Unconditional basic sequences

A series $\sum_{n=1}^{\infty} x_n$ in a Banach space $X$ is said to be unconditionally convergent if

$$\sum_{n=1}^{\infty} x_{\pi(n)}$$

converges for all permutations $\pi$ of $\mathbb{N}$. It is absolutely convergent if $\sum_{n=1}^{\infty} \| x_n \|$ converges. Notice that by Riemann's theorem, unconditional and absolute convergence coincide in $\mathbb{R}$ and hence in all finite-dimensional spaces.

Notice that if $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent, then $\sum_{n=1}^{\infty} x_{\pi(n)}$ has the same limit for all $\pi$. For let $x^* \in X^*$

then $\sum_{n=1}^{\infty} x^*(x_n) = x^* \left( \sum_{n=1}^{\infty} x_n \right)$ is unconditionally and hence absolutely convergent and thus

$$x^* \left( \sum_{n=1}^{\infty} x_n \right) = x^* \left( \sum_{n=1}^{\infty} x_{\pi(n)} \right)$$

for all $\pi$. 
Proposition (Orlicz) TFAE for \((x_n)_{n=1}^\infty\) in a Banach space \(X\):

(i) \(\sum_{n=1}^\infty x_n\) is unconditionally convergent.

(ii) \(\sum_{i=1}^\infty x_{\pi(i)}\) converges for all \(\pi_1 < \pi_2 < \ldots\).

(iii) \(\sum_{n=1}^\infty \Theta_n x_n\) converges for all choices of signs \(\Theta_n \in \{-1, 1\}\).

Proof (iii) \(\Rightarrow\) (ii): Set \(\Theta_n = 1\) if \(n = n_i\) for some \(i\) and otherwise \(\Theta_n = -1\). Then

\[
\sum_{i=1}^\infty x_{\pi(i)} = \frac{1}{2} \left( \sum_{n=1}^\infty x_n + \sum_{n=1}^\infty \Theta_n x_n \right)
\]

and hence converges.

(ii) \(\Rightarrow\) (i): Suppose \(\Pi\) is a permutation of \(N\) but \(\sum_{n=1}^\infty x_{\Pi(n)}\) diverges. Then Cauchy's criterion cannot be satisfied and we can therefore find intervals of integers \(I_1 < I_2 < \ldots\) such that for some \(\varepsilon > 0\) and all \(i\),

\[
\left\| \sum_{n \in I_i} x_{\Pi(n)} \right\| \geq \varepsilon.
\]
But then by passing to a subsequence we can suppose that for \( i < j \) and \( m \in I_i, k \in I_j \),
\[ \pi(m) < \pi(k), \]
Thus if we enumerate
\[ \{ \pi(m) : m \in \bigcup_{i=1}^{\infty} I_i \} \]
increasingly
\[ \pi(n_1) < \pi(n_2) < \pi(n_3) \ldots \]
we see that
\[ \{ \pi(m) : m \in I_i \} \]
does not converge in
\[ (\mathbb{N}, \leq) \]
and since
\[ \sum_{j=1}^{\infty} x_{n_j} \]
diverges contrary to our assumption.

(ii) \implies (iii) : If \[ \sum_{n=1}^{\infty} \Theta_n x_n \]
does not converge
then we can find intervals \( I_1 < I_2 < \ldots \)
of bounded length and some \( \varepsilon > 0 \) such that
\[ \| \sum_{n \in I_i} \Theta_n x_n \| \geq 2 \varepsilon, \]
whereby either
\[ \| \sum_{n \in I_i} x_n \| \geq \varepsilon \]
or
\[ \| \sum_{n \in I_i} x_n \| \geq \varepsilon. \]

Let \( \pi \) be a permutation of \( \mathbb{N} \) such that on \( I_i \) \( \pi \) first enumerates the \( n \) such that \( \Theta_n = 1 \).
Then
\[ \sum_{n=1}^{\infty} x_{\pi(n)} \]
fails Cauchy's criterion,
contending our assumption. \( \Box \)
Definition. A basis \((e_n)_{n=1}^{\infty}\) is called unconditional if for all \(x \in X\), 
\[
\sum_{n=1}^{\infty} e_n(x) e_n
\]
converges unconditionally (to \(x\)).

Notice we can then define for all \(\Theta \in \ell^1_1, Q^\infty\) an operator on \(X\) by
\[
M(\Theta) \left( \sum_{n=1}^{\infty} a_n e_n \right) = \sum_{n=1}^{\infty} \Theta_n a_n e_n
\]
By the closed graph theorem, \(M(\Theta)\) is continuous.

