ISOMORPHISM OF BOREL FULL GROUPS

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Abstract. Suppose that $G$ and $H$ are Polish groups which act in a Borel fashion on Polish spaces $X$ and $Y$. Let $E^X_G$ and $E^Y_H$ denote the corresponding orbit equivalence relations, and $[G]$ and $[H]$ the corresponding Borel full groups. Modulo the obvious counterexamples, we show that $[G] \cong [H] \iff E^X_G \cong B E^Y_H$.

1. Introduction

Suppose that a Polish group $G$ acts in a Borel fashion on a Polish space $X$. The orbit equivalence relation induced by the action of $G$ on $X$ is given by

$$x_1 E^X_G x_2 \iff \exists g \in G \ (g \cdot x_1 = x_2).$$

The (Borel) full group associated with the action of $G$ on $X$ is the group $[G]$ of Borel automorphisms $f : X \to X$ such that $\forall x \in X \ (x E^X_G f(x))$.

Suppose that $E$ and $F$ are (not necessarily Borel) equivalence relations on Polish spaces $X$ and $Y$. An isomorphism of $E$ and $F$ is a bijection $\pi : X \to Y$ such that

$$\forall x_1, x_2 \in X \ (x_1 E x_2 \iff \pi(x_1) F \pi(x_2)).$$

We say that $E$ and $F$ are Borel isomorphic, or $E \cong_B F$, if there is a Borel isomorphism of $E$ and $F$. Here we establish the connection between Borel isomorphism of orbit equivalence relations and algebraic isomorphism of their full groups.

Theorem 1.1. Suppose that $G$ and $H$ are Polish groups which act in a Borel fashion on Polish spaces $X$ and $Y$, and the following conditions hold:

1. The actions of $G$ and $H$ have the same number of singleton orbits.
2. If the actions of $G$ and $H$ both have infinitely many doubleton orbits, then they have the same number of doubleton orbits.

Then $[G] \cong [H] \iff E^X_G \cong_B E^Y_H$.

2. Implementing isomorphisms via point maps

Here we describe how to build isomorphisms of the aperiodic parts of equivalence relations which implement a given algebraic isomorphism of their full groups.
Suppose that $E$ is a (not necessarily Borel) equivