Theorem Let $X$ be an infinite-dimensional Banach space. Then for every $\varepsilon > 0$, $X$ either contains an unconditional basic sequence of constant $\leq \frac{4}{5}$ or an $\text{HI}(\varepsilon)$ subspace.

Proof We can suppose that $X$ is separable and $\varepsilon > 0$ is fixed. For all finite dimensional $E, F \subseteq X$ and infinite dimensional $Z \subseteq X$ we put

$$A(E, F; Z) := \sup_{V, U \subseteq Z \text{ inf. dim.}} \alpha (E + U, F + V)$$

$$\leq \alpha (E, F).$$

We say that $(E, F)$ accepts $Z$ if $A(E, F; Z) < \varepsilon$. Thus for any choice of $U, V$ there are $z_1 \in S_{E+U}$, $z_2 \in S_{F+V}$ such that $\|z_1 - z_2\| < \varepsilon$, meaning that $E+U$ and $F+V$ are close to being parallel in $Z$.

Notice that $(\mathbb{R}^2, \mathbb{R}^2)$ accepts $Z$ if and only if $Z$ is $\text{HI}(\varepsilon')$ for some $\varepsilon' < \varepsilon$. 


We remark that $(E,F)$ accepts $Z$ if and only if $(E,F)$ accepts $Z'$ and accepts if $Z' \subseteq Z$.

Thus $A(E,F;Z') \leq A(E,F;Z)$ and so if $(E,F)$ accepts $Z$ it also accepts $Z'$.

Moreover, if $(E,F)$ accepts $Z$, then it also accepts $Z + G$ for all finite-dimensional $G \subseteq X$. For if $U, V \subseteq Z + G$ are infinite-dimensional, then so are $U' = U \cap Z$ and $V' = V \cap Z$ and clearly $A(E + U', F + V') \leq A(E + U, F + V) \leq A(E,F;Z) < \varepsilon$.

So $A(E,F;Z + G) < \varepsilon$ too.

In fact, this argument shows that

$$A(E,F;Z) = A(E,F;Z + G)$$

for all finite-dimensional $G \subseteq X$.

Notice that if $(E,F)$ does not accept $Z$, then in particular $A(E,F) > \varepsilon$.

We say that $(E,F)$ rejects $Z$ if no infinite-dimensional subspace $Z' \subseteq Z$ is accepted by $(E,F)$. 
There is an infinite-dimensional subspace $Z_0 \subseteq X$ such that for all finite-dimensional $E, F \subseteq X$, either $(E, F)$ accepts or rejects $Z_0$.

Proof. Let $F$ denote the collection of all finite-dimensional subspaces of $X$ and let $\delta(E, E')$ be the Hausdorff distance between $F_E$ and $F_{E'}$.

Since $X$ is separable so is $(\mathcal{F}, \delta)$ and so we can pick a countable dense subset $\mathbb{E} \subseteq \mathcal{F}$ in the $\delta$-metric. We first prove the lemma for pairs $(E, F)$ in $\mathbb{E}$:

Sublemma. There is a subspace $Z_0$ of $X$ of infinite dimension such that for all $(E, F) \in \mathbb{E}^2$ and reals $a \in \mathbb{R}, \forall E$ we have
- either $\Lambda(E, F; Z_0) < a$
- or $\Lambda(E, F; Y) \geq a$ for all infinite-dimensional $Y \subseteq Z_0$. 

...
Proof: Enumerate all triples \((E, F, a) \in \mathbb{R}^2 \times (\mathbb{N} \cup \{0\}, \mathbb{E})\)

de \((E_n, F_n, a_n)\). We construct \(X = X_0 \supseteq X_1 \supseteq \ldots\)

does follow:

- if \(A(E_n, F_n; Y) \geq a_n\) for every infinite-dimensional \(Y \subseteq X_n\), let \(X_{n+1} = X_n\).