For if \(\sum_{n=1}^{\infty} a_n^{(k)} e_n \rightarrow \sum_{n=1}^{\infty} a_n e_n\) and
\[
\sum_{n=1}^{\infty} \Theta_n a_n^{(k)} e_n \rightarrow \sum_{n=1}^{\infty} b_n e_n
\]
then for every \(n\),
\[
a_n^{(k)} \rightarrow a_n \quad \text{and} \quad \Theta_n a_n^{(k)} e_n \rightarrow b_n e_n \quad \text{for all} \quad b_n \in \Theta_n a_n e_n.
\]
We claim else that \(\|M(\Theta)\| \in \ell_1, Q^\infty\) is uniformly bounded. If not, by the principle of uniform boundedness, it cannot be pointwise bounded and thus for some \(x = \sum_{n=1}^{\infty} a_n e_n\) and \(\Theta^{(k)} \in \ell_1, Q^\infty\),
\[
M(\Theta^{(k)}) x \rightarrow \infty.
\]
Using the fact that \(\ell_1, Q^\infty\) is finite for all \(N\), this easily implies that we can find intervals \(I_0 < I_1 < \ldots\) of integers
and \( k_i \) such that \( \| \sum_{n \in I_{k_i}} \theta_n^{(k_i)} a_n e_n \| \to \infty \). Letting

\[
\theta_n = \begin{cases} 
\theta_n^{(k_i)} & \text{if } n \in I_{k_i} \\
1 & \text{otherwise}
\end{cases}
\]

we see that

\[
\sum_{n=1}^{\infty} \theta_n a_n e_n
\]

diverges, which is a contradiction.

The number \( K_\theta = \sup \| M_\theta \| \) is called the unconditioned basis constant of \((e_n)\).

**Proposition.** For all \( \lambda = (\lambda_n) \) with \( |\lambda_n| \leq 1 \) and

\[
x = \sum_{n=1}^{\infty} \lambda_n a_n e_n
\]

we have

\[
\| \sum_{n=1}^{\infty} \lambda_n a_n e_n \| \leq K_\theta \| \sum_{n=1}^{\infty} \lambda_n a_n e_n \|
\]

**Proof.** Define \( M_\lambda : \text{span} \{ e_n \}_{n=1}^{\infty} \to \text{span} \{ e_n \}_{n=1}^{\infty} \) by

\[
M_\lambda \left( \sum_{n=1}^{N} \lambda_n e_n \right) = \sum_{n=1}^{N} \lambda_n e_n.
\]

It suffices to show that \( \| M_\lambda \| \leq K_\theta \). So suppose \( x = \sum_{n=1}^{N} \lambda_n e_n \) is given and find \( x^* \in X \) of norm 1 such that

\[
x^* \left( \sum_{n=1}^{N} \lambda_n e_n \right) = \left\| \sum_{n=1}^{N} \lambda_n e_n \right\|
\]

and \( \theta_n = 1 \) if \( \lambda_n x^*(e_n) \geq 0 \) and \( -1 \) otherwise.
Thus,

\[ \left\| \sum_{n=1}^{N} \lambda_n a_n e_n \right\| \leq \sum_{n=1}^{N} |\lambda_n| \left| a_n x^*(e_n) \right| \leq \sup_{1 \leq n \leq N} |\lambda_n| \sum_{n=1}^{N} |a_n x^*(e_n)| \]

\[ \leq \sum_{n=1}^{N} \Theta_n a_n x^*(e_n) = x^* \left( \sum_{n=1}^{N} \Theta_n a_n e_n \right) \]

\[ \leq \|x^*\| \|M\| \| \sum_{n=1}^{N} a_n e_n \| \leq K_n \| \sum_{n=1}^{N} a_n e_n \|. \]

In particular if \( \Lambda \subseteq \mathbb{N} \) we can set \( \lambda_n = \frac{1}{n} \) if \( n \in \Lambda \) and \( \lambda_n = 0 \) if \( n \not\in \Lambda \) and we have that if \( P_{\Lambda} : X \to X \) is the projection

\[ P_{\Lambda} \left( \sum_{n=1}^{\infty} a_n e_n \right) = \sum_{n \in \Lambda} a_n e_n \quad \text{then} \quad \|P_{\Lambda}\| \leq K_n. \]

The number \( \sup_{\Lambda \subseteq \mathbb{N}} \|P_{\Lambda}\| = K_S \) is the \underline{supersum} constant of \((e_n)\) and \((e_n)\) is \underline{supersummable} unconditional if \( K_S = 1 \).