BANACH SPACES WITHOUT MINIMAL SUBSPACES - EXAMPLES

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Abstract. We analyse several examples of spaces, some of them new, and relate them to several dichotomies obtained in [5] by classifying them according to which side of the dichotomies they fall.

Contents

1. Introduction

In this article we give several new examples of Banach spaces, corresponding to different classes of a list defined in [5]. This paper may be seen as a more empirical continuation of [5] in which our stress is on the study of examples for the new classes of Banach spaces considered in that work.

1.1. **Gowers' list of inevitable classes.** In the paper [9], W.T. Gowers had defined a program of isomorphic classification of Banach spaces. The aim of this program is a *loose classification of Banach spaces up to subspaces*, by producing a list of classes of Banach spaces such that:

(a) if a space belongs to a class, then every subspace belongs to the same class, or maybe, in the case when the properties defining the class depend on a basis of the space, every block subspace belongs to the same class,

(b) the classes are *inevitable*, i.e., every Banach space contains a subspace in one of the classes,

(c) any two classes in the list are disjoint,

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(d) belonging to one class gives a lot of information about operators that may be defined on the space or on its subspaces.

We shall refer to such a list as a *list of inevitable classes of Gowers*. For the motivation of Gowers' program as well as the relation of this program to classical problems in Banach space theory we refer to [5]. Let us just say that the class of spaces c_0 and ℓ_p is seen as the nicest or most regular class, and so, the objective of Gowers' program really is the classification of those spaces (such as Tsirelson's space T) which do not contain a copy of c_0 or ℓ_p. Actually, in [5], mainly spaces without *minimal subspaces* are classified, and so in this article, we shall consider various examples of Banach spaces without minimal subspaces. We shall first give a summary of the classification obtained in [5] and of the results that led to that classification.

After the construction by Gowers and Maurey of a *hereditarily indecomposable* (or HI) space GM, i.e., a space such that no subspace may be written as the direct sum of infinite dimensional subspaces [10], Gowers proved that every Banach space contains either an HI subspace or a subspace with an unconditional basis [8]. This dichotomy is called *first dichotomy* of Gowers in [5]. These were the first two examples of inevitable classes. He then refined the list by proving a *second dichotomy*: any Banach space contains a subspace with a basis such that either no two disjointly supported block subspaces are isomorphic, or such that any two subspaces have further subspaces which are isomorphic. He called the second property *quasi minimality*. Finally, H. Rosenthal had defined a space to be *minimal* if it embeds into any of its subspaces. A quasi minimal space which does not contain a minimal subspace is called *strictly quasi minimal*, so Gowers again divided the class of quasi minimal spaces into the class of strictly quasi minimal spaces and the class of minimal spaces.

Gowers therefore produced a list of four classes of Banach spaces, corresponding to classical examples, or more recent couterexamples to classical questions: HI spaces, such as GM; spaces with bases such that no disjointly supported subspaces are isomorphic, such as the couterexample G_u of Gowers to the hyperplane's problem of Banach; strictly quasi minimal spaces with an unconditional basis, such as T; and finally, minimal spaces, such as c_0 or ℓ_p, but also T^*, Schlumprecht's space S, or as proved recently in [12], its dual S^*.

In [5] three new dichotomies for Banach spaces were obtained. The first one of these new dichotomies, the *third dichotomy*, concerns the property of minimality defined by Rosenthal. Recall that a Banach space is minimal if it embeds into any of its infinite dimensional subspaces. On the other hand, a space Y is *tight* in a basic sequence (e_i) if there is a sequence of successive intervals $I_0 < I_1 < I_2 < \dots$ of $\mathbb{N}$ such that for all infinite subsets $A \subseteq \mathbb{N}$, we have

$$Y \not\sqsubseteq [e_n \mid n \notin \bigcup_{i \in A} I_i].$$

A *tight basis* is a basis such that every subspace is tight in it, and a *tight space* is a space with a tight basis.

It is then observed in [5] that the tightness property is hereditary, incompatible with minimality, and it is proved that:

Theorem 1.1 (3rd dichotomy, Ferenczi-Rosendal 2007)**.** *Let E be a Banach space without minimal subspaces. Then E has a tight subspace.*

Actual examples of tight spaces in [5] turn out to satisfy one of two stronger forms of tightness. The first was called *tightness with constants.* A basis (e_n) is tight with constants when for for every infinite dimensional space Y, the sequence of successive intervals $I_0 < I_1 < \dots$ of $\mathbb{N}$ witnessing the tightness of Y in (e_n) may be chosen so that $Y \not\sqsubseteq_K [e_n \mid n \notin I_K]$ for each K. This is the case for Tsirelson's space.

The second kind of tightness was called *tightness by range.* Here the range, range x, of a vector x is the smallest interval of integers containing its support, and the range of a block subspace $[x_n]$ is $\bigcup_n$ range x_n. A basis (e_n) is tight by range when for every block subspace $Y = [y_n]$, the sequence of successive intervals $I_0 < I_1 < \dots$ of $\mathbb{N}$ witnessing the tightness of Y in (e_n) may be defined by $I_k =$ range y_k for each k. This is equivalent to no two block subspaces with disjoint ranges being comparable.

When the definition of tightness may be checked with $I_k =$ supp y_k instead of range y_k, then a stronger property is obtained which is called tightness by support, and is equivalent to the property defined by Gowers in the second dichotomy that no disjointly supported block subspaces are isomorphic, Therefore G_u is an example of space with a basis which is tight by support and therefore by range.

As we shall see, one of the aims of this paper is to present various examples of tight spaces of these two forms.

In [5] it was proved that there are natural dichotomies between each of these strong forms of tightness and respective weak forms of minimality. For the first notion, recall that given two Banach spaces X and Y, we say that X is *crudely finitely representable* in Y if there is a constant K such that for any finite-dimensional subspace $F \subseteq X$ there is an embedding $T\colon F \to Y$ with constant K, i.e., $\|T\| \cdot \|T^{-1}\| \leqslant K$. A space X is said to be *locally minimal* if for some constant K, X is K-crudely finitely representable in any of its subspaces.

Theorem 1.2 (5th dichotomy, Ferenczi-Rosendal 2007)**.** *Any Banach space E contains a subspace with a basis that is either tight with constants or is locally minimal.*

There is also a dichotomy concerning tightness by range. A space X with a basis (x_n) is said to be *subsequentially minimal* if every subspace of X contains an isomorphic copy of a subsequence of (x_n). Essentially this notion had been previously considered by Kutzarova, Leung, Manoussakis and Tang in the context of modified partially mixed Tsirelson spaces [11].

Theorem 1.3 (4th dichotomy, Ferenczi-Rosendal 2007)**.** *Any Banach space E contains a subspace with a basis that is either tight by range or is subsequentially minimal.*

The second case in Theorem 1.3 may be improved to the following hereditary property of a basis (x_n), that we call *sequential minimality*: (x_n) is quasi minimal and every block sequence of $[x_n]$ has a subsequentially minimal block sequence.

Finally there exists a sixth dichotomy theorem due to A. Tcaciuc [16], stated here in a slightly strengthened form. A space X is *uniformly inhomogeneous* when

$$\forall M \geqslant 1\ \exists n \in \mathbb{N}\ \forall Y_1, \ldots, Y_{2n} \subseteq Y\ \exists y_i \in \mathcal{S}_{Y_i}\ (y_i)_{i=1}^{n} \not\sim_M (y_i)_{i=n+1}^{2n}.$$

On the contrary, a basis (e_n) is said to be *strongly asymptotically* ℓ_p, $1 \leqslant p \leqslant +\infty$, [3], if there exists a constant C and a function $f : \mathbb{N} \to \mathbb{N}$ such that for any n, any family of n unit vectors which are disjointly supported in $[e_k \mid k \geqslant f(n)]$ is C-equivalent to the canonical basis of ℓ_p^n. Tcaciuc then proves [16] :

Theorem 1.4 (Tcaciuc's dichotomy, 2005)**.** *Any Banach space contains a subspace with a basis which is either uniformly inhomogeneous or strongly asymptotically* ℓ_p *for some* $1 \leqslant p \leqslant +\infty$.

The six dichotomies and the interdependence of the properties involved can be visualised in the following diagram, see [5].

$$\begin{array}{ccc}
\text{Strongly asymptotic } \ell_p & ** \text{ Tcaciuc's dichotomy } ** & \text{Uniformly inhomogeneous} \\
\Downarrow & & \Uparrow \\
\text{Unconditional basis} & ** \text{ 1st dichotomy } ** & \text{Hereditarily indecomposable} \\
\Uparrow & & \Downarrow \\
\text{Tight by support} & ** \text{ 2nd dichotomy } ** & \text{Quasi minimal} \\
\Downarrow & & \Uparrow \\
\text{Tight by range} & ** \text{ 4th dichotomy } ** & \text{Sequentially minimal} \\
\Downarrow & & \Uparrow \\
\text{Tight} & ** \text{ 3rd dichotomy } ** & \text{Minimal} \\
\Uparrow & & \Downarrow \\
\text{Tight with constants} & ** \text{ 5th dichotomy } ** & \text{Locally minimal}
\end{array}$$

Combining the six dichotomies and the relations between them, the following list of 19 classes of Banach spaces contained in any Banach space is obtained in [5]:

Theorem 1.5 (Ferenczi - Rosendal 2007)**.** *Any infinite dimensional Banach space contains a subspace of one of the types listed in the following diagram:*

Type	*Properties*	*Examples*
(1a)	*HI, tight by range and with constants*	*?*
(1b)	*HI, tight by range, locally minimal*	G^*
(2)	*HI, tight, sequentially minimal*	*?*
(3a)	*tight by support and with constants, uniformly inhomogeneous*	*?*
(3b)	*tight by support, locally minimal, uniformly inhomogeneous*	G_u^*
(3c)	*tight by support, strongly asymptotically* ℓ_p, $1 \leqslant p < \infty$	X_u
(3d)	*tight by support, strongly asymptotically* ℓ_∞	X_u^*
(4)	*unconditional basis, quasi minimal, tight by range*	*?*
(5a)	*unconditional basis, tight with constants, sequentially minimal, uniformly inhomogeneous*	*?*
(5b)	*unconditional basis, tight, sequentially and locally minimal, uniformly inhomogeneous*	*?*
(5c)	*tight with constants, sequentially minimal, strongly asymptotically* ℓ_p, $1 \leqslant p < \infty$	T, $T^{(p)}$
(5d)	*tight, sequentially minimal, strongly asymptotically* ℓ_∞	*?*
(6a)	*unconditional basis, minimal, uniformly inhomogeneous*	S, S^*
(6b)	*minimal, reflexive, strongly asymptotically* ℓ_∞	T^*
(6c)	*isomorphic to* c_0 *or* l_p, $1 \leqslant p < \infty$	c_0, ℓ_p

In [5] the properties of the spaces S, G, G_u and the existence and properties of X_u are mentioned without proof. It is the main objective of this paper to prove the results about these spaces which appear in the above diagram.

So in what follows various (and for some of them new) examples of "pure" tight spaces are analysed combining some of the properties of tightness or minimality associated to each dichotomy. We shall provide several examples of tight spaces from the two main families of exotic Banach spaces: spaces of the type of Gowers and Maurey [10] and spaces of the type of Argyros and Deliyanni [1]. Recall that both types of spaces are defined using a coding procedure to "conditionalise" the norm of some ground space defined by induction. In spaces of the type of Gowers and Maurey, the ground space is the space S of Schlumprecht, and in spaces of the type of Argyros and Deliyanni, it is a mixed (in further versions modified or partly modified) Tsirelson space associated to the sequence of Schreier families. The space S is far from being asymptotic ℓ_p and is actually uniformly inhomogeneous, and this is the case for our examples of the type of Gowers-Maurey as well. On the other hand, we use a space in the second family, inspired by an example of Argyros, Deliyanni, Kutzarova and Manoussakis [2], to produce strongly asymptotically ℓ_1 and ℓ_∞ examples with strong tightness properties.

2. Tight unconditional spaces of the type of Gowers and Maurey

In this section we prove that the dual of the type (3) space G_u constructed by Gowers in [6] is locally minimal of type (3), that Gowers' hereditarily indecomposable and asymptotically unconditional space G defined in [7] is of type (1), and that its dual G^* is locally minimal of type (1). These spaces are natural variations on Gowers and Maurey's space GM, and so familiarity with that construction will be assumed: we shall not redefine the now classical notation relative to GM, such as the sets of integers K and L, R.I.S. sequences, the set $\mathbf{Q}$ of functionals, special functionals, etc., instead we shall try to give details on the parts in which G_u or G differ from GM.

The idea of the proofs is similar to [6]. The HI property for Gowers-Maurey's spaces is obtained as follows. Some vector x is constructed such that $\|x\|$ is large, but so that if x' is obtained from x by changing signs of the components of x, then $x^*(x')$ is small for any norming functional x^*, and so $\|x'\|$ is small. The upper bound for $x^*(x')$ is obtained by a combination of unconditional estimates (not depending on the signs) and of conditional estimates (i.e., based on the fact that $|\sum_{i=1}^n \epsilon_i|$ is much smaller than n if $\epsilon_i = (-1)^i$ for all i).

For our examples we shall need to prove that some operator T is unbounded. Thus we shall construct a vector x such that say Tx has large norm, and such that $x^*(x)$ is small for any norming x^*. The upper bound for $x^*(x)$ will be obtained by the same unconditional estimates as in the HI case, while conditional estimates will be trivial due to disjointness of supports of the corresponding component vectors and functionals. The method will be similar for the dual spaces.

Recall that if X is a space with a bimonotone basis, an ℓ_{1+}^n-average with constant $1+\epsilon$ is a normalised vector of the form $\sum_{i=1}^n x_i$, where $x_1 < \cdots < x_n$ and $\|x_i\| \leqslant \frac{1+\epsilon}{n}$ for all i. An $\ell_{\infty+}^n$-average with constant $1+\epsilon$ is a normalised vector of the form $\sum_{i=1}^n x_i$, where $x_1 < \cdots < x_n$ and $\|x_i\| \geqslant \frac{1}{1+\epsilon}$ for all i. An ℓ_{1+}^n-vector (resp. $\ell_{\infty+}^n$-vector) is a non zero multiple of an ℓ_{1+}^n-average (resp. $\ell_{\infty+}^n$-average). The function f is defined by $f(n) = \log_2(n+1)$. The space X is said to satisfy a lower f-estimate if for any $x_1 < \cdots < x_n$,

$$\frac{1}{f(n)} \sum_{i=1}^n \|x_i\| \leqslant \|\sum_{i=1}^n x_i\|.$$

Lemma 2.1. *Let X be a reflexive space with a bimonotone basis and satisfying a lower f-estimate. Let (y_k^*) be a normalised block sequence of X^*, $n \in \mathbb{N}$, $\epsilon, \alpha > 0$. Then there exists a constant $N(n,\epsilon)$, successive subsets F_i of $[1, N(n,\epsilon)]$, $1 \leqslant i \leqslant n$, and $\lambda > 0$ such that if $x_i^* := \lambda \sum_{k \in F_i} y_k^*$ for all i, then $x^* = \sum_{i=1}^n x_i^*$ is an $\ell_{\infty+}^n$-average with constant $1+\epsilon$. Furthermore, if for each i, x_i is such that $\|x_i\| \leqslant 1$, $\mathrm{range}\, x_i \subseteq \mathrm{range}\, x_i^*$ and $x_i^*(x_i) \geqslant \alpha \|x_i^*\|$, then $x = \sum_{i=1}^n x_i$ is an ℓ_{1+}^n-vector with constant $\frac{1+\epsilon}{\alpha}$ such that $x^*(x) \geqslant \frac{\alpha}{1+\epsilon}\|x\|$.*

Proof. Since X satisfies a lower f-estimate, it follows by duality that any sequence of successive functionals $x_1^* < \cdots < x_n^*$ in G_u^* satisfies the following upper estimate:

$$1 \leqslant \|\sum_{i=1}^n x_i^*\| \leqslant f(n) \max_{1 \leqslant i \leqslant n} \|x_i^*\|.$$

Let $N = n^k$ where k is such that $(1+\epsilon)^k > f(n^k)$. Assume towards a contradiction that the result is false for $N(n,\epsilon) = N$, then

$$y^* = (y_1^* + \ldots + y_{n^{k-1}}^*) + \ldots + (y_{(n-1)n^{k-1}+1}^* + \ldots + y_{n^k}^*)$$

is not an $\ell_{\infty+}^n$-vector with constant $1+\epsilon$, and therefore, for some i,

$$\|y_{in^{k-1}+1}^* + \ldots + y_{(i+1)n^{k-1}}^*\| \leqslant \frac{1}{1+\epsilon}\|y^*\|.$$

Applying the same reasoning to the above sum instead of y^*, we obtain, for some j,

$$\|y^*_{jn^{k-2}+1}+\ldots+y^*_{(j+1)n^{k-2}}\| \leqslant \frac{1}{(1+\epsilon)^2}\|y^*\|.$$

By induction we obtain that

$$1 \leqslant \frac{1}{(1+\epsilon)^k}\|y^*\| \leqslant \frac{1}{(1+\epsilon)^k} f(n^k),$$

a contradiction.

Let therefore x^* be such an $\ell_{\infty+}^n$-average with constant $1+\epsilon$ of the form $\sum_i x_i^*$. Let for each i, x_i be such that $\|x_i\| \leqslant 1$, range $x_i \subseteq$ range x_i^* and $x_i^*(x_i) \geqslant \alpha\|x_i^*\|$. Then

$$\|\sum_i x_i\| \geqslant x^*(\sum_i x_i) \geqslant \frac{\alpha n}{1+\epsilon} \geqslant \frac{\alpha}{1+\epsilon}\|\sum_i x_i\|,$$

and in particular for each i,

$$\|x_i\| \leqslant \frac{1+\epsilon}{\alpha n}\|\sum_i x_i\|.$$

□

The following lemma is fundamental and therefore worth stating explicitly. It appears for example as Lemma 4 in [7]. Recall that an (M, g)-form is a functional of the form $g(M)^{-1}(x_1^*+\ldots+x_M^*)$, with $x_1^* < \cdots < x_M^*$ of norm at most 1.

Lemma 2.2 (Lemma 4 in [7])**.** *Let $f, g \in \mathcal{F}$ with $g \geqslant \sqrt{f}$, let X be a space with a bimonotone basis satisfying a lower f-estimate, let $\epsilon > 0$, let $x_1, \ldots, x_N$ be a R.I.S. in X for f with constant $1+\epsilon$ and let $x = \sum_{i=1}^N x_i$. Suppose that*

$$\|Ex\| \leqslant \sup\Big\{|x^*(Ex)| : M \geqslant 2, x^* \textit{ is an } (M,g)\textit{-form}\Big\}$$

for every interval E such that $\|Ex\| \geq 1/3$. Then $\|x\| \leqslant (1+\epsilon+\epsilon')Ng(N)^{-1}$.

2.1. **A locally minimal space tight by support.** Let G_u be the space defined in [6]. This space has a suppression unconditional basis, is tight by support and therefore reflexive, and its norm is given by the following implicit equation, for all $x \in c_{00}$:

$$\|x\| = \|x\|_{c_0} \vee \sup\Big\{f(n)^{-1}\sum_{i=1}^n \|E_i x\| \;\Big|\; 2 \leqslant n, E_1 < \ldots < E_n\Big\}$$
$$\vee \sup\Big\{|x^*(x)| \;\Big|\; k \in K, x^* \text{ special of length } k\Big\}$$

where $E_1, \ldots, E_n$ are successive subsets (not necessarily intervals) of $\mathbb{N}$.

Proposition 2.3. *The dual G_u^* of G_u is tight by support and locally minimal.*

Proof. Given $n \in \mathbb{N}$ and $\epsilon = 1/10$ we may by Lemma 2.1 find some N such that there exists in the span of any $x_1^* < \ldots < x_N^*$ an $\ell_{\infty+}^n$-average with constant $1+\epsilon$. By unconditionality we deduce that any block-subspace of G_u^* contains ℓ_∞^n's uniformly, and therefore G_u^* is locally minimal.

Assume now towards a contradiction that (x_n^*) and (y_n^*) are disjointly supported and equivalent block sequences in G_u^*, and let $T : [x_n^*] \to [y_n^*]$ be defined by $Tx_n^* = y_n^*$.

We may assume that each x_n^* is an $\ell_{\infty+}^n$-average with constant $1+\epsilon$. Using Hahn-Banach theorem, the 1-unconditionality of the basis, and Lemma 2.1, we may also find for each n an ℓ_{1+}^n-average x_n with constant $1+\epsilon$ such that $\operatorname{supp} x_n \subseteq \operatorname{supp} x_n^*$ and $x_n^*(x_n) \geqslant 1/2$. By construction, for each n, Tx_n^* is disjointly supported from $[x_k]$, and up to modifying T, we may assume that Tx_n^* is in $\mathbf{Q}$ and of norm at most 1 for each n.

If $z_1, \ldots, z_m$ is a R.I.S. of these ℓ_{1+}^n-averages x_n with constant $1+\epsilon$, with $m \in [\log N, \exp N]$, $N \in L$, and $z_1^*, \ldots, z_m^*$ are the functionals associated to $z_1, \ldots, z_m$, then by [6] Lemma 7, the (m,f)-form $z^* = f(m)^{-1}(z_1^* + \ldots + z_m^*)$ satisfies

$$z^*(z_1 + \ldots + z_m) \geqslant \frac{m}{2f(m)} \geqslant \frac{1}{4}\|z_1 + \ldots + z_m\|,$$

and furthermore Tz^* is also an (m,f)-form. Therefore we may build R.I.S. vectors z with constant $1+\epsilon$ of arbitrary length m in $[\log N, \exp N]$, $N \in L$, so that z is 4^{-1}-normed by an (m,f)-form z^* such that Tz^* is also an (m,f)-form. We may then consider a sequence $z_1, \ldots, z_k$ of length $k \in K$ of such R.I.S. vectors of length m_i, and some corresponding (m_i,f)-forms $z_1^*, \ldots, z_k^*$ (i.e z_i^* 4^{-1}-norms z_i and Tz_i^* is also an (m_i,f)-form for all i), such that $Tz_1^*, \ldots, Tz_k^*$ is a special sequence. Then we let $z = z_1 + \cdots + z_k$ and $z^* = f(k)^{-1/2}(z_1^* + \ldots + z_k^*)$. Since $Tz^* = f(k)^{-1/2}(Tz_1^* + \ldots + Tz_k^*)$ is a special functional it follows that

$$\|Tz^*\| \leqslant 1.$$

Our aim is now to show that $\|z\| \leqslant 3kf(k)^{-1}$. It will then follow that

$$\|z^*\| \geqslant z^*(z)/\|z\| \geqslant f(k)^{1/2}/12.$$

Since k was arbitrary in K this will imply that T^{-1} is unbounded and provide the desired contradiction.

The proof is almost exactly the same as in [6]. Let $K_0 = K \setminus \{k\}$ and let g be the corresponding function given by [6] Lemma 6. To prove that $\|z\| \leqslant 3kf(k)^{-1}$ it is enough by [6] Lemma 8 and Lemma 2.2 to prove that for any interval E such that $\|Ez\| \geqslant 1/3$, Ez is normed by some (M,g)-form with $M \geqslant 2$.

By the discussion in the proof of the main theorem in [6], the only possible norming functionals apart from (M,g)-forms are special functionals of length k. So let $w^* = f(k)^{-1/2}(w_1^* + \cdots + w_k^*)$ be a special functional of length k, and E be an interval such that $\|Ez\| \geqslant 1/3$. We need to show that w^* does not norm Ez.

Let t be minimal such that $w_t^* \neq Tz_t^*$. If $i \neq j$ or $i = j > t$ then by definition of special sequences there exist $M \neq N \in L$, $\min(M,N) \geqslant j_{2k}$, such that w_i^* is an (M,f)-form and z_j is an R.I.S. vector of size N and constant $1+\epsilon$. By [6] Lemma 8, z_j is an $\ell_{1+}^{N^{1/10}}$-average with constant 2. If $M < N$ then $2M < \log\log\log N$ so, by [6] Corollary 3, $|w_i^*(Ez_j)| \leqslant 6f(M)^{-1}$. If $M > N$ then $\log\log\log M > 2N$ so, by [6] Lemma 4, $|w_i^*(Ez_j)| \leqslant 2f(N)/N$. In both cases it follows that $|w_i^*(Ez_j)| \leqslant k^{-2}$.

If $i = j = t$ we have $|w_i^*(Ez_j)| \leqslant 1$. Finally if $i = j < t$ then $w_i^* = Tz_i^*$. Since Tz_i^* is disjointly supported from $[x_k]$ and therefore from z_j, it follows simply that $w_i^*(Ez_j) = 0$ in that case.

Summing up we have obtained that

$$|w^*(Ez)| \leqslant f(k)^{-1/2}(k^2.k^{-2} + 1) = 2f(k)^{-1/2} < 1/3 \leqslant \|Ez\|.$$

Therefore w^* does not norm Ez and this finishes the proof. □

It may be observed that G_u^* is uniformly inhomogeneous. We state this in a general form which implies the result for G_u, Schlumprecht's space S and its dual S^*. This is also true Gowers-Maurey's space GM and its dual GM^*, as well as for G and G^*, where G is the HI asymptotically unconditional space of Gowers from [7], which we shall redefine and study later on. As HI spaces are always uniformly inhomogeneous however, we need to observe that a slightly stronger result is obtained by the proof of the next statement to see that Proposition 2.4 is not trivial in the case of GM, G or their duals - see the three paragraphs after Proposition 2.4.

Proposition 2.4. *Let $f \in \mathcal{F}$ and let X be a space with a bimonotone basis satisfying a lower f-estimate. Let $\epsilon_0 = 1/10$, and assume that for every $n \in [\log N, \exp N], N \in L$, $x_1, \ldots, x_n$ a R.I.S. in X with constant $1+\epsilon_0$ and $x = \sum_{i=1}^N x_i$,*

$$\|Ex\| \leqslant \sup\Big\{|x^*(Ex)| : M \geqslant 2, x^* \text{ is an } (M,f)\text{-form}\Big\}$$

for every interval E such that $\|Ex\| \geq 1/3$. Then X and X^* are uniformly inhomogeneous.

Proof. Given $\epsilon > 0$, let $m \in L$ be such that $f(m) \geqslant 24\epsilon^{-1}$. Let $Y_1, \ldots, Y_{2m}$ be arbitrary block subspaces of X. By the classical method for spaces with a lower f estimate, we may find a R.I.S. sequence $y_1 < \cdots < y_m$ with constant $1+\epsilon_0$ with $y_i \in Y_{2i-1}, \forall i$. By Lemma 2.2,

$$\|\sum_{i=1}^m y_i\| \leqslant 2mf(m)^{-1}.$$

Let on the other hand $n \in [m^{10}, \exp m]$ and $E_1 < \cdots < E_m$ be sets such that $\bigcup_{j=1}^m E_j = \{1, \ldots, n\}$ and $|E_j|$ is within 1 of $\frac{n}{m}$ for all j. We may construct a R.I.S. sequence $x_1, \ldots, x_n$ with constant $1+\epsilon_0$ such that $x_i \in Y_{2j}$ whenever $i \in E_j$.

By Lemma 2.2,

$$\|\sum_{i \in E_j} x_i\| \leqslant (1+2\epsilon_0)(\frac{n}{m}+1)f(\frac{n}{m}-1)^{-1} \leqslant 2nf(n)^{-1}m^{-1}.$$

Let $z_j = \|\sum_{i \in E_j} x_i\|^{-1} \sum_{i \in E_j} x_i$. Then $z_j \in Y_{2j}$ for all j and

$$\|\sum_{j=1}^m z_j\| \geqslant f(n)^{-1} \sum_{j=1}^m \big(\|\sum_{i \in E_j} x_i\|^{-1} \sum_{i \in E_j} \|x_i\|\big) \geqslant m/2.$$

Therefore

$$\|\sum_{i=1}^m y_i\| \leqslant 4f(m)^{-1} \|\sum_{i=1}^m z_i\| \leqslant \epsilon \|\sum_{i=1}^m z_i\|.$$

Obviously $(y_i)_{i=1}^m$ is not ϵ^{-1}-equivalent to $(z_i)_{i=1}^m$, and this means that X is uniformly inhomogeneous.

The proof concerning the dual is quite similar and uses the same notation. Let $Y_{1*}, \ldots, Y_{2m*}$ be arbitrary block subspaces of X^*. By Lemma 2.1 we may find a R.I.S. sequence $y_1 < \cdots < y_m$ with constant $1+\epsilon_0$ and functionals $y_i^* \in Y_{2i-1*}$ such that range $y_i^* \subseteq$ range y_i and $y_i^*(y_i) \geqslant 1/2$ for all i. Since $\|\sum_{i=1}^m y_i\| \leqslant 2mf(m)^{-1}$, it follows that

$$\|\sum_{i=1}^m y_i^*\| \geqslant \|\sum_{i=1}^m y_i\|^{-1} \sum_{i=1}^m y_i^*(y_i) \geqslant f(m)/4.$$

On the other hand we may construct a R.I.S. sequence $x_1, \ldots, x_n$ with constant $1+\epsilon_0$ and functionals x_i^* such that range $x_i^* \subseteq$ range x_i, $x_i^*(x_i) \geqslant 1/2$ for all i, and such that $x_i^* \in Y_{2j*}$ whenever $i \in E_j$. Since $\|\sum_{i\in E_j} x_i\| \leqslant 2nf(n)^{-1}m^{-1}$, it follows that

$$\|\sum_{i\in E_j} x_i^*\| \geqslant \frac{n}{3m}\frac{mf(n)}{2n} = f(n)/6.$$

Let $z_j^* = \|\sum_{i\in E_j} x_i^*\|^{-1}\sum_{i\in E_j} x_i^*$. Then $z_j^* \in Y_{2j*}$ for all j and

$$\|\sum_{j=1}^m z_j^*\| \leqslant \frac{6}{f(n)}f(n) = 6.$$

Therefore

$$\|\sum_{i=1}^m z_i^*\| \leqslant 24f(m)^{-1}\|\sum_{i=1}^m y_i^*\| \leqslant \epsilon\|\sum_{i=1}^m y_i^*\|.$$

□

Corollary 2.5. *The spaces S, S^*, GM, GM^*, G, G^*, G_u, and G_u^* are uniformly inhomogeneous.*

A slightly stronger statement may be obtained by the proof of Proposition 2.4, in the sense that the vectors y_i in the definition of uniform inhomogeneity may be chosen to be successive. More explicitely, the conclusion may be replaced by the statement that

$$\forall M \geqslant 1\ \exists n \in \mathbb{N}\ \forall Y_1, \ldots, Y_{2n} \subseteq Y\ \exists y_i \in \mathcal{S}_{Y_i}\ (y_i)_{i=1}^n \not\sim_M (y_i)_{i=n+1}^{2n}.$$

where $y_1 < \cdots < y_n$ and $y_{n+1} < \cdots < y_{2n}$.

This property is therefore a block version of the property of uniform inhomogeneity. It was observed in [5] that the sixth dichomoty had the following "block" version: any Schauder basis of a Banach space contains a block sequence which is either block uniformly inhomogeneous in the above sense or asymptotically ℓ_p for some $p \in [1, +\infty]$.

It is interesting to observe that either side of this dichotomy corresponds to one of the two main families of HI spaces, namely spaces of the type of Gowers-Maurey, based on the example of Schlumprecht, and spaces of the type of Argyros-Deliyanni, based on Tsirelson's type spaces. More precisely, spaces of the type of Gowers-Maurey are block uniformly inhomogeneous, while spaces of the type of Argyros-Deliyanni are asymptotically ℓ_1. Observe that the original dichotomy of Tcaciuc fails to distinguish between these two families, since any HI space is trivially uniformly inhomogeneous, see [5].

3. Tight HI spaces of the type of Gowers and Maurey

In this section we show that Gowers' space G constructed in [7] and its dual are of type (1). The proof is a refinement of the proof that G_u or G_u^* is of type (3), in which we observe that the hypothesis of unconditionality may be replaced by asymptotic unconditionality. The idea is to produce constituent parts of vectors or functionals in Gowers' construction with sufficient control on their supports (and not just on their ranges, as would be enough to obtain the HI property for example).

The space G has a norm defined by induction as in GM, with the addition of a new term which guarantees that its basis (e_n) is 2-asymptotically unconditional, that is for any sequence of normalised vectors $N < x_1 < \ldots < x_N$, any sequence of scalars $a_1, \ldots, a_N$ and any sequence of signs $\epsilon_1, \ldots, \epsilon_N$,

$$\|\sum_{n=1}^{N} \epsilon_n a_n x_n\| \leqslant 2\|\sum_{n=1}^{N} a_n x_n\|.$$

The basis is bimonotone and, although this is not stated in [7], it may be proved as for GM that G is reflexive. It follows that the dual basis of (e_n) is also 2-asymptotically unconditional. The norm on G is defined by the implicit equation, for all $x \in c_{00}$:

$$\begin{aligned} \|x\| = \|x\|_{c_0} \vee \ \sup\Big\{ f(n)^{-1} \sum_{i=1}^{n} \|E_i x\| \ \Big| \ 2 \leqslant n, E_1 < \ldots < E_n \Big\} \\ \vee \ \sup\Big\{ |x^*(Ex)| \ \Big| \ k \in K, x^* \text{ special of length } k, E \subseteq \mathbb{N} \Big\} \\ \vee \ \sup\Big\{ \|Sx\| \ \Big| \ S \text{ is an admissible operator} \Big\}, \end{aligned}$$

where E, $E_1, \ldots, E_n$ are intervals of integers, and S is an *admissible operator* if $Sx = \frac{1}{2} \sum_{n=1}^{N} \epsilon_n E_n x$ for some sequence of signs $\epsilon_1, \ldots, \epsilon_N$ and some sequence $E_1, \ldots, E_N$ of intervals which is *admissible*, i.e. $N < E_1$ and $1 + \max E_i = \min E_{i+1}$ for every $i < N$.

R.I.S. pairs and *special pairs* are considered in [7]; first we shall need a more general definition of these. Let $x_1, \ldots, x_m$ be a R.I.S. with constant C, $m \in [\log N, \exp N]$, $N \in L$, and let $x_1^*, \ldots, x_m^*$ be successive normalised functionals. Then we call *generalised R.I.S. pair with constant* C the pair (x, x^*) defined by $x = \|\sum_{i=1}^{m} x_i\|^{-1} (\sum_{i=1}^{m} x_i)$ and $x^* = f(m)^{-1} \sum_{i=1}^{m} x_i^*$.

Let $z_1, \ldots, z_k$ be a sequence of successive normalised R.I.S. vectors with constant C, and let $z_1^*, \ldots, z_k^*$ be a special sequence such that (z_i, z_i^*) is a generalized R.I.S. pair for each i. Then we shall call *generalised special pair with constant* C the pair (z, z^*) defined by $z = \sum_{i=1}^{k} z_i$ and $z^* = f(k)^{-1/2} (\sum_{i=1}^{k} z_i^*)$. The pair $(\|z\|^{-1} z, z^*)$ will be called *normalised generalised special pair.*

Lemma 3.1. *Let (z, z^*) be a generalised special pair in G, of length $k \in K$, with constant* 2 *and such that* $\operatorname{supp} z^* \cap \operatorname{supp} z = \emptyset$. *Then*

$$\|z\| \leqslant \frac{5k}{f(k)}.$$

Proof. The proof follows classically the methods of [10] or [6]. Let $K_0 = K \setminus \{k\}$ and let g be the corresponding function given by [7] Lemma 5. To prove that $\|z\| \leqslant 5k f(k)^{-1}$ it is enough by Lemma 2.2 to prove that for any interval E such that $\|Ez\| \geqslant 1/3$, Ez is normed by some (M, g)-form with $M \geqslant 2$.

By the discussion in [7] after the definition of the norm, the only possible norming functionals apart from (M, g)-forms are of the form Sw^* where w^* is a special functional of length k and S is an "acceptable" operator. We shall not state the definition of an acceptable operator S, we shall just need to know that since such an operator is diagonal of norm at most 1, it preserves support and (M, g)-forms, [7] Lemma 6. So let $w^* = f(k)^{-1/2}(w_1^* + \cdots + w_k^*)$ be a special functional of length

k, S be an acceptable operator, and E be an interval such that $\|Ez\| \geqslant 1/3$. We need to show that Sw^* does not norm Ez.

Let t be minimal such that $w_t^* \neq z_t^*$. If $i \neq j$ or $i = j > t$ then by definition of special sequences there exist $M \neq N \in L$, $\min(M, N) \geqslant j_{2k}$, such that w_i^* and therefore Sw_i^* is an (M, f)-form and z_j is an R.I.S. vector of size N and constant 2. By [7] Lemma 8, z_j is an $\ell_{1+}^{N^{1/10}}$-average with constant 4. If $M < N$ then $2M < \log\log\log N$ so, by [7] Lemma 2, $|Sw_i^*(Ez_j)| \leqslant 12f(M)^{-1}$. If $M > N$ then $\log\log\log M > 2N$ so, by [7] Lemma 3, $|Sw_i^*(Ez_j)| \leqslant 3f(N)/N$. In both cases it follows that $|Sw_i^*(Ez_j)| \leqslant k^{-2}$.

If $i = j = t$ we simply have $|Sw_i^*(Ez_j)| \leqslant 1$. Finally if $i = j < t$ then $w_i^* = z_i^*$. and since $\operatorname{supp} Sz_i^* \subseteq \operatorname{supp} z_i^*$ and $\operatorname{supp} Ez_i \subseteq \operatorname{supp} z_i$, it follows that $Sw_i^*(Ez_j) = 0$ in this case.

Summing up we have obtained that

$$|Sw^*(Ez)| \leqslant f(k)^{-1/2}(k^2.k^{-2} + 1) = 2f(k)^{-1/2} < 1/3 \leqslant \|Ez\|.$$

Therefore Sw^* does not norm Ez and this finishes the proof. □

The next lemma is expressed in a version which may seem technical but this will make the proof that G is of type (1) more pleasant to read. At first reading, the reader may simply assume that $T = Id$ in its hypothesis.

Lemma 3.2. *Let $n \in \mathbb{N}$ and let $\epsilon > 0$. Let $(x_i)_i$ be a normalised block basis in G of length n^k and supported after $2n^k$, where $k = \min\{i \mid f(n^i) < (1+\epsilon)^i\}$, and $T : [x_i] \to G$ be an isomorphism such that (Tx_i) is also a normalised block basis. Then for any $n \in \mathbb{N}$ and $\epsilon > 0$, there exist a finite interval F and a multiple x of $\sum_{i \in F} x_i$ such that Tx is an ℓ_{1+}^n-average with constant $1 + \epsilon$, and a normalised functional x^* such that $x^*(x) > 1/2$ and $\operatorname{supp} x^* \subseteq \bigcup_{i \in F} \operatorname{range} x_i$.*

Proof. The proof from [7] that the block basis (Tx_i) contains an ℓ_{1+}^n-average with constant $1+\epsilon$ is the same as for GM, and gives that such a vector exists of the form $Tx = \lambda \sum_{i \in F} Tx_i$, thanks to the condition on the length of (x_i). We may therefore deduce that $2|F| - 1 < \operatorname{supp} x$. Let y^* be a unit functional which norms x and such that $\operatorname{range} y^* \subseteq \operatorname{range} x$. Let $x^* = Ey^*$ where E is the union of the $|F|$ intervals $\operatorname{range} x_i, i \in F$. Then $x^*(x) = y^*(x) = 1$ and by unconditional asymptoticity of G^*, $\|x^*\| \leqslant \frac{3}{2}\|y^*\| < 2$. □

The proof that G is HI requires defining “extra-special sequences” after having defined special sequences in the usual GM way. However, to prove that G is tight by range, we shall not need to enter that level of complexity and shall just use special sequences.

Proposition 3.3. *The space G is of type (1).*

Proof. Assume some normalised block-sequence (x_n) is such that $[x_n]$ embeds into $Y = [e_i, i \notin \bigcup_n \operatorname{range} x_n]$ and look for a contradiction. Passing to a subsequence and by reflexivity we may assume that there is some isomorphism $T : [x_n] \to Y$ satisfying the hypothesis of Lemma 3.2, that is, (Tx_n) is a normalised block basis in Y. Fixing $\epsilon = 1/10$ we may construct by Lemma 3.2 some block-sequence of vectors in $[x_n]$ which are 1/2-normed by functionals in $\mathbf{Q}$ of support included in $\bigcup_n \operatorname{range} x_n$, and whose images by T form a sequence of increasing length ℓ_{1+}^n-averages with constant $1 + \epsilon$. If $Tz_1, \ldots, Tz_m$ is a R.I.S. of these ℓ_{1+}^n-averages with constant $1 + \epsilon$, with $m \in [\log N, \exp N]$, $N \in L$, and $z_1^*, \ldots, z_m^*$ are the functionals associated to $z_1, \ldots, z_m$, then by [7] Lemma 7, the (m,f)-form $z^* = f(m)^{-1}(z_1^* + \ldots + z_m^*)$ satisfies

$$z^*(z_1 + \ldots + z_m) \geqslant \frac{m}{2f(m)} \geqslant \frac{1}{4}\|Tz_1 + \ldots + Tz_m\| \geqslant (4\|T^{-1}\|)^{-1}\|z_1 + \cdots + z_m\|.$$

Therefore we may build R.I.S. vectors Tz with constant $1+\epsilon$ of arbitrary length m in $[\log N, \exp N]$, $N \in L$, so that z is $(4\|T^{-1}\|)^{-1}$-normed by an (m,f)-form z^* of support included in $\bigcup_n \text{range}\, x_n$. For such (z, z^*), (Tz, z^*) is a generalised R.I.S. pair. We then consider a sequence $Tz_1, \ldots, Tz_k$ of length $k \in K$ of such R.I.S. vectors, such that there exists some special sequence of corresponding functionals $z_1^*, \ldots, z_k^*$, and finally the pair (z, z^*) where $z = z_1 + \cdots + z_k$ and $z^* = f(k)^{-1/2}(z_1^* + \ldots + z_k^*)$: observe that the support of z^* is still included in $\bigcup_n \text{range}\, x_n$. Since (Tz, z^*) is a generalised special pair, it follows from Lemma 3.1 that

$$\|Tz\| \leqslant 5kf(k)^{-1}.$$

On the other hand,

$$\|z\| \geqslant z^*(z) \geqslant (4\|T^{-1}\|)^{-1}kf(k)^{-1/2}.$$

Since k was arbitrary in K this implies that T^{-1} is unbounded and provides the desired contradiction. $\square$

As we shall now prove, the dual G^* of G is of type (1) as well, but also locally minimal.

Lemma 3.4. *Let (x_i^*) be a normalised block basis in G^*. Then for any $n \in \mathbb{N}$ and $\epsilon > 0$, there exists $N(n,\epsilon)$, a finite interval $F \subseteq [1, N(n,\epsilon)]$, a multiple x^* of $\sum_{i \in F} x_i^*$ which is an $\ell_{\infty+}^n$-average with constant $1+\epsilon$ and an ℓ_{1+}^n-average x with constant 2 such that $x^*(x) > 1/2$ and $\text{supp}\, x \subseteq \bigcup_{i \in F} \text{range}\, x_i^*$.*

Proof. We may assume that $\epsilon < 1/6$. By Lemma 2.1 we may find for each $i \leqslant n$ an interval F_i, with $|F_i| \leqslant 2 \min F_i$, and a vector y_i^* of the form $\lambda \sum_{k \in F_i} x_k^*$, such that $y^* = \sum_{i=1}^n y_i^*$ is an $\ell_{\infty+}^n$-average with constant $1+\epsilon$. Let, for each i, x_i be normalised such that $y_i^*(x_i) = \|y_i^*\|$ and $\text{range}\, x_i \subseteq \text{range}\, y_i^*$. Let $y_i = E_i x_i$, where E_i denotes the canonical projection on $[e_m, m \in \bigcup_{k \in F_i} \text{range}\, x_k^*]$. By the asymptotic unconditionality of (e_n), we have that $\|y_i\| \leqslant 3/2$. Let $y_i' = \|y_i\|^{-1} y_i$, then

$$y_i^*(y_i') = \|y_i\|^{-1} y_i^*(y_i) = \|y_i\|^{-1} y_i^*(x_i) \geqslant \frac{2}{3}\|y_i^*\|.$$

By Lemma 2.1, the vector $x = \sum_i y_i'$ is an ℓ_{1+}^n-vector with constant 2, such that $x^*(x) > \|x\|/2$, and clearly $\text{supp}\, x \subseteq \bigcup_{i \in F} \text{range}\, x_i^*$. $\square$

Proposition 3.5. *The space G^* is locally minimal and tight by range.*

Proof. By Lemma 3.4 we may find in any finite block subspace of G^* of length $N(n,\epsilon)$ and supported after $N(n,\epsilon)$ an $\ell_{\infty+}^n$-average with constant $1+\epsilon$. By asymptotic unconditionality we deduce that uniformly, any block-subspace of G^* contains ℓ_∞^n's, and therefore G^* is locally minimal.

We prove that G^* is tight by range. Assume towards a contradiction that some normalised block-sequence (x_n^*) is such that $[x_n^*]$ embeds into $Y = [e_i^*, i \notin \bigcup_n \text{range}\, x_n^*]$ and look for a contradiction. If T is the associated isomorphism, we may by passing to a subsequence and perturbating T assume that Tx_n^* is successive.

Let $\epsilon = 1/10$. By Lemma 3.4, we find in $[x_k^*]$ and for each n, an $\ell_{\infty+}^n$-average y_n^* with constant $1+\epsilon$ and an ℓ_{1+}^n-average y_n with constant 2, such that $y_n^*(y_n) > 1/2$ and $\operatorname{supp} y_n \subseteq \bigcup_k \operatorname{range} x_k^*$. By construction, for each n, Ty_n^* is disjointly supported from $[x_k^*]$, and up to modifying T, we may assume that Ty_n^* is in $\mathbf{Q}$ and of norm at most 1 for each n.

If $z_1, \ldots, z_m$ is a R.I.S. of these ℓ_{1+}^n-averages y_n with constant 2, with $m \in [\log N, \exp N]$, $N \in L$, and $z_1^*, \ldots, z_m^*$ are the $\ell_{\infty+}^n$-averages associated to $z_1, \ldots, z_m$, then by [6] Lemma 7, the (m, f)-form $z^* = f(m)^{-1}(z_1^* + \ldots + z_m^*)$ satisfies

$$z^*(z_1 + \ldots + z_m) \geqslant \frac{m}{2f(m)} \geqslant \frac{1}{6}\|z_1 + \ldots + z_m\|,$$

and furthermore Tz^* is also an (m, f)-form. Therefore we may build R.I.S. vectors z with constant 2 of arbitrary length m in $[\log N, \exp N]$, $N \in L$, so that z is 6^{-1}-normed by an (m, f)-form z^* such that Tz^* is also an (m, f)-form. We may then consider a sequence $z_1, \ldots, z_k$ of length $k \in K$ of such R.I.S. vectors of length m_i, and some corresponding functionals $z_1^*, \ldots, z_k^*$ (i.e., z_i^* 6^{-1}-norms z_i and Tz_i^* is also an (m_i, f)-form for all i), such that $Tz_1^*, \ldots, Tz_k^*$ is a special sequence. Then we let $z = z_1 + \cdots + z_k$ and $z^* = f(k)^{-1/2}(z_1^* + \ldots + z_k^*)$, and observe that (z, Tz^*) is a generalised special pair. Since $Tz^* = f(k)^{-1/2}(Tz_1^* + \ldots + Tz_k^*)$ is a special functional it follows that

$$\|Tz^*\| \leqslant 1.$$

But it follows from Lemma 3.1 that $\|z\| \leqslant 5kf(k)^{-1}$. Therefore

$$\|z^*\| \geqslant z^*(z)/\|z\| \geqslant f(k)^{1/2}/30.$$

Since k was arbitrary in K this implies that T^{-1} is unbounded and provides the desired contradiction. □

It remains to check that G^* is HI. The proof is very similar to the one in [7] that G is HI, and we shall therefore not give all details. There are two main differences between the two proofs. In [7] some special vectors and functionals are constructed, and the vectors are taken alternatively in arbitrary block subspaces Y and Z of G. In our case we need to take the functionals in arbitrary subspaces Y_* and Z_* of G^* instead. This is possible because of Lemma 3.4. We also need to correct what seems to be a slight imprecision in the proof of [7] about the value of some normalising factors, and therefore we also get worst constants for our estimates.

Let $\epsilon = 1/10$. Following Gowers we define an *R.I.S. pair* of size N to be a generalised R.I.S. pair (x, x^*) with constant $1+\epsilon$ of the form $(\|x_1+\ldots+x_N\|^{-1}(x_1+\ldots+x_N), f(N)^{-1}(x_1^*+\cdots+x_N^*))$, where $x_n^*(x_n) \geqslant 1/3$ and $\operatorname{range} x_n^* \subset \operatorname{range} x_n$ for each n. A *special pair* is a normalised generalised special pair with constant $1+\epsilon$ of the form (x, x^*) where $x = \|x_1 + \ldots + x_k\|^{-1}(x_1 + \ldots + x_k)$ and $x^* = f(k)^{-1/2}(x_1^* + \cdots + x_k^*)$ with $\operatorname{range} x_n^* \subseteq \operatorname{range} x_n$ and for each n, $x_n^* \in \mathbf{Q}$, $|x_n^*(x_n) - 1/2| < 10^{-\min \operatorname{supp} x_n}$. By [7] Lemma 8, z is a R.I.S. vector with constant 2 whenever (z, z^*) is a special pair. We shall also require that $k \leqslant \min \operatorname{supp} x_1$, which will imply by [7] Lemma 9 that for $m < k^{1/10}$, z is a ℓ_{1+}^m-average with constant 8 (see the beginning of the proof of Proposition 3.6).

Going up a level of "specialness", a *special R.I.S.-pair* is a generalised R.I.S.-pair with constant 8 of the form $(\|x_1+\ldots+x_N\|^{-1}(x_1+\ldots+x_N), f(N)^{-1}(x_1^*+\cdots+x_N^*))$, where $\operatorname{range} x_n^* \subset \operatorname{range} x_n$ for each n, and with the additional condition that (x_n, x_n^*) is a special pair of length at least $\min \operatorname{supp} x_n$. Finally, an *extra-special* *pair* of size k is a normalised generalised special pair (x,x^*) with constant 8 of the form $x = \|x_1+\ldots+x_k\|^{-1}(x_1+\ldots+x_k)$ and $x^* = f(k)^{-1/2}(x_1^*+\cdots+x_k^*)$ with range $x_n^* \subseteq$ range x_n, such that, for each n, (x_n,x_n^*) is a special R.I.S.-pair of length $\sigma(x_1^*,\ldots,x_{n-1}^*)$.

Given Y_*, Z_* block subspaces of G^* we shall show how to find an extra-special pair (x,x^*) of size k, with x^* built out of vectors in Y_* or Z_*, such that the signs of these constituent parts of x^* can be changed according to belonging to Y_* or Z_* to produce a vector x'^* with $\|x'^*\| \leqslant 12f(k)^{-1/2}\|x^*\|$. This will then prove the result.

Consider then an extra-special pair (x,x^*). Then x splits up as

$$\nu^{-1}\sum_{i=1}^{k}\nu_i^{-1}\sum_{j=1}^{N_i}\nu_{ij}^{-1}\sum_{r=1}^{k_{ij}}x_{ijr}$$

and x^* as

$$f(k)^{-1/2}\sum_{i=1}^{k}f(N_i)^{-1}\sum_{j=1}^{N_i}f(k_{ij})^{-1}\sum_{r=1}^{k_{ij}}x_{ijr}^*$$

where the numbers ν, ν_i and ν_{ij} are the norms of what appears to the right. These special sequences are chosen far enough "to the right" so that $k_{ij} \leqslant \min\operatorname{supp} x_{ij1}$, and also so that $(\max\operatorname{supp}\, x_{i\,j-1})^2 k_{ij}^{-1} \leqslant 4^{-(i+j)}$. We shall also write x_i for $\nu_i^{-1}\sum_{j=1}^{N_i}\nu_{ij}^{-1}\sum_{r=1}^{k_{ij}}x_{ijr}$ and x_{ij} for $\nu_{ij}^{-1}\sum_{r=1}^{k_{ij}}x_{ijr}$.

We define a vector x' by

$$\sum_{i=1}^{k}\nu_i'^{-1}\sum_{j=1}^{N_i}\nu_{ij}'^{-1}\sum_{r=1}^{k_{ij}}(-1)^r x_{ijr},$$

where the numbers ν_i' and ν_{ij}' are the norms of what appears to the right. We shall write x_i' for $\nu_i'^{-1}\sum_{j=1}^{N_i}\nu_{ij}'^{-1}\sum_{r=1}^{k_{ij}}(-1)^r x_{ijr}$ and x_{ij}' for $\nu_{ij}'^{-1}\sum_{r=1}^{k_{ij}}(-1)^r x_{ijr}$.

Finally we define a functional x'^* as

$$f(k)^{-1/2}\sum_{i=1}^{k}f(N_i)^{-1}\sum_{j=1}^{N_i}f(k_{ij})^{-1}\sum_{r=1}^{k_{ij}}(-1)^k x_{ijr}^*.$$

Proposition 3.6. *The space G^* is HI.*

Proof. Fix Y_* and Z_* block subspaces of G^*. By Lemma 3.4 we may construct an extra-special pair (x,x^*) so that x_{ijr}^* belongs to Y_* when r is odd and to Z_* when r is even.

We first discuss the normalisation of the vectors involved in the definition of x'. By the increasing condition on k_{ij} and x_{ijr} and by asymptotic unconditionality, we have that

$$\|\sum_{r=1}^{k_{ij}}(-1)^r x_{ijr}\| \leqslant 2\|\sum_{r=1}^{k_{ij}}x_{ijr}\|,$$

which means that $\nu_{ij}' \leqslant 2\nu_{ij}$. Furthermore it also follows that the functional $(1/2)f(k_{ij})^{-1/2}\sum_{r=1}^{k_{ij}}(-1)^r x_{ijr}^*$ is of norm at most 1, and therefore we have that $\|\sum_{r=1}^{k_{ij}}(-1)^r x_{ijr}\| \geqslant (1/2)k_{ij}f(k_{ij})^{-1/2}$. Lemma 9 from [7] therefore tells us that, for every i,j, x_{ij}' is an $\ell_{1+}^{m_{ij}}$-average with constant 8, if $m_{ij} < k_{ij}^{1/10}$. But the k_{ij}

increase so fast that, for any i, this implies that the sequence $x'_{i1}, \ldots, x'_{i\,N_i}$ is a rapidly increasing sequence with constant 8. By [7] Lemma 7, it follows that

$$\|\sum_{j=1}^{N_i} x'_{ij}\| \leqslant 9N_i/f(N_i).$$

Therefore by the f-lower estimate in G we have that $\nu'_i \leqslant 9\nu_i$.

We shall now prove that $\|x'\| \leqslant 12kf(k)^{-1}$. This will imply that

$$\|x'_*\| \geqslant \frac{x'_*(x')}{\|x'\|} \geqslant \frac{f(k)}{12k}[f(k)^{-1/2}\sum_{i=1}^{k} f(N_i)^{-1}\nu_i'^{-1}\sum_{j=1}^{N_i} f(k_{ij})^{-1}\nu_{ij}'^{-1}\sum_{r=1}^{k_{ij}} x^*_{ijr}(x_{ijr})]$$

$$\geqslant f(k)^{1/2}(12k)^{-1}.18^{-1}[\sum_{i=1}^{k} f(N_i)^{-1}\nu_i^{-1}\sum_{j=1}^{N_i} f(k_{ij})^{-1}\nu_{ij}^{-1}\sum_{r=1}^{k_{ij}} x^*_{ijr}(x_{ijr})]$$

$$= f(k)^{1/2}(216k)^{-1}\sum_{i=1}^{k} x_i^*(x_i) \geqslant 648^{-1}f(k)^{1/2}.$$

By construction of x^* and x'^* this will imply that

$$\|y^* - z^*\| \geqslant 648^{-1}f(k)^{1/2}\|y^* + z^*\|$$

for some non zero $y^* \in Y_*$ and $z^* \in Z_*$, and since $k \in K$ was arbitrary, as well as Y_* and Z_*, this will prove that G^* is HI.

The proof that $\|x'\| \leqslant 12kf(k)^{-1}$ is given in three steps:

Step 1. *The vector x' is a R.I.S. vector with constant 11.*

Proof. We already know the sequence $x'_{i1}, \ldots, x'_{i\,N_i}$ is a rapidly increasing sequence with constant 8. Then by [7] Lemma 8 we get that x'_i is also an $\ell_{1+}^{M_i}$-average with constant 11, if $M_i < N_i^{1/10}$. Finally, this implies that x' is an R.I.S.-vector with constant 11, as claimed. □

Step 2. *Let $K_0 = K \setminus \{k\}$, let $g \in \mathcal{F}$ be the corresponding function given by* [7] *Lemma 5. For every interval E such that $\|Ex'\| \geqslant 1/3$, Ex' is normed by an (M, g)-form.*

Proof. The proof is exactly the same as the one of Step 2 in the proof of Gowers concerning G, apart from some constants which are modified due to the change of constant in Step 1 and to the normalising constants relating ν_i and ν_{ij} respectively to ν'_i and ν'_{ij}. The reader is therefore referred to [7]. □

Step 3. *The norm of x' is at most $12kg(k)^{-1} = 12kf(k)^{-1}$*

Proof. This is an immediate consequence of Step 1, Step 2 and of Lemma 2.2. □

We conclude that the space G^* is HI, and thus locally minimal of type (1). □

Let us observe that the examples of locally minimal, non-minimal, spaces we have produced so far could be said to be so for trivial reasons: since they hereditarily contain ℓ_∞^n's uniformly, any Banach space is crudely finitely representable in any of their subspaces. It remains open whether there exists a tight and locally minimal Banach space which does not contain ℓ_∞^n's uniformly, i.e., which has finite cotype.

4. Unconditional tight spaces of the type of Argyros and Deliyanni

By Proposition 2.4, unconditional or HI spaces built on the model of Gowers-Maurey's spaces are uniformly inhomogeneous (and even block uniformly inhomogeneous). We shall now consider a space of Argyros-Deliyanni type, more specifically of the type of a space constructed by Argyros, Deliyanni, Kutzarova and Manoussakis [2], with the opposite property, i.e., with a basis which is strongly asymptotically ℓ_1. This space will also be tight by support. By Proposition dddddd from [5] this basis will therefore be tight with constants as well, making this example the "worst" known so far in terms of minimality.

Again in this section block vectors will not necessarily be normalized and some familiarity with the construction in [2] will be assumed.

4.1. **A strongly asymptotically ℓ_1 space tight by support.** In [2] an example of HI space X_{hi} is constructed, based on a "boundedly modified" mixed Tsirelson space $X_{M(1),u}$. We shall construct an unconditional version X_u of X_{hi} in a similar way as G_u is an unconditional version of GM. The proof that X_u is of type (3) will be based on the proof that X_{hi} is HI, conditional estimates in the proof of [2] becoming essentially trivial in our case due to disjointness of supports.

Fix a basis (e_n) and $\mathcal{M}$ a family of finite subsets of $\mathbb{N}$. Recall that a family $x_1, \ldots, x_n$ is $\mathcal{M}$-*admissible* if $x_1 < \cdots < x_n$ and $\{\min \text{supp } x_1, \ldots, \min \text{supp } x_n\} \in \mathcal{M}$, and $\mathcal{M}$-*allowable* if $x_1, \ldots, x_n$ are vectors with disjoint supports such that $\{\min \text{supp } x_1, \ldots, \min \text{supp } x_n\} \in \mathcal{M}$. Let $\mathcal{S}$ denote the family of Schreier sets, i.e., of subsets F of $\mathbb{N}$ such that $|F| \leqslant \min F$, $\mathcal{M}_j$ be the subsequence of the sequence $(\mathcal{F}_k)$ of Schreier families associated to sequences of integers t_j and k_j defined in [2] p 70.

We need to define a new notion. For W a set of functionals which is stable under projections onto subsets of $\mathbb{N}$, we let $\text{conv}_{\mathbb{Q}} W$ denote the set of rational convex combinations of elements of W. By the stability property of W we may write any $c^* \in \text{conv}_{\mathbb{Q}} W$ as a rational convex combination of the form $\sum_i \lambda_i x_i^*$ where $x_i^* \in W$ and supp $x_i^* \subseteq$ supp c^* for each i. In this case the set $\{x_i^*\}_i$ will be called a W-compatible decomposition of c^*, and we let $W(c^*) \subseteq W$ be the union of all W-compatible decompositions of c^*. Note that if $\mathcal{M}$ is a family of finite subsets of $\mathbb{N}$, $(c_1^*, \ldots, c_d^*)$ is $\mathcal{M}$-admissible, and $x_i^* \in W(c_i^*)$ for all i, then $(x_1^*, \ldots, x_d^*)$ is also $\mathcal{M}$-admissible.

Let $\mathcal{B} = \{\sum_n \lambda_n e_n : (\lambda_n)_n \in c_{00}, \lambda_n \in \mathbb{Q} \cap [-1, 1]\}$ and let Φ be a 1-1 function from $\mathcal{B}^{<\mathbb{N}}$ into $2\mathbb{N}$ such that if $(c_1^*, \ldots, c_k^*) \in \mathcal{B}^{<\mathbb{N}}$, j_1 is minimal such that $c_1^* \in \text{conv}_{\mathbb{Q}} \mathcal{A}_{j_1}$, and $j_l = \Phi(c_1^*, \ldots, c_{l-1}^*)$ for each $l = 2, 3, \ldots$, then $\Phi(c_1^*, \ldots, c_k^*) > \max\{j_1, \ldots, j_k\}$ (the set $\mathcal{A}_j$ is defined in [2] p 71 by $\mathcal{A}_j = \cup_n (K_j^n \setminus K^0)$ where the K_j^n's are the sets corresponding to the inductive definition of $X_{M(1),u}$).

For $j = 1, 2, \ldots$, we set $L_j^0 = \{\pm e_n : n \in \mathbb{N}\}$. Suppose that $\{L_j^n\}_{j=1}^{\infty}$ have been defined. We set $L^n = \cup_{j=1}^{\infty} L_j^n$ and

$$L_1^{n+1} = \pm L_1^n \cup \{\frac{1}{2}(x_1^* + \ldots + x_d^*) : d \in \mathbb{N}, x_i^* \in L^n,$$

$$(x_1^*, \ldots, x_d^*) \text{ is } \mathcal{S}-\text{allowable}\},$$

and for $j \geqslant 1$,

$$L_{2j}^{n+1} = \pm L_{2j}^n \cup \{\frac{1}{m_{2j}}(x_1^* + \ldots + x_d^*) : d \in \mathbb{N}, x_i^* \in L^n,$$

$$(x_1^*, \ldots, x_d^*) \text{ is } \mathcal{M}_{2j} - \text{admissible}\},$$

$$L'^{\,n+1}_{2j+1} = \pm L^n_{2j+1} \cup \{\frac{1}{m_{2j+1}}(x_1^* + \ldots + x_d^*) : d \in \mathbb{N} \text{ such that}$$

$$\exists (c_1^*, \ldots, c_d^*)\ \mathcal{M}_{2j+1} - \text{admissible and } k > 2j+1 \text{ with } c_1^* \in \mathrm{conv}_{\mathbb{Q}} L^n_{2k}, x_1^* \in L^n_{2k}(c_1^*),$$

$$c_i^* \in \mathrm{conv}_{\mathbb{Q}} L^n_{\Phi(c_1^*, \ldots, c_{i-1}^*)}, x_i^* \in L^n_{\Phi(c_1^*, \ldots, c_{i-1}^*)}(c_i^*) \text{ for } 1 < i \leqslant d\},$$

$$L^{n+1}_{2j+1} = \{Ex^* : x^* \in L'^{\,n+1}_{2j+1}, E \text{ subset of } \mathbb{N}\}.$$

We set $\mathcal{B}_j = \cup_{n=1}^{\infty}(L_j^n \setminus L^0)$ and we consider the norm on c_{00} defined by the set $L = L^0 \cup (\cup_{j=1}^{\infty} \mathcal{B}_j)$. The space X_u is the completion of c_{00} under this norm.

In [2] the space X_{hi} is defined in the same way except that E is an **interval** of integers in the definition of L^{n+1}_{2j+1}, and the definition of $L'^{\,n+1}_{2j+1}$ is simpler, i.e., the coding Φ is defined directly on $\mathcal{M}_{2j+1}$-admissible families $x_1^*, \ldots, x_d^*$ in $L^{<\mathbb{N}}$ and in the definition each x_i^* belongs to $L^n_{\Phi(x_1^*, \ldots, x_{i-1}^*)}$. To prove the desired properties for X_u one could use the simpler definition of $L'^{\,n+1}_{2j+1}$; however this definition doesn't seem to provide enough special functionals to obtain interesting properties for the dual as well.

The ground space for X_{hi} and for X_u is the space $X_{M(1),u}$ associated to a norming set K defined by the same procedure as L, except that K^n_{2j+1} is defined in the same way as K^n_{2j}, i.e.

$$K^{n+1}_{2j} = \pm K^n_{2j} \cup \{\frac{1}{m_{2j}}(x_1^* + \ldots + x_d^*) : d \in \mathbb{N}, x_i^* \in K^n,$$

$$(x_1^*, \ldots, x_d^*) \text{ is } \mathcal{M}_{2j+1} - \text{admissible}\}.$$

For $n = 0, 1, 2, \ldots$, we see that L_j^n is a subset of K_j^n, and therefore $L \subseteq K$. The norming set L is closed under projections onto **subsets** of $\mathbb{N}$, from which it follows that its canonical basis is unconditional, and has the property that for every j and every $\mathcal{M}_{2j}$–admissible family $f_1, f_2, \ldots f_d$ contained in L, $f = \frac{1}{m_{2j}}(f_1 + \cdots + f_d)$ belongs to L. The *weight* of such an f is defined by $w(f) = 1/m_{2j}$. It follows that for every $j = 1, 2, \ldots$ and every $\mathcal{M}_{2j}$–admissible family $x_1 < x_2 < \ldots < x_n$ in X_u,

$$\|\sum_{k=1}^n x_k\| \geqslant \frac{1}{m_{2j}} \sum_{k=1}^n \|x_k\|.$$

Likewise, for $\mathcal{S}$–allowable families $f_1, \ldots, f_n$ in L, we have $f = \frac{1}{2}(f_1 + \cdots + f_d) \in L$, and we define $w(f) = 1/2$. The weight is defined similarly in the case $2j+1$.

Lemma 4.1. *The canonical basis of X_u is strongly asymptotically ℓ_1.*

Proof. Fix $n \leqslant x_1, \ldots, x_n$ where $x_1, \ldots, x_n$ are normalised and disjointly supported. Fix $\epsilon > 0$ and let for each i, $f_i \in L$ be such that $f_i(x_i) \geqslant (1+\epsilon)^{-1}$ and supp $f_i \subseteq$ supp x_i. The condition on the supports may be imposed because L is stable under projections onto subsets of $\mathbb{N}$. Then $\frac{1}{2}\sum_{i=1}^n \pm f_i \in L$ and therefore

$$\|\sum_{i=1}^n \lambda_i x_i\| \geqslant \frac{1}{2}\sum_{i=1}^n |\lambda_i| f_i(x_i) \geqslant \frac{1}{2(1+\epsilon)} \sum_{i=1}^n |\lambda_i|,$$

for any λ_i's. Therefore $x_1, \ldots, x_n$ is 2-equivalent to the canonical basis of ℓ_1^n. □

It remains to prove that X_u has type (3). Recall that an analysis $(K^s(f))_s$ of $f \in K$ is a decomposition of f corresponding to the inductive definition of K, see the precise definition in Definition 2.3 [2]. We shall combine three types of arguments. First L was constructed so that $L \prec K$, which means essentially that each $f \in L$ has an analysis $(K^s(f))_s$ whose elements actually belong to L (see the definition on page 74 of [2]); so all the results obtained in Section 2 of [2] for spaces defined through arbitrary $\tilde{K} \prec K$ (and in particular the crucial Proposition 2.9) are valid in our case. Then we shall produce estimates similar to those valid for X_{hi} and which are of two forms: unconditional estimates, in which case the proofs from [2] may be applied directly up to minor changes of notation, and thus we shall refer to [2] for details of the proofs; and conditional estimates, which are different from those of X_{hi}, but easier due to hypotheses of disjointness of supports.

Recall that if $\mathcal{F}$ is a family of finite subsets of $\mathbb{N}$, then

$$\mathcal{F}' = \{A \cup B : A, B \in \mathcal{F}, A \cap B = \emptyset\}.$$

Given $\varepsilon > 0$ and $j = 2, 3, \ldots$, an (ε, j)-*basic special convex combination* ((ε, j)-*basic s.c.c.) (relative to* $X_{M(1),u}$) is a vector of the form $\sum_{k \in F} a_k e_k$ such that: $F \in \mathcal{M}_j, a_k \geqslant 0, \sum_{k \in F} a_k = 1$, $\{a_k\}_{k \in F}$ is decreasing, and, for every $G \in \mathcal{F}'_{t_j(k_{j-1}+1)}$, $\sum_{k \in G} a_k < \varepsilon$.

Given a block sequence $(x_k)_{k \in \mathbf{N}}$ in X_u and $j \geqslant 2$, a convex combination $\sum_{i=1}^n a_i x_{k_i}$ is said to be an (ε, j)-*special convex combination* of $(x_k)_{k \in \mathbf{N}}$ ((ε, j)-s.c.c), if there exist $l_1 < l_2 < \ldots < l_n$ such that $2 < \operatorname{supp} x_{k_1} \leqslant l_1 < \operatorname{supp} x_{k_2} \leqslant l_2 < \ldots < \operatorname{supp} x_{k_n} \leqslant l_n$, and $\sum_{i=1}^n a_i e_{l_i}$ is an (ε, j)-basic s.c.c. An (ε, j)-s.c.c. $\sum_{i=1}^n a_i x_{k_i}$ is called *seminormalised* if $\|x_{k_i}\| = 1,\ i = 1, \ldots, n$ and

$$\|\sum_{i=1}^n a_i x_{k_i}\| \geqslant \frac{1}{2}.$$

Rapidly increasing sequences and (ε, j)–R.I. special convex combinations in X_u are defined by [2] Definitions 2.8 and 2.16 respectively, with $\tilde{K} = L$.

Using the lower estimate for $\mathcal{M}_{2j}$-admissible families in X_u we get as in [2] Lemma 3.1.

Lemma 4.2. *For $\epsilon > 0$, $j = 1, 2, \ldots$ and every normalised block sequence $\{x_k\}_{k=1}^\infty$ in X_u, there exists a finite normalised block sequence $(y_s)_{s=1}^n$ of (x_k) and coefficients $(a_s)_{s=1}^n$ such that $\sum_{s=1}^n a_s y_s$ is a seminormalised $(\epsilon, 2j)$–s.c.c..*

The following definition is inspired from some of the hypotheses of [2] Proposition 3.3.

Definition 4.3. *Let $j > 100$. Suppose that $\{j_k\}_{k=1}^n$, $\{y_k\}_{k=1}^n$, $\{c_k^*\}_{k=1}^n$ and $\{b_k\}_{k=1}^n$ are such that*

(i) *There exists a rapidly increasing sequence*

$$\{x_{(k,i)} :\ k = 1, \ldots, n,\ i = 1, \ldots, n_k\}$$

with $x_{(k,i)} < x_{(k,i+1)} < x_{(k+1,l)}$ for all $k < n$, $i < n_k$, $l \leqslant n_{k+1}$, such that:
(a) *Each $x_{(k,i)}$ is a seminormalised $(\frac{1}{m^4_{j_{(k,i)}}}, j_{(k,i)})$–s.c.c. where, for each k, $2j_k + 2 < j_{(k,i)}$, $i = 1, \ldots n_k$.*
(b) *Each y_k is a $(\frac{1}{m^4_{2j_k}}, 2j_k)$– R.I.s.c.c. of $\{x_{(k,i)}\}_{i=1}^{n_k}$ of the form $y_k = \sum_{i=1}^{n_k} b_{(k,i)} x_{(k,i)}$.*
(c) *The sequence $\{b_k\}_{k=1}^n$ is decreasing and $\sum_{k=1}^n b_k y_k$ is a $(\frac{1}{m^4_{2j+1}}, 2j+1)$–s.c.c.*

(ii) $c_k^* \in \mathrm{conv}_{\mathbb{Q}} L_{2j_k}$, *and* $\max(\mathrm{supp}\ c_{k-1}^* \cup \mathrm{supp}\ y_{k-1}) < \min(\mathrm{supp}\ c_k^* \cup \mathrm{supp}\ y_k)$, $\forall k$.

(iii) $j_1 > 2j+1$ *and* $2j_k = \Phi(c_1^*, \dots, c_{k-1}^*)$, $k = 2, \dots, n$.

Then $(j_k, y_k, c_k^*, b_k)_{k=1}^n$ *is said to be a* j-quadruple.

The following proposition is essential. It is the counterpart of Lemma 3.1 for the space X_u.

Proposition 4.4. *Assume that* $(j_k, y_k, c_k^*, b_k)_{k=1}^n$ *is a j-quadruple in* X_u *such that* $\mathrm{supp}\ c_k^* \cap \mathrm{supp}\ y_k = \emptyset$ *for all* $k = 1, \dots, n$. *Then*

$$\|\sum_{k=1}^n b_k m_{2j_k} y_k\| \leqslant \frac{75}{m_{2j+1}^2}.$$

Proof. Our aim is to show that for every $\varphi \in \cup_{i=1}^\infty \mathcal{B}_i$,

$$\varphi(\sum_{k=1}^n b_k m_{2j_k} y_k) \leqslant \frac{75}{m_{2j+1}^2}.$$

The proof is given in several steps.

`1st Case`: $w(\varphi) = \frac{1}{m_{2j+1}}$. Then φ has the form $\varphi = \frac{1}{m_{2j+1}}(Ey_1^* + \dots + Ey_{k_2}^* + Ey_{k_2+1}^* + \cdots Ey_d^*)$ where E is a subset of $\mathbb{N}$ and where $y_k^* \in L_{2j_k}(c_k^*)$ $\forall k \leqslant k_2$ and $y_k^* \in L_{2j_k}(d_k^*)$ $\forall k \geqslant k_2+1$, with $d_{k_2+1}^* \neq c_{k_2+1}^*$ (this is similar to the form of such a functional in X_{hi} but with the integer k_1 defined there equal to 1 in our case).

If $k \leqslant k_2$ then c_s^* and therefore y_s^* is disjointly supported from y_k, so $Ey_s^*(y_k) = 0$ for all s, and therefore $\varphi(y_k) = 0$. If $k = k_2+1$ then we simply have $|\varphi(y_k)| \leqslant \|y_k\| \leqslant 17 m_{2j_k}^{-1}$, [2] Corollary 2.17. Finally if $k > k_2+1$ then since Φ is 1-1, we have that $j_{k_2+1} \neq j_k$ and for all $s = k_2+1, \dots, d$, d_s^* and therefore y_s^* belong to $\mathcal{B}_{2t_s}$ with $t_s \neq j_k$. It is then easy to check that we may reproduce the proof of [2] Lemma 3.5, applied to $Ey_1^*, \dots, Ey_d^*$, to obtain the unconditional estimate

$$|\varphi(m_{2j_k} y_k)| \leqslant \frac{1}{m_{2j+1}^2}.$$

In particular instead of [2] Proposition 3.2, which is a reformulation of [2] Corollary 2.17 for X_{hi}, we simply use [2] Corollary 2.17 with $\tilde{K} = L$.

Summing up these estimates we obtain the desired result for the 1st Case.

`2nd Case`: $w(\varphi) \leqslant \frac{1}{m_{2j+2}}$. Then we get an unconditional estimate for the evaluation of $\varphi(\sum_{k=1}^n b_k m_{2j_k} y_k)$ directly, reproducing the short proof of [2] Lemma 3.7, using again [2] Corollary 2.17 instead of [2] Proposition 3.2. Therefore

$$|\varphi(\sum_{k=1}^n b_k m_{2j_k} y_k)| \leqslant \frac{35}{m_{2j+2}} \leqslant \frac{35}{m_{2j+1}^2}.$$

`3rd Case`: $w(\varphi) > \frac{1}{m_{2j+1}}$. We have $y_k = \sum_{i=1}^{n_k} b_{(k,i)} x_{(k,i)}$ and the sequence $\{x_{(k,i)}, k = 1, \dots n, i = 1, \dots n_k\}$ is a R.I.S. w.r.t. L. By [2] Proposition 2.9 there exist a functional $\psi \in K'$ (see the definition in [2] p 71) and blocks of the basis $u_{(k,i)}$, $k = 1, \dots, n$, $i = 1, \dots, n_k$ with $\mathrm{supp}\ u_{(k,i)} \subseteq \mathrm{supp}\ x_{(k,i)}$, $\|u_k\|_{\ell_1} \leqslant 16$ and such that

$$|\varphi(\sum_{k=1}^n b_k m_{2j_k}(\sum_{i=1}^{n_k} b_{(k,i)} x_{(k,i)}))| \leqslant m_{2j_1} b_1 b_{(1,1)} + \psi(\sum_{k=1}^n b_k m_{2j_k}(\sum_{i=1}^{k_n} b_{(k,i)} u_{(k,i)})) + \frac{1}{m_{2j+2}^2}$$

$$\leqslant \psi(\sum_{k=1}^{n} b_k m_{2j_k}(\sum_{i=1}^{k_n} b_{(k,i)}u_{(k,i)})) + \frac{1}{m_{2j+2}}.$$

Therefore it suffices to estimate

$$\psi(\sum_{k=1}^{n} b_k m_{2j_k}(\sum_{i=1}^{n_k} b_{(k,i)}u_{(k,i)})).$$

In [2] ψ is decomposed as $\psi_1 + \psi_2$ and different estimates are applied to ψ_1 and ψ_2. Our case is easier as we may simply assume that $\psi_1 = 0$ and $\psi_2 = \psi$. We shall therefore refer to some arguments of [2] concerning some ψ_2 keeping in mind that $\psi_2 = \psi$.

Let $D_1^k, \dots, D_4^k$ be defined as in [2] Lemma 3.11 (a). Then as in [2],

$$\bigcup_{p=1}^{4} D_p^k = \bigcup_{i=1}^{n_k} \text{supp } u_{(k,i)} \cap \text{supp } \psi.$$

The proof that

$$\psi|_{\bigcup_k D_2^k}(\sum_k b_k m_{2j_k}(\sum_i b_{(k,i)}u_{(k,i)})) \leqslant \frac{1}{m_{2j+2}}, \tag{1}$$

$$\psi|_{\bigcup_k D_3^k}(\sum_k b_k m_{2j_k}(\sum_i b_{(k,i)}u_{(k,i)})) \leqslant \frac{16}{m_{2j+2}}, \tag{2}$$

and

$$\psi|_{\bigcup_k D_1^k}(\sum_k b_k m_{2j_k}(\sum_i b_{(k,i)}u_{(k,i)})) \leqslant \frac{1}{m_{2j+2}}. \tag{3}$$

may be easily reproduced from [2] Lemma 3.11. The case of D_4^k is slightly different from [2] and therefore we give more details. We claim

Claim: Let $D = \bigcup_k D_4^k$. Then

$$\psi|_D(\sum_k b_k m_{2j_k}(\sum_i b_{(k,i)}u_{(k,i)})) \leqslant \frac{64}{m_{2j+2}}, \tag{4}$$

Once the claim is proved it follows by adding the estimates that the 3rd Case is proved, and this concludes the proof of the Proposition.

Proof of the claim: Recall that D_4^k is defined by

$$D_4^k = \{m \in \bigcup_{i=1}^{n_k} A_{(k,i)} : \text{for all } f \in \bigcup_s K^s(\psi) \text{ with } m \in \text{supp} f, w(f) \geqslant \frac{1}{m_{2j_k}} \text{ and}$$

$$\text{there exists } f \in \bigcup_s K^s(\psi) \text{ with } m \in \text{supp} f, w(f) = \frac{1}{m_{2j_k}} \text{and}$$

$$\text{for every } g \in \bigcup_s K^s(\psi) \text{ with supp } f \subset \text{supp } g \text{ strictly}, w(g) \geqslant \frac{1}{m_{2j+1}}\}.$$

For every $k = 1, \dots, n$, $i = 1, \dots, n_k$ and every $m \in \text{supp } u_{(k,i)} \cap D_4^k$, there exists a unique functional $f^{(k,i,m)} \in \bigcup_s K^s(\psi)$ with $m \in \text{supp } f$, $w(f) = \frac{1}{m_{2j_k}}$ and such that, for all $g \in \bigcup_s K^s(\psi)$ with supp $f \subseteq$ supp g strictly, $w(g) \geqslant \frac{1}{m_{2j+1}}$. By definition, for $k \neq p$ and $i = 1, \ldots, n_k$, $m \in \text{supp } u_{(k,i)}$, we have $\text{supp} f^{(k,i,m)} \cap D_4^p = \emptyset$. Also, if $f^{(k,i,m)} \neq f^{(k,r,n)}$, then $\text{supp } f^{(k,i,m)} \cap \text{supp } f^{(k,r,n)} = \emptyset$.

For each $k = 1, \ldots, n$, let $\{f^{k,t}\}_{t=1}^{r_k} \subseteq \bigcup K^s(\varphi)$ be a selection of mutually disjoint such functionals with $D_4^k = \bigcup_{t=1}^{r_k} \text{supp } f^{k,t}$. For each such functional $f^{k,t}$, we set

$$a_{f^{k,t}} = \sum_{i=1}^{n_k} b_{(k,i)} \sum_{m \in \text{supp } f^{k,t}} a_m.$$

Then,

$$f^{k,t}(b_k m_{2j_k}(\sum_i b_{(k,i)} u_{(k,i)})) \leqslant b_k a_{f^{k,t}}. \tag{5}$$

We define as in [2] a functional $g \in K'$ with $|g|_{2j}^* \leqslant 1$ (see definition [2] p 71), and blocks u_k of the basis so that $\|u_k\|_{\ell_1} \leqslant 16$, $\text{supp } u_k \subseteq \bigcup_i \text{supp } u_{(k,i)}$ and

$$\psi|_{D_4}(\sum_k b_k m_{2j_k}(\sum_i b_{(k,i)} u_{(k,i)})) \leqslant g(2 \sum_k b_k u_k),$$

hence by [2] Lemma 2.4(b) we shall have the result.

For $f = \frac{1}{m_q} \sum_{p=1}^d f_p \in \bigcup_s K^s(\psi|_{D_4})$ we set

$$J = \{1 \leqslant p \leqslant d : f_p = f^{k,t} \text{ for some } k = 1 \ldots, n, \ t = 1, \ldots, r_k\},$$

$$T = \{1 \leqslant p \leqslant d : \text{ there exists } f^{k,t} \text{ with } \text{supp} f^{k,t} \subseteq \text{supp} f_p \text{ strictly}\}.$$

For every $f \in \bigcup_s K^s(\psi|_{D_4})$ we shall define by induction a functional g_f, by $g_f = 0$ when $J \cup T = \emptyset$, while if $J \cup T \neq \emptyset$ we shall construct g_f with the following properties. Let $D_f = \bigcup_{p \in J \cup T} \text{supp} f_p$ and $u_k = \sum a_{f^{k,t}} e_{f^{k,t}}$, where $e_{f^{k,t}} = e_{\min \text{supp} f^{k,t}}$, then:

(a) $\text{supp } g_f \subseteq \text{supp } f$.
(b) $g_f \in K'$ and $w(g_f) \geqslant w(f)$,
(c) $f|_{D_f}(\sum_k b_k m_{2j_k}(\sum_i b_{(k,i)} u_{(k,i)})) \leqslant g_f(2 \sum_k b_k u_k)$.

Let $s > 0$ and suppose that g_f have been defined for all $f \in \bigcup_{t=0}^{s-1} K^t(\psi|_{D_4})$ and let $f = \frac{1}{m_q}(f_1 + \ldots + f_d) \in K^s(\psi|_{D_4}) \backslash K^{s-1}(\psi|_{D_4})$ where the family $(f_p)_{p=1}^d$ is $\mathcal{M}_q$-admissible if $q > 1$, or $\mathcal{S}$-allowable if $q = 1$. The proofs of case (i) ($1/m_q = 1/m_{2j_k}$ for some $k \leqslant n$) and case (ii) ($1/m_q > 1/m_{2j+1}$) are identical with [2] p 106. Assume therefore that case (iii) holds, i.e., $1/m_q = 1/m_{2j+1}$. For the same reasons as in [2] we have that $T = \emptyset$.

Summing up we assume that $f \in K^s(\psi|_{D_4}) \backslash K^{s-1}(\psi|_{D_4})$ is of the form

$$f = \frac{1}{m_{2j+1}} \sum_{p=1}^d f_p = \frac{1}{m_{2j+1}}(Ey_1^* + \ldots + Ey_{k_2}^* + Ey_{k_2+1}^* + \ldots + Ey_d^*),$$

where $(y_i^*)_i$ is associated to $(c_1^*, \ldots, c_{k_2}^*, d_{k_2+1}^*, \ldots)$ with $d_{k_2+1}^* \neq c_{k_2+1}^*$, that $T = \emptyset$ and $J \neq \emptyset$, and it only remains to define g_f satisfying (a)(b)(c).

Now by the proof of [2] Proposition 2.9, $\psi = \psi_\varphi$ was defined through the analysis of φ, in particular by [2] Remark 2.19 (a),

$$\psi = \frac{1}{m_{2j+1}} \sum_{k \in I} \psi_{Ey_k^*}$$

for some subset I of $\{1, \ldots, d\}$. Furthermore, for $l \in I$, $l \leqslant k_2$ and $1 \leqslant k \leqslant d$, $\text{supp } Ey_l^* \cap \text{supp } x_k = \emptyset$, therefore there is no functional in a family of type I and II w.r.t. $\overline{x_k}$ of support included in $\text{supp } Ey_l^*$ (see [2] Definition 2.11 p 77).

This implies that $D_{Ey_l^*} = \emptyset$ ([2] Definition p 85), and therefore that $\psi_{Ey_l^*} = 0$ ([2] bottom of p 85).

For $l \in I$, $l > k_2 + 1$, then since Φ is $1-1$, $w(Ey_l^*) = w(Ed_l^*) \neq 1/m_{2j_k} \forall k$. Therefore $w(\psi_{Ey_l^*}) \neq 1/m_{2j_k} \forall k$, [2] Remark 2.19 (a). Then by the definition of D_4^k, supp $\psi_{Ey_l^*} \cap D_4^k = \emptyset$ for all k.

Finally this means that $\psi_{|D_4} = \frac{1}{m_{2j+1}} \psi_{Ey^*_{k_2+1}|D_4}$ and $J = \{k_2+1\}$, $D_f =$ supp f_{k_2+1}. Write then $f_{k_2+1} = f^{k_0,t}$ and set $g_f = \frac{1}{2} e^*_{f_{k_2+1}}$, therefore (a)(b) are trivially verified. It only remains to check (c). But by (5),

$$f|_{D_f}(\sum_k b_k m_{2j_k}(\sum_i b_{(k,i)} u_{(k,i)})) \leqslant b_{k_0} a_{f_{k_2+1}}$$
$$= b_{k_0} a_{f_{k_2+1}} e^*_{f_{k_2+1}}(e_{f_{k_2+1}}) = g_f(2 b_{k_0} a_{f_{k_2+1}} e_{f_{k_2+1}})$$
$$= g_f(2 \sum_t b_{k_0} a^{f_{k,t}} e_{f^{k,t}}) = g_f(2 \sum_k b_k u_k).$$

So (c) is proved. Therefore g_f is defined for each f by induction, and the Claim is verified. This concludes the proof of the Proposition. ☐

Proposition 4.5. *The space X_u is of type (3).*

Proof. Assume towards a contradiction that T is an isomorphism from some block-subspace $[x_n]$ of X_u into the subspace $[e_i, i \notin \bigcup_n \text{supp } x_n]$. We may assume that max(supp x_n, supp Tx_n) $<$ min(supp x_{n+1}, supp Tx_{n+1}) and min supp $x_n <$ min supp Tx_n for each n, and by Lemma 4.2, that each x_n is a $(\frac{1}{m_{2n}^4}, 2n)$ R.I.s.c.c. ([2] Definition 2.16). We may write

$$x_n = \sum_{t=1}^{p_n} a_{n,t} x_{n,t}$$

where $(x_{n,1}, \ldots, x_{n,p_n})$ is $\mathcal{M}_{2n}$-admissible. Let for each n, t, $x^*_{n,t} \in L$ be such that supp $x^*_{n,t} \subseteq$ supp $Tx_{n,t}$ and such that

$$x^*_{n,t}(Tx_{n,t}) \geqslant \frac{1}{2}\|Tx_{n,t}\| \geqslant \frac{1}{4\|T^{-1}\|},$$

and let $x^*_n = \frac{1}{m_{2n}}(x^*_{n,1} + \ldots + x^*_{n,p_n}) \in L_{2n}$. Note that supp $x^*_n \cap$ supp $x_n = \emptyset$ and that

$$x^*_n(Tx_n) \geqslant \frac{1}{m_{2n}} \sum_{t=1}^{p_n} \frac{a_{n,t}}{4\|T^{-1}\|} = (4\|T^{-1}\| m_{2n})^{-1}.$$

We may therefore for any $j > 100$ construct a j-quadruple $(j_k, y_k, c^*_k, b_k)_{k=1}^n$ satisfying the hypotheses of Proposition 4.4 and such that $y_k \in [x_i]_i$ and $c^*_k(Ty_k) \geqslant (4\|T^{-1}\| m_{2j_k})^{-1}$ for each k (note that we may assume that $c^*_k \in L_{j_{2k}}$ for each k). From Proposition 4.4 we deduce

$$\|\sum_{k=1}^n b_k m_{2j_k} y_k\| \leqslant \frac{75}{m^2_{2j+1}}.$$

On the other hand $\psi = \frac{1}{m_{2j+1}} \sum_{k=1}^n c^*_k$ belongs to L therefore

$$\|T(\sum_{k=1}^n b_k m_{2j_k} y_k)\| \geqslant \psi(\sum_{k=1}^n b_k m_{2j_k} Ty_k) \geqslant \frac{1}{4\|T^{-1}\| m_{2j+1}}.$$

We deduce finally that

$$m_{2j+1} \leqslant 300\|T\|\|T^{-1}\|,$$

which contradicts the boundedness of T. □

4.2. **A strongly asymptotically ℓ_∞ space tight by support.** Since the canonical basis of X_u is tight and unconditional, it follows that X_u is reflexive. In particular this implies that the dual basis of the canonical basis of X_u is a strongly asymptotically ℓ_∞ basis of X_u^*. It remains to prove that this basis is tight with support.

It is easy to prove by duality that for any $\mathcal{M}_{2j}$-admissible sequence of functionals $f_1, \ldots, f_n$ in X_u^*, we have the upper estimate

$$\|\sum_i f_i\| \leqslant m_{2j} \sup_i \|f_i\|.$$

We use this observation to prove a lemma about the existence of s.c.c. normed by functionals belonging to an arbitrary subspace of X_u^*. The proof is standard except that estimates have to be taken in X_u^* instead of X_u.

Lemma 4.6. *For $\epsilon > 0$, $j = 1, 2, \ldots$ and every normalised block sequence $\{f_k\}_{k=1}^\infty$ in X_u^*, there exists a normalised functional $f \in [f_k]$ and a seminormalised $(\epsilon, 2j)$–s.c.c. x in X_u such that* supp $f \subseteq$ supp x *and $f(x) \geqslant 1/2$.*

Proof. For each k let y_k be normalised such that supp y_k = supp f_k and $f_k(y_k) = 1$. Recall that the integers k_n and t_n are defined by $k_1 = 1$, $2^{t_n} \geqslant m_n^2$ and $k_n = t_n(k_{n-1} + 1) + 1$, and that $\mathcal{M}_j = \mathcal{F}_{k_j}$ for all j.

Applying Lemma 4.2 we find a successive sequence of $(\epsilon, 2j)$–s.c.c. of (y_k) of the form $(\sum_{i \in I_k} a_i y_i)_k$ with $\{f_i, i \in I_k\}$ $\mathcal{F}_{k_{2j-1}+1}$-admissible. If $\|\sum_{i \in I_k} f_i\| \leqslant 2$ for some k, we are done, for then

$$(\sum_{i \in I_k} f_i)(\sum_{i \in I_k} a_i y_i) \geqslant \frac{1}{2}\|\sum_{i \in I_k} f_i\|.$$

So assume $\|\sum_{i \in I_k} f_i\| > 2$ for all k, apply the same procedure to the sequence $f_k^1 = \|\sum_{i \in I_k} f_i\|^{-1} \sum_{i \in I_k} f_i$, and obtain a successive sequence of $(\epsilon, 2j)$–s.c.c. of the sequence $(y_k^1)_k$ associated to $(f_k^1)_k$, of the form $(\sum_{i \in I_k^1} a_i^1 y_i^1)_k$, with $\{f_l : \text{supp } f_l \subseteq \sum_{i \in I_k^1} f_i^1\}$ a $\mathcal{F}_{k_{2j-1}+1}[\mathcal{F}_{k_{2j-1}+1}]$-admissible, and therefore $\mathcal{M}_{2j}$-admissible set. Then we are done unless $\|\sum_{i \in I_k^1} f_i^1\| > 2$ for all k, in which case we set

$$f_k^2 = \|\sum_{j \in I_k^1} f_j^1\|^{-1} \sum_{j \in I_k^1} f_j^1$$

and observe by the upper estimate in X_u^* that

$$1 = \|f_k^2\| = \|\sum_{j \in I_k^1} \sum_{i \in I_j} \|\sum_{j \in I_k^1} f_j^1\|^{-1} \|\sum_{i \in I_j} f_i\|^{-1} f_i\| \leqslant m_{2j}/4.$$

Repeating this procedure we claim that we are done in at most t_{2j} steps. Otherwise we obtain that the set

$$A = \{f_l : \text{supp } f_l \subseteq \sum_{i \in I_k^{t_{2j-1}}} f_i^{t_{2j-1}}\}$$

is $\mathcal{M}_{2j}$-admissible. Since $f_k^{t_{2j}} = \sum_{f_l \in A} \alpha_l f_l$, where the normalising factor α_l is less than $(1/2)^{t_{2j}}$ for each l, we deduce from the upper estimate that

$$1 = \|f_k^{t_{2j}}\| \leqslant 2^{-t_{2j}} m_{2j},$$

a contradiction by definition of the integers t_i's. □

To prove the last proposition of this section we need to make two observations. First if $(f_1, \ldots, f_n) \in \mathrm{conv}_{\mathbb{Q}} L$ is $\mathcal{M}_{2j}$-admissible, then $\frac{1}{m_{2j}} \sum_{k=1}^n f_k \in \mathrm{conv}_{\mathbb{Q}} L_{2j}$. Indeed using the stability of L under projections onto subsets of $\mathbb{N}$ we may easily find convex rational coefficients λ_i such that each f_k is of the form

$$f_k = \sum_i \lambda_i f_i^k, \;\; f_i^k \in L, \text{ supp } f_i^k \subseteq \text{supp } f_k \; \forall i.$$

Then $\frac{1}{m_{2j}} \sum_{k=1}^n f_k = \sum_i \lambda_i (\frac{1}{m_{2j}} \sum_{k=1}^n f_i^k)$ and each $\frac{1}{m_{2j}} \sum_{k=1}^n f_i^k$ belongs to L_{2j}.

Likewise if $\psi = \frac{1}{m_{2j+1}}(c_1^* + \ldots + c_d^*)$, $k > 2j+1$, $c_1^* \in \mathrm{conv}_{\mathbb{Q}} L_{2k}$ and $c_l^* \in \mathrm{conv}_{\mathbb{Q}} L_{\Phi(c_1^*, \ldots, c_{l-1}^*)}$ $\forall l \geqslant 2$, then $\psi \in \mathrm{conv}_{\mathbb{Q}} L$. Indeed as above we may write

$$\psi = \sum_i \lambda_i (\frac{1}{m_{2j+1}} \sum_{l=1}^d f_i^l), \;\; f_i^1 \in L_{2k}, f_i^l \in L_{\Phi(c_1^*, \ldots, c_{i-1}^*)}(c_i^*) \; \forall l \geqslant 2,$$

and each $\frac{1}{m_{2j+1}} \sum_{l=1}^d f_i^l$ belongs to $L_{2j+1}'^{n+1} \subseteq L$.

Proposition 4.7. *The space X_u^* is of type (3).*

Proof. Assume towards a contradiction that T is an isomorphism from some block-subspace $[f_n]$ of X_u^* into the subspace $[e_i^*, i \notin \cup_n \text{supp } f_n]$. We may assume that $\max(\text{supp } f_n, \text{supp } Tf_n) < \min(\text{supp } f_{n+1}, \text{supp } Tf_{n+1})$ and $\min \text{supp } Tf_n < \min \text{supp } f_n$ for each n. Since the closed unit ball of X_u^* is equal to $\overline{\mathrm{conv}_{\mathbb{Q}} L}$ we may also assume that $f_n \in \mathrm{conv}_{\mathbb{Q}} L$ for each n. Applying Lemma 4.6, we may also suppose that each f_n is associated to a $(\frac{1}{m_{2n}^4}, 2n)$ s.c.c. x_n with $Tf_n(x_n) \geqslant 1/3$ and $\text{supp } x_n \subset \text{supp } Tf_n$, and we shall also assume that $\|Tf_n\| = 1$ for each n. Build then for each k a $(\frac{1}{m_{2k}^4}, 2k)$ R.I.s.c.c. $y_k = \sum_{n \in A_k} a_n x_n$ such that $(Tf_n)_{n \in A_k}$ and therefore $(f_n)_{n \in A_k}$ is $\mathcal{M}_{2k}$-admissible. Then note that by the first observation before this proposition,

$$c_k^* := m_{2k}^{-1} \sum_{n \in A_k} f_n \in \mathrm{conv}_{\mathbb{Q}} L_{2k},$$

and observe that $\text{supp } c_k^* \cap \text{supp } y_k = \emptyset$ and that $Tc_k^*(y_k) \geqslant (3m_{2k})^{-1}$.

We may therefore for any $j > 100$ construct a j-quadruple $(j_k, y_k, c_k^*, b_k)_{k=1}^n$ satisfying the hypotheses of Proposition 4.4 and such that $c_k^* \in [f_i]_i$ and $Tc_k^*(y_k) \geqslant (3m_{2j_k})^{-1}$ for each k. From Proposition 4.4 we deduce

$$\|\sum_{k=1}^n b_k m_{2j_k} y_k\| \leqslant \frac{75}{m_{2j+1}^2}.$$

Therefore

$$\|\sum_{k=1}^d Tc_k^*\| \geqslant \frac{\sum_{k=1}^d b_k m_{2j_k} Tc_k^*(y_k)}{\|\sum_{k=1}^n b_k m_{2j_k} y_k\|} \geqslant \frac{m_{2j+1}^2}{225},$$

but on the other hand

$$\|\sum_{k=1}^{d} c_k^*\| \leqslant m_{2j+1}$$

since by the second observation the functional $m_{2j+1}^{-1} \sum_{k=1}^{d} c_k^*$ belongs to $\mathrm{conv}_{\mathbb{Q}} L$. We deduce finally that

$$m_{2j+1} \leqslant 225\|T\|,$$

which contradicts the boundedness of T. □

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