with A as exceptional set. Let

$$g^{(1,1)}(0) := \dim \mathcal{G}_0^{(1,1)} = \dim G^{(1,1)}(M \setminus A). \quad (3.2.14)$$

We will omit 0 in $g^{(1,1)}(0)$ if there is no confusion from the context.

Let $\pi : (M, A_1, \dots, A_k) \rightarrow (V, 0_1, \dots, 0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i)$, $1 \leq i \leq k$, as exceptional sets. In this case we still let

$$G^{(1,1)}(M \setminus A) := \Gamma(M \setminus A, \Omega_M^2) / \langle \Gamma(M \setminus A, \Omega_M^1) \wedge \Gamma(M \setminus A, \Omega_M^1) \rangle .$$

Definition 3.2.2.10. *If X is a compact connected strictly pseudoconvex CR manifold of real dimension 3, in the boundary of a bounded strictly pseudoconvex domain D in \mathbb{C}^N . Suppose V in \mathbb{C}^N such that the boundary of V is X . Let $\pi : (M, A = \bigcup_i A_i) \rightarrow (V, 0_1, \dots, 0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i)$, $1 \leq i \leq k$, as exceptional sets. Let*

$$G^{(1,1)}(M \setminus A) := \Gamma(M \setminus A, \Omega_M^2) / \langle \Gamma(M \setminus A, \Omega_M^1) \wedge \Gamma(M \setminus A, \Omega_M^1) \rangle \quad (3.2.15)$$

and

$$G^{(1,1)}(X) := \mathcal{S}^2(X) / \langle \mathcal{S}^1(X) \wedge \mathcal{S}^1(X) \rangle \quad (3.2.16)$$

where \mathcal{S}^p are holomorphic cross sections of $\wedge^p(\widehat{T}(X)^*)$. Then we set

$$g^{(1,1)}(M \setminus A) := \dim G^{(1,1)}(M \setminus A), \quad (3.2.17)$$

$$g^{(1,1)}(X) := \dim G^{(1,1)}(X). \quad (3.2.18)$$

Lemma 3.2.2.11. *If X is a compact connected strictly pseudoconvex CR manifold of real dimension 3, in the boundary of a bounded strictly pseudoconvex domain D in \mathbb{C}^N . Suppose V in \mathbb{C}^N such that the boundary of V is X . Let $\pi : (M, A_1, \dots, A_k) \rightarrow (V, 0_1, \dots, 0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i)$, $1 \leq i \leq k$, as exceptional sets. Then $g^{(1,1)}(X) = g^{(1,1)}(M \setminus A)$.*

Proof. Take a one-convex exhausting function ϕ on M such that $\phi \geq 0$ on M and $\phi(y) = 0$ if and only if $y \in A$. Set $M_r = \{y \in M, \phi(y) \geq r\}$. Since $X = \partial M$ is strictly pseudoconvex, any holomorphic p -form $\theta \in \mathcal{S}^p(X)$ can be extended to a one side neighborhood of X in M . Hence, θ can be thought of as holomorphic p -form on M_r , i.e. an element in $\Gamma(M_r, \Omega_{M_r}^p)$. By Andreotti and Grauert ([An-Gr]), $\Gamma(M_r, \Omega_{M_r}^p)$ is isomorphic to $\Gamma(M \setminus A, \Omega_M^p)$. So $g^{(1,1)}(X) = g^{(1,1)}(M \setminus A)$. Q.E.D.

By Lemma 3.2.2.8 and 3.2.2.11, we can get the following lemma easily.

Lemma 3.2.2.12. *If X is a compact connected strictly pseudoconvex CR manifold of real dimension 3, in the boundary of a bounded strictly pseudoconvex domain D in \mathbb{C}^N . Suppose V in \mathbb{C}^N such that the boundary of V is X . Let $\pi : (M, A_1, \dots, A_k) \rightarrow (V, 0_1, \dots, 0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i)$, $1 \leq i \leq k$, as exceptional sets. Then $g^{(1,1)}(X) = \sum_i g^{(1,1)}(0_i) = \sum_i \dim \mathcal{G}_{0_i}^{(1,1)}$.*

The following proposition is to show that $g^{(1,1)}$ is bounded above.

Proposition 3.2.2.13. *Let V be a 2-dimensional Stein space with 0 as its only singular point.*

Then $g^{(1,1)} \leq p_g + g^{(2)}$.

Proof. Since

$$g^{(1,1)} = \dim \Gamma(M \setminus A, \Omega_M^2) / \langle \Gamma(M \setminus A, \Omega_M^1) \wedge \Gamma(M \setminus A, \Omega_M^1) \rangle,$$

$$p_g = \dim \Gamma(M \setminus A, \Omega_M^2) / \Gamma(M, \Omega_M^2),$$

$$g^{(2)} := \dim \Gamma(M, \Omega^2) / \pi^* \Gamma(V, \Omega_V^2).$$

and

$$\begin{aligned} \pi^* \Gamma(V, \Omega_V^2) &= \langle \pi^* \Gamma(V, \Omega_V^1) \wedge \pi^* \Gamma(V, \Omega_V^1) \rangle \\ &\subseteq \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \\ &\subseteq \Gamma(M \setminus A, \Omega_M^1) \wedge \Gamma(M \setminus A, \Omega_M^1), \end{aligned} \tag{3.2.19}$$

the result follows.

Q.E.D.

The following two theorems are the crucial parts for the classical Plateau problem.

Theorem 3.2.2.14. *Let V be a 2-dimensional Stein space with 0 as its only normal singular point with \mathbb{C}^* -action. Let $\pi : (M, A) \rightarrow (V, 0)$ be a minimal good resolution of the singularity with A as exceptional set, then $f^{(1,1)} \geq 1$.*

Proof. It suffices to prove there is a holomorphic 2-form ω on M which is not contained in $\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle$. Embed $(V, 0)$ locally into \mathbb{C}^m and let z_1, \dots, z_m be coordinate function of \mathbb{C}^m .

If $p_g > 0$, there exists a holomorphic 2-form ω_0 on $M \setminus A$ but not on M , i.e. $\omega_0 \in \Gamma(M \setminus A, \Omega_M^2) \setminus \Gamma(M, \Omega_M^2)$. So ω_0 must have pole along some irreducible component A_1 of A . Suppose ω_1 has the lowest order of pole along A_1 such that $\omega_1 \in \Gamma(M \setminus A, \Omega_M^2) \setminus \Gamma(M, \Omega_M^2)$. Then $\pi^*(z_j)\omega_1$ is holomorphic along A_1 for all j , $1 \leq j \leq m$. If $\pi^*(z_j)\omega_1 \notin \Gamma(M, \Omega_M^2)$, it must have pole along another irreducible component A_2 of A . Suppose $\omega_2 \in \Gamma(M \setminus A, \Omega_M^2)$ has the lowest order of poles along A_2 and holomorphic along A_1 . Then $\pi^*(z_j)\omega_2$ is holomorphic along A_1 and A_2 , for all j , $1 \leq j \leq m$. Since the number of irreducible components of A is finite, by induction, there exists a holomorphic 2-form $\omega_k \in \Gamma(M \setminus A, \Omega_M^2)$ which has the lowest order of pole along some irreducible component A_k and holomorphic along A_1, \dots, A_{k-1} , and $\pi^*(z_j)\omega_k \in \Gamma(M, \Omega_M^2)$ for all j , $1 \leq j \leq m$. Choose a point b in A_k which is a smooth point of A . Let (x_1, x_2) be a coordinate system centered at b such that A_k is given locally by $x_1 = 0$ at b . Take the power series expansion of $\pi^*(z_j)$ around b :

$$\pi^*(z_j) \doteq x_1^{T_j} f_j, 1 \leq j \leq m, \quad (3.2.20)$$

where f_j is holomorphic function such that $f_j(0, x_2) \neq 0$ and “ \doteq ” means local equality around b . Without loss of generality, we can suppose $r_1 = \min\{r_1, \dots, r_m\}$. By local calculation, we know that for any element ψ in $\pi^*\Gamma(V, \Omega_V^2)$, the vanishing order of ψ along A_k , which is denoted by $Ord_{A_k}\psi$, is at least $2r_1 - 1$. However $Ord_{A_k}\pi^*(z_1)\omega_k$ is at most $r_1 - 1$. So $\pi^*(z_1)\omega_k \in \Gamma(M, \Omega_M^2) \setminus \pi^*\Gamma(V, \Omega_V^2)$. We pick such kind $\omega \in \Gamma(M, \Omega_M^2) \setminus \pi^*\Gamma(V, \Omega_V^2)$ which has the lowest order of zeros, r , along A_k , i.e. $Ord_{A_k}\omega = r$. So $r < r_1$.

Let $\xi_V \in \Gamma(V, \Theta_V)$, where $\Theta_V := \mathcal{H}om_{\mathcal{O}_V}(\Omega_V^1, \mathcal{O}_V)$, denote the generating vector field of the \mathbb{C}^* -action and let i_{ξ_V} denote the contraction map. For some $\alpha \in \Gamma(V, \bar{\Omega}_V^1)$, write α as a sum $\sum \alpha^j$ of quasi-homogeneous elements where α^j is a quasi-homogeneous element of degree $l_j > 0$. Let $L_{\xi_V} = i_{\xi_V}d + di_{\xi_V}$ be the Lie derivation. Then

$$l_j\alpha^j = L_{\xi_V}\alpha^j = i_{\xi_V}d(\alpha^j) + di_{\xi_V}(\alpha^j).$$

So

$$\Gamma(V, \bar{\Omega}_V^1) = d(\Gamma(V, \mathcal{O}_V)) + i_{\xi_V}(\Gamma(V, \bar{\Omega}_V^2)). \quad (3.2.21)$$

Since for minimal good resolution, we have $\pi_*\Theta_M = \Theta_V$ (cf (42)), where Θ_M is the vector field on M . Thus, there exists ξ_M which is a lift of ξ_V , i.e. $\pi_*\xi_M = \xi_V$. We know that ξ_M is tangential to the exceptional set, so

$$\xi_M \doteq x_1^{a_1}p \frac{\partial}{\partial x_1} + x_1^{a_2}q \frac{\partial}{\partial x_2}, a_1 \geq 1, a_2 \geq 0$$

where p and q are holomorphic functions.

Let $i_{\xi_M} : \Gamma(M, \Omega_M^2) \longrightarrow \Gamma(M, \Omega_M^1)$ be the contraction map corresponding to i_{ξ_V} . If $\zeta \in \Gamma(M, \Omega_M^2)$ and $\zeta \doteq x_1^u g dx_1 \wedge dx_2$, then

$$i_{\xi_M}(\zeta) \doteq i_{\xi_M}(x_1^u g dx_1 \wedge dx_2) = -x_1^{u+a_2} q g dx_1 + x_1^{u+a_1} p g dx_2.$$

From Equation 3.2.21,

$$\Gamma(M, \Omega_M^1) = d(\Gamma(M, \mathcal{O}_M)) + i_{\xi_M}(\Gamma(M, \Omega_M^2)).$$

Since V is normal, $g^{(0)} = 0$, i.e. $\Gamma(M, \mathcal{O}_M) = \pi^*(\Gamma(V, \mathcal{O}_V))$.

We now prove that ω is not contained in $\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle$. Consider $\eta, \varphi \in \Gamma(M, \Omega_M^1)$ locally around b .

Suppose $\eta = \eta_1 + \eta_2$ and $\varphi = \varphi_1 + \varphi_2$, where $\eta_1, \varphi_1 \in d(\Gamma(M, \mathcal{O}_M))$, $\eta_2, \varphi_2 \in i_{\xi_M}(\Gamma(M, \Omega_M^2))$.

Let

$$\eta_2 = i_{\xi_M}(\zeta), \quad \zeta \doteq x_1^u g dx_1 \wedge dx_2, \quad g(0, x_2) \neq 0,$$

and

$$\varphi_2 = i_{\xi_M}(\varsigma), \quad \varsigma \doteq x_1^v h dx_1 \wedge dx_2, \quad h(0, x_2) \neq 0.$$

Then

$$\eta \wedge \varphi = \eta_1 \wedge \varphi_1 + (\eta_1 \wedge \varphi_2 + \eta_2 \wedge \varphi_1) + \eta_2 \wedge \varphi_2.$$

Since

$$d\pi^*(z_i) \wedge d\pi^*(z_j) = (r_i x_1^{r_i+r_j-1} f_i \frac{\partial f_j}{\partial x_2} - r_j x_1^{r_i+r_j-1} f_j \frac{\partial f_i}{\partial x_2}) dx_1 \wedge dx_2,$$

$$Ord_{A_k} \eta_1 \wedge \varphi_1 \geq 2 \cdot r_1 - 1 > r.$$

Write η_2 and φ_2 locally around b:

$$\eta_2 \doteq -x_1^{u+a_2} qg dx_1 + x_1^{u+a_1} pg dx_2,$$

$$\varphi_2 \doteq -x_1^{v+a_2} qh dx_1 + x_1^{v+a_1} ph dx_2.$$

$$\text{So } \eta_2 \wedge \varphi_2 \doteq 0.$$

Also notice that

$$d\pi^*(z_j) = r_j x_1^{r_j-1} f_j dx_1 + x_1^{r_j} \frac{\partial f_j}{\partial x_2} dx_2$$

So

$$Ord_{A_k} \eta_1 \wedge \varphi_2 \geq r_1 + v > r$$

and

$$Ord_{A_k} \eta_2 \wedge \varphi_1 \geq r_1 + u > r.$$

From the discussion above, we can get $Ord_{A_k} \eta \wedge \varphi > r$.

Therefore ω is not a linear combination of elements in $\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle$

If $p_g = 0$, from (43), the canonical bundle K_M is generated by its global sections in a neighborhood of the exceptional set. So there exists $\omega \in \Gamma(M, \Omega_M^2)$ such that ω does not vanish

along some irreducible component A_k of A . The rest of the argument is same as those in the case of $p_g > 0$, with r is 0. Q.E.D.

Theorem 3.2.2.15. *Let V be a 2-dimensional Stein space with 0 as its only normal singular point with \mathbb{C}^* -action. Let $\pi : (M, A) \rightarrow (V, 0)$ be a minimal good resolution of the singularity with A as exceptional set, then $g^{(1,1)} \geq 1$.*

Proof. If $\dim \Gamma(M \setminus A, \Omega_M^2) / \Gamma(M, \Omega_M^2) > 0$, then there exists

$$\omega_0 \in \Gamma(M \setminus A, \Omega_M^2) \setminus \Gamma(M, \Omega_M^2).$$

So ω_0 must have pole along some irreducible component A_k of A . Suppose ω has the highest order of pole along A_k and $\omega \in \Gamma(M \setminus A, \Omega_M^2)$. Denote $Ord_{A_k} \omega = r < 0$. Let z_1, \dots, z_m be coordinate function of \mathbb{C}^m . Choose a point b in A_k which is a smooth point of A . Let (x_1, x_2) be a coordinate system centered at b such that A_k is given locally by $x_1 = 0$ at b . Take the power series expansion of $\pi^*(z_j)$ around b .

$$\pi^*(z_j) = x_1^{r_j} f_j, 1 \leq j \leq m, \tag{3.2.22}$$

where f_j is holomorphic function such that $f_j(0, x_2) \neq 0$. So by the choice of ω , $\min\{r_1, \dots, r_m\} > 0 > r$.

Let $\xi_V \in \Gamma(V, \Theta_V)$, where $\Theta_V := \mathcal{H}om_{\mathcal{O}_V}(\Omega_V^1, \mathcal{O}_V)$, denote the generating vector field of the \mathbb{C}^* -action, i_{ξ_V} the contraction map. For some $\alpha \in \Gamma(V, \bar{\Omega}_V^1)$, write α as a sum $\sum \alpha^j$ of

quasi-homogeneous elements where α^j is a quasi-homogeneous element of degree $l_j > 0$. Let

$L_{\xi_V} = i_{\xi_V}d + di_{\xi_V}$ be the Lie derivation. Then

$$l_j\alpha^j = L_{\xi_V}\alpha^j = i_{\xi_V}d(\alpha^j) + di_{\xi_V}(\alpha^j).$$

So

$$\Gamma(V, \bar{\Omega}_V^1) = d(\Gamma(V, \mathcal{O}_V)) + i_{\xi_V}(\Gamma(V, \bar{\Omega}_V^2)). \quad (3.2.23)$$

Let ξ_M be a lift of ξ_V which is mentioned in the last theorem.

$$\xi_M \doteq x_1^{a_1}p \frac{\partial}{\partial x_1} + x_1^{a_2}q \frac{\partial}{\partial x_2}, a_1 \geq 1, a_2 \geq 0$$

where p and q are holomorphic functions.

Let $i_{\xi_M} : \Gamma(M \setminus A, \Omega_M^2) \longrightarrow \Gamma(M \setminus A, \Omega_M^1)$ be the contraction map corresponding to i_{ξ_V} . If

$\zeta \in \Gamma(M \setminus A, \Omega_M^2)$ and $\zeta \doteq x_1^u g dx_1 \wedge dx_2$, then

$$i_{\xi_M}(\zeta) \doteq i_{\xi_M}(x_1^u g dx_1 \wedge dx_2) = -x_1^{u+a_2} q g dx_1 + x_1^{u+a_1} p g dx_2.$$

From Equation 3.2.23,

$$\Gamma(M \setminus A, \Omega_M^1) = d(\Gamma(M \setminus A, \mathcal{O}_M)) + i_{\xi_M}(\Gamma(M \setminus A, \Omega_M^2)).$$

Since V is normal, $g^{(0)} = 0$, i.e. $\Gamma(M, \mathcal{O}_M) = \pi^*(\Gamma(V, \mathcal{O}_V))$. Moreover, by the normality of $(V, 0)$, $\Gamma(M, \mathcal{O}_M) = \Gamma(M \setminus A, \mathcal{O}_M)$.

We now prove that ω is not contained in $\langle \Gamma(M \setminus A, \Omega_M^1) \wedge \Gamma(M \setminus A, \Omega_M^1) \rangle$. Consider $\eta, \varphi \in \Gamma(M \setminus A, \Omega_M^1)$ locally around b

Suppose $\eta = \eta_1 + \eta_2$ and $\varphi = \varphi_1 + \varphi_2$, where $\eta_1, \varphi_1 \in d(\Gamma(M, \mathcal{O}_M))$, $\eta_2, \varphi_2 \in i_{\xi_M}(\Gamma(M \setminus A, \Omega_M^2))$.

Let

$$\eta_2 = i_{\xi_M}(\zeta), \quad \zeta \doteq x_1^u g dx_1 \wedge dx_2, \quad g(0, x_2) \neq 0,$$

and

$$\varphi_2 = i_{\xi_M}(\varsigma), \quad \varsigma \doteq x_1^v h dx_1 \wedge dx_2, \quad h(0, x_2) \neq 0.$$

So u and v are bounded lower by r .

Then

$$\eta \wedge \varphi = \eta_1 \wedge \varphi_1 + (\eta_1 \wedge \varphi_2 + \eta_2 \wedge \varphi_1) + \eta_2 \wedge \varphi_2.$$

Since

$$d\pi^*(z_i) \wedge d\pi^*(z_j) = (r_i x_1^{r_i+r_j-1} f_i \frac{\partial f_j}{\partial x_2} - r_j x_1^{r_i+r_j-1} f_j \frac{\partial f_i}{\partial x_2}) dx_1 \wedge dx_2,$$

$Ord_{A_k} \eta_1 \wedge \varphi_1 \geq 2 \cdot \min\{r_1, \dots, r_m\} - 1 > r$.

Write η_2 and φ_2 locally around b :

$$\eta_2 \doteq -x_1^{u+a_2} q g dx_1 + x_1^{u+a_1} p g dx_2,$$

$$\varphi_2 \doteq -x_1^{v+a_2} q h dx_1 + x_1^{v+a_1} p h dx_2.$$

So $\eta_2 \wedge \varphi_2 \stackrel{\circ}{=} 0$.

Also notice that

$$d\pi^*(z_j) = r_j x_1^{r_j-1} f_j dx_1 + x_1^{r_j} \frac{\partial f_j}{\partial x_2} dx_2$$

So

$$Ord_{A_k} \eta_1 \wedge \varphi_2 \geq \min\{r_1, \dots, r_m\} + v > r$$

and

$$Ord_{A_k} \eta_2 \wedge \varphi_1 \geq \min\{r_1, \dots, r_m\} + u > r.$$

From the discussion above, we can get $Ord_{A_k} \eta \wedge \varphi > r$.

Therefore ω is not a linear combination of elements in $\langle \Gamma(M \setminus A, \Omega_M^1) \wedge \Gamma(M \setminus A, \Omega_M^1) \rangle$

If $\dim \Gamma(M \setminus A, \Omega_M^2) / \Gamma(M, \Omega_M^2) = 0$, the singularity is rational. So irregularity $q = 0$ (cf.

(44)). Then

$$\frac{\Gamma(M \setminus A, \Omega_M^2)}{\langle \Gamma(M \setminus A, \Omega_M^1) \wedge \Gamma(M \setminus A, \Omega_M^1) \rangle} = \frac{\Gamma(M, \Omega_M^2)}{\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle},$$

$$g^{(1,1)} = \dim \frac{\Gamma(M, \Omega_M^2)}{\langle \Gamma(M, \Omega_M^1) \wedge \Gamma(M, \Omega_M^1) \rangle} = f^{(1,1)} \geq 1.$$

Q.E.D.

3.3 The complex Plateau problem

In 1981, Yau (26) solved the classical complex Plateau problem for the case $n \geq 3$.

Theorem 3.3.0.1. (26) *Let X is a compact connected strictly pseudoconvex CR manifold of real dimension $2n - 1$, $n \geq 3$, in the boundary of a bounded strictly pseudoconvex domain D in \mathbb{C}^{n+1} . Then X is a boundary of the complex sub-manifold $V \subset D - X$ if and only if Kohn-Rossi cohomology groups $H_{KR}^{p,q}(X)$ are zeros for $1 \leq q \leq n - 2$*

Next we want to use our new invariants introduced in § 3 to solve the classical complex Plateau problem for the case $n = 2$.

First, we present some known results in the paper (27).

Theorem 3.3.0.2. (27) *Let X be a compact connected $(2n - 1)$ -dimensional ($n \geq 2$) strictly pseudoconvex CR manifold. Suppose that X is the boundary of a n -dimensional strictly pseudoconvex manifold M which is a modification of a Stein space V with only isolated singularities $\{0_1, \dots, 0_m\}$. Let A be the maximal compact analytic set in M which can be blown down to $\{0_1, \dots, 0_m\}$. Then*

1. $H_h^q(X) \cong H_h^q(M \setminus A) \cong H_h^q(M) \quad 1 \leq q \leq n - 1$
2. $H_h^n(X) \cong H_h^n(M \setminus A)$, $\dim H_h^n(M \setminus A) = \dim H_h^n(M) + s$, where $s = s_1 + \dots + s_m$, s_i is the s -invariant of the singularity $(V, 0_i)$.

Theorem 3.3.0.3. (27) *Let $(V, 0)$ be a Gorenstein surface singularity. Let $\pi : M \rightarrow V$ be a good resolution with $A = \pi^{-1}(0)$ as exceptional set. Assume that M is contractible to A . If $s = 0$, then $(V, 0)$ is a quasi-homogeneous singularity, $H^1(A, \mathbb{C}) = 0$, $\dim H^1(M, \Omega^1) = \dim H^2(A, \mathbb{C}) + \dim H^1(M, \mathcal{O})$, and $H_h^1(M) = H_h^2(M) = 0$. Conversely, if $(V, 0)$ is a two*

dimensional quasi-homogeneous Gorenstein singularity and $H^1(A, \mathbb{C}) = 0$, then the s -invariant vanishes.

Theorem 3.3.0.4. (27) *Let X be a strictly pseudoconvex compact Calabi-Yau CR manifold of dimension 3. Suppose that X is contained in the boundary of a strictly pseudoconvex bounded domain D in \mathbb{C}^N . If the holomorphic De Rham cohomology $H_h^2(X) = 0$, then X is a boundary of a complex variety V in D with boundary regularity and V has only isolated singularities in the interior and the normalizations of these singularities are Gorenstein surface singularities with vanishing s -invariant.*

Corollary 3.3.0.5. (27) *Let X be a strictly pseudoconvex compact CR manifold of dimension 3. Suppose that X is contained in the boundary of a strictly pseudoconvex bounded domain D in \mathbb{C}^3 . If the holomorphic De Rham cohomology $H_h^2(X) = 0$, then X is a boundary of a complex variety V in D with boundary regularity and V has only isolated quasi-homogeneous singularities such that the dual graphs of the exceptional sets in the resolution are star shaped and all the curves are rational.*

So from several theorems above we can see that, in the paper (27), Luk and Yau give a sufficient condition $H_h^2(X) = 0$ to determine when X can bound some special singularities. However, even if both $H_h^1(X)$ and $H_h^2(X)$ vanish, V still can be singular.

We use CR invariants given in the last section to get sufficient and necessary conditions for the variety bounded by X being smooth.

Lemma 3.3.0.6. *Let $(V, 0)$ be the germ of a 2-dimensional isolated Gorenstein singularity, then the singularity is normal.*

Proof. For a variety, the dualizing sheaf of module is always reflexive, which means that the depth of the module at the singularity is at least 2. If $(V, 0)$ is an isolated non-normal singularity, then the local ring at the singular point has depth 1. So the dualizing sheaf can not be invertible which contradicts to the singularity being Gorenstein. Q.E.D.

Theorem 3.3.0.7. *Let X be a strictly pseudoconvex compact Calabi-Yau CR manifold of dimension 3. Suppose that X is contained in the boundary of a strictly pseudoconvex bounded domain D in \mathbb{C}^N . Then X is a boundary of the complex sub-manifold $V \subset D - X$ with boundary regularity if and only if s -invariant $s(X)$ and $f^{(1,1)}(X)$ vanish.*

Proof. (\Rightarrow): By Theorem 3.3.0.2 s -invariant $s(X) = 0$. Since every holomorphic 2 form can be expressed by linear combination of exterior product of two holomorphic 1 forms locally in manifold V , $f^{(1,1)}(X) = 0$ follows from Lemma 3.2.2.5.

(\Leftarrow): It is well known that X is a boundary of a variety V in D with boundary regularity (21). Since $s = 0$, X is a boundary of the complex sub-manifold $V \subset D - X$ with only isolated Gorenstein quasi-homogeneous singularities $\{0_1, \dots, 0_k\}$. Recall the singularities are normal by Lemma 3.3.0.6. From Theorem 3.3.0.3, $H_h^1(M_i) = 0$, where M_i is the minimal good resolution of a sufficiently small neighborhood of 0_i in V , $1 \leq i \leq k$. From Theorem 3.2.2.14, $\dim F^{(1,1)}(M_i) > 0$ which contradict to $f^{(1,1)}(X) = 0$. So V is smooth. Q.E.D.

Corollary 3.3.0.8. *Let X be a strictly pseudoconvex compact CR manifold of dimension 3. Suppose that X is contained in the boundary of a strictly pseudoconvex bounded domain D in \mathbb{C}^3 . Then X is a boundary of the complex sub-manifold $V \subset D - X$ if and only if s -invariant $s(X)$ and $f^{(1,1)}(X)$ vanish.*

Theorem 3.3.0.9. *Let X be a strictly pseudoconvex compact Calabi-Yau CR manifold of dimension 3. Suppose that X is contained in the boundary of a strictly pseudoconvex bounded domain D in \mathbb{C}^N . Then X is a boundary of the complex sub-manifold $V \subset D - X$ with boundary regularity and the variety is smooth if and only if s -invariant and $g^{(1,1)}(X)$ vanish.*

Proof. (\Rightarrow) : Since V is smooth, $g^{(1,1)}(X) = f^{(1,1)}(X)$. The result is from Theorem 3.3.0.7.

(\Leftarrow) : It is well known that X is a boundary of a variety V in D with boundary regularity (21). Since $s = 0$, X is a boundary of the complex sub-manifold $V \subset D - X$ with only isolated Gorenstein quasi-homogeneous singularities $\{0_1, \dots, 0_k\}$. Recall the singularities are normal by Lemma 3.3.0.6. Let $\pi_i : M_i \rightarrow V_i$ be the minimal good resolution of a sufficiently small neighborhood V_i of 0_i in V , $1 \leq i \leq k$. From Theorem 3.2.2.15, $\dim G^{(1,1)}(M_i) > 0$ which contradict to $g^{(1,1)}(X) = 0$. So V is smooth. Q.E.D.

Corollary 3.3.0.10. *Let X be a strictly pseudoconvex compact CR manifold of dimension 3. Suppose that X is contained in the boundary of a strictly pseudoconvex bounded domain D in \mathbb{C}^3 . Then X is a boundary of the complex sub-manifold $V \subset D - X$ if and only if s -invariant and $g^{(1,1)}(X)$ vanish.*

From Theorem 3.3.0.2, we know that if $H_h^2(X) = 0$, then $s = 0$. So we can get a necessary and sufficient condition in terms of boundary X with $H_h^2(X) = 0$ to determine when X is a boundary of a manifold.

Corollary 3.3.0.11. *Let X be a strictly pseudoconvex compact Calabi-Yau CR manifold of dimension 3. Suppose that X is contained in the boundary of a strictly pseudoconvex bounded domain D in \mathbb{C}^N with $H_h^2(X) = 0$. Then X is a boundary of the complex sub-manifold $V \subset D - X$ with boundary regularity if and only if $g^{(1,1)}(X) = 0$.*

Corollary 3.3.0.12. *Let X be a strictly pseudoconvex compact CR manifold of dimension 3. Suppose that X is contained in the boundary of a strictly pseudoconvex bounded domain D in \mathbb{C}^3 with $H_h^2(X) = 0$. Then X is a boundary of the complex sub-manifold $V \subset D - X$ if and only if $g^{(1,1)}(X) = 0$.*

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- UIC-Purdue Workshop on Algebraic Geometry and Commutative Algebra (Purdue University, West Lafayette, Indiana, USA, 2006).
- Give a talk “Rational Double Points” at graduate Algebraic Geometry seminar (UIC, Chicago, Illinois, USA, 2006).
- Give a talk “One-parameter subgroups and limit points in toric variety” at graduate Algebraic Geometry seminar (UIC, Chicago, Illinois, USA, 2007).
- Give a talk “Cohomology of line bundles for toric variety I, II” at graduate Algebraic Geometry seminar (UIC, Chicago, Illinois, USA, 2007).

- University of Michigan-Ohio State-UIC algebraic geometry workshop (University of Michigan, Ann Arbor, Michigan, USA, 2007).
 - Midwest Workshop on Complex Analysis and Geometry (UIC, Chicago, Illinois, USA, 2007).
 - The 4th International Congress of Chinese Mathematicians (Hangzhou, China, 2007).
 - Give a talk “Deformations in Algebraic Geometry” at graduate Algebraic Geometry seminar (UIC, Chicago, Illinois, USA, 2007).
 - Algebraic Geometry and Commutative Algebra Conference to celebrate to Robin Hartshorne’s 70th birthday (UIC, Chicago, Illinois, USA, 2008).
 - The 2nd Bluegrass Algebra Conference (University of Kentucky, Lexington, Kentucky, USA, 2009)
- Honors and Awards:
 - China Undergraduate Mathematical Contest in Modeling, Shanghai district, second prize, 1999, 2000.
 - Normal University Scholarship, East China Normal University, China, 1999, 2000, 2001.
 - Sony Scholarship, China, 2000.
 - Shanghai Excellent Graduates, China, 2002.
 - Master Scholarship, East China Normal University, China, 2003.

- Invited to give a talk “The Equations of Rational Triple Points” at The 9th National Algebraic Conference, Huang Shan, China, 2004.
- Master Thesis silver medal of New World Mathematics Awards in The 4th International Congress of Chinese Mathematicians, Hangzhou, China, 2007.

- Publications:

- Cubic equations of rational triple points of dimension two (with Zhijie Chen, Sheng-Li Tan, and Fei Yu), Proceedings of the International Conference on Complex Geometry and Related Fields, 63–76, Amer. Math. Soc., Providence, RI, 2007.
- Higher order Bergman functions and complete numerical invariants for a family of pseudoconvex CR manifolds (with Stephen Yau), 45 pp. in ms. (to appear) Journal of Differential Geometry.
- Local holomorphic De Rham cohomology (with Stephen Yau), 7 pp. in ms., submitted for publication.
- New CR invariants and its application to the complex Plateau problem (with Stephen Yau), 18 pp. in ms., submitted for publication.
- Explicit construction of moduli space of bounded complete Reinhardt domains in \mathbb{C}^n and Hilbert 14th problems (with Yun Gao and Stephen Yau), 21 pp. in ms., submitted for publication.
- Complete invariants for special domains in D , E varieties (with Yun Gao and Stephen Yau), in preparation.