Exercises in Descriptive set theory I, September 2007

1 Exercise

Consider Cantor space $2^{\mathbb{N}}$ that we identify with the power set of $\mathbb{N}$, $\mathcal{P}(\mathbb{N})$, via the natural bijection $\chi_A \mapsto A$.

Show that if $G \subseteq 2^{\mathbb{N}}$ is comeagre, then there is a sequence of successive finite intervals of $\mathbb{N}$, $I_0 < I_1 < \ldots$, and subsets $a_n \subseteq I_n$ such that for all $A \subseteq \mathbb{N}$

$$(\exists^\infty n\ A \cap I_n = a_n) \Longrightarrow A \in G.$$

2 Exercise

Show that the following sets are Borel inside the indicated spaces and use the Kuratowski-Tarski algorithm to decide their Borel complexity (i.e., show they belong to some $\mathbf{\Sigma}^0_\xi$, $\mathbf{\Pi}^0_\xi$ or $\mathbf{\Delta}^0_\xi$, but you don't have to prove that this is the best possible).

- c_0 seen as a subset of the Polish space $\mathbb{R}^{\mathbb{N}}$.
- ℓ_1 seen as a subset of $\mathbb{R}^{\mathbb{N}}$.

3 Exercise

Suppose that X and Y are Polish spaces with X compact. Let $F \subseteq X \times Y$ be closed. Show that

$$\mathrm{proj}_Y(F) = \{y \in Y \mid \exists x \in X\ (x, y) \in F\}$$

is closed.

Now suppose that X is K_σ, i.e., a countable union of compact sets, and F is F_σ. Show that $\mathrm{proj}_Y(F)$ is F_σ.

4 Exercise

Suppose that X and Y are Polish spaces. Show that if $A \subseteq X \times Y$ is $\mathbf{\Sigma}^0_\xi$, then for all $x \in X$, $A_x = \{y \in Y \mid (x, y) \in A\}$ is $\mathbf{\Sigma}^0_\xi$. Similarly for $\mathbf{\Sigma}^1_1$.

5 Exercise

Suppose X is a Polish space and d a compatible complete metric. Denote by $\mathcal{K}(X)$ the collection of all compact subsets of X and let d_H be the Hausdorff metric on $\mathcal{K}(X)$ defined by

$$d_H(K, L) = \max\{\max_{x \in K} d(x, L), \max_{y \in L} d(y, K)\}.$$

Show that $(\mathcal{K}(X), d_H)$ is a separable complete metric space.

6 Exercise

Show the following reduction principle for open sets in zero-dimensional spaces.

If X is a zero-dimensional Polish space and $U, V \subseteq X$ are open subsets, show that there are disjoint open $U^* \subseteq U$ and $V^* \subseteq V$ such that $U^* \cup V^* = U \cup V$.