- otherwise there is \(X_{n+1} \subseteq X_n\) with

\[
A(E_n, F_n; X_{n+1}) < a_n.
\]

We diagonalize \(X_0 \supseteq X_1 \supseteq \ldots\) by choosing by induction on \(n\), \(X_{n+1} \subseteq X_n \setminus \text{span} \{x_0, \ldots, x_n\}\). Let \(Z_0 = \text{span} \{x_0, \ldots, x_n\}\). We have

\[
Z_0 \subseteq X_n + \text{span} \{x_0, \ldots, x_n\} \quad \text{for all } n \quad \text{and}
\]

\[
A(E_n, F_n; Z_0) \leq A(E_n, F_n; X_n + \text{span} \{x_0, \ldots, x_n\})
\]

\[
= A(E_n, F_n; X_n).
\]

Thus, for all \(n\), either \(A(E_n, F_n; Z_0) < a_n\)
or \(A(E_n, F_n; Y) \geq a_n\) for all \(Y \subseteq Z_0\).

Sublemma2: \(A(E, F; Z) \leq A(E', F'; Z) + 2\delta(E, E') + 2\delta(F, F')\).
Proof. It is enough to show that if $E, E' \subseteq X$ are finite-dimensional and $M, Z \subseteq X$ infinite-dimensional then

$$\sup_{U \subseteq Z} \langle (E' + U, M) \rangle \leq \sup_{U \subseteq Z} \langle (E + U, M) \rangle + 2\delta(E', E).$$

So suppose $s > \sup_{U \subseteq Z} \langle (E' + U, M) \rangle$ and let $U \subseteq Z$ be an arbitrary infinite-dimensional subspace.

Then for all $t > 1$ we can find $\phi_1, ..., \phi_k \in X^*$ such that

$$t \cdot \max \{ \| \phi_i \| : i = 1, ..., k \} \geq \| \ell \|$$

for all $e \in E$ (since $\dim E < \infty$).

Let $U' = U \cap \ker \phi_1 \cap \ker \phi_2 \cap \ker \phi_k$. Then

$$t \| u + e' \| \geq \max \{ \| \phi_i (u + e) \| : i = 1, ..., k \} \geq \| \ell \|$$

for all $e \in E$ and $u' \in U'$. Also, as

$s > \langle (E' + U', M) \rangle$, we can find $e + u'$, $m'$ of norm 1 with $\| e + u' + m' \| < s$, but

$\| e + m' \| = 1$ implies that $\| m' \| \leq t$ and so

for some $e' \in E'$, $\| e - e' \| \leq t\delta$, whereby

for some $x \in S_{E' + U'}$, $\| x - (e' + u') \| \leq t\delta$. 
Proof of Lemma: Let $Z_0$ be the subspace of $U$ that contains $V$. We claim that $Z_0$ works. For if $(E,F)$ does not reject $Z_0$, then it accepts some subspace $Y \subseteq Z_0$ and hence for some radius $\alpha < \varepsilon$ we have $A(E,F;Y) < \alpha$. Let $b \leq \alpha$ and $c \in \mathbb{R}$ be such that $A(E,E') < b$ and $A(F,F') < c$. Then $A(E',F';Y) < \alpha + b < \varepsilon - 4b$ and hence also $A(E',F';Z_0) < \alpha + b$ by closure at $Z_0$. Therefore, $A(E,F;Z_0) < \alpha + 2b < \varepsilon$ and $(E,F)$ accepts $Z_0$. \[ \square \]
Lemma: If $(E,F)$ rejects $Z_0$, then for all infinite-dimensional $Y \subseteq Z_0$, there is an infinite-dimensional $W \subseteq Y$ such that for all finite-dimensional $G \subseteq W$, $(E+G,F)$ rejects $Z_0$.

Proof: Assume not. Then there is $Y \subseteq Z_0$ such that for all $W \subseteq Y$ there is $G \subseteq W$ such that $(E+G,F)$ does not reject $Z_0$, whence, by assumption on $Z_0$, $(E+G,F)$ accepts $Z_0$.

So for all $Y \subseteq Z_0$ infinite-dimensional, since $E+G+W = E+W$,

$$\kappa(E+W,F+V) = \kappa(E+G+W,F+V) \leq \lambda(E+G,F,Z_0) < \varepsilon,$$

and hence $\sup \limits_{W,V \subseteq Y} \kappa(E+W,F+V) < \varepsilon$, since $Y$ infinite dim.

So $(E,F)$ accepts $Y$ contradicting the hypothesis. \qed

Iterating this lemma we get:
Lemma. Suppose $(E_l, F_l)_{l=1}^N$ reject $Z_0$.

Then for all infinite-dimensional $Y \subseteq Z_0$ there is an infinite-dimensional $W \subseteq Y$ such that for all infinite-dimensional $G \subseteq W$ and all $l = 1, \ldots, N$

$$(E_l + G, F_l) \text{ rejects } Z_0.$$ 

In particular:

Lemma. If $(E_l, F_l)_{l=1}^N$ reject $Z_0$, then for all infinite-dimensional $Y \subseteq Z_0$ there is a normalised vector $y \in Y$ such that for all $l = 1, \ldots, N$

$$(E_l + [y], F_l) \text{ rejects } Z_0.$$ 

Construction of the unconditional basic sequence:

We can suppose that $(\xi_0, \xi_0)$ rejects $Z_0$.

From this, we wish to construct by induction on $n$ a sequence $(\xi_n)_{n=1}^\infty$ of normalised vectors such that if $E_l = \text{span}\{\xi_i : i \in I_l\}$, then for all partitions $I \cup J = \{1, \ldots, n\}$

$$(E_I, E_J) \text{ rejects } Z_0.$$ Moreover, $\dim \text{span}\{\xi_i : i \in I\} = n$.

The choice of $\xi_1$ is given directly by the preceding lemma.
Now, suppose \( e_1, \ldots, e_n \) have been chosen. List all partitions \( I \cup J = \{1, \ldots, n\} \) as \( (I, J) \) \( i=1 \) and let \( (E_l, F_l) = (E_{I_l}, E_{J_l}) \). Then every pair \((E_l, F_l)\) rejects \( Z_0 \). Fix a subspace \( Y \subseteq Z_0 \) of constant dimension such that \( Y \cap \text{span}\{e_1, \ldots, e_n\} = \{0\} \) and find by the lemma some normalised \( y \in Y \) such that for all \( l \), 
\[
(E_l + [y], F_l) \text{ rejects } Z_0.
\]
Now, notice that else \((F_l + [y], E_l)\) can be written as \((E_{l'}, [y], F_{l'})\) for some \( l' \), so else \((E_{l'} + [y], F_{l'})\) rejects \( Z_0 \), so we can take \( e_{l'} = y \).

**Lemma** \(
\| (E, F) \| \geq E, \) then for all \( xeE, yeF, \)
\[
\| x+y \| \leq \frac{1}{E} \| x-y \|.
\]

**Proof** By and homogeneity we can suppose \( \| y \| \leq \| x \| = 1 \). Then \( \| y + x \| \leq 2 \) and we need only show that \( \| x - y \| \geq \frac{1}{2} \).
For all \( y \notin \mathbb{B} \), let \( \| y \| \leq 1 - \frac{\theta}{2} \), whereby

\[
\| x - y \| \geq 1 - \left( 1 - \frac{\theta}{2} \right) \geq \frac{\theta}{2}
\]

\[
\Rightarrow 1 - \frac{\theta}{2} < \| y \| \leq 1 \quad \text{and hence}
\]

\[
\varepsilon \leq \| x - \frac{y}{\| y \|} \| \leq \| x - y \| + \| y - \frac{y}{\| y \|} \|
\]

\[
= \| x - y \| + \left( 1 - \| y \| \right) \leq \| x - y \| + \frac{\theta}{2}
\]

and

\[
\| x - y \| \geq \frac{\theta}{2}.
\]

Suppose now that \( a_1, a_2, \ldots, a_n \) are real and

\[
e_i \in \{-1, 1\}. \quad \text{Let } \mathcal{E} = \{ i \in n \mid e_i = 1 \} \quad \text{and}
\]

\[
\mathcal{F} = \{ i \in n \mid e_i = -1 \}. \quad \text{Then } x := \sum_{i \in \mathcal{E}} a_i e_i \in E_{\mathcal{E}}
\]

and

\[
y := \sum_{i \in \mathcal{F}} a_i e_i \in E_{\mathcal{F}}
\]

and

\[
\left( \sum_{i=1}^n e_i a_i e_i \right) = \| x + y \| \leq \frac{\theta}{\varepsilon} \| x - y \| = \frac{\theta}{\varepsilon} \left( \sum_{i=1}^n \| a_i e_i \| \right).
\]

This shows that \( (e_n) \) is unconditional with constant of unconditionality \( \leq \frac{\theta}{\varepsilon} \). \( \square \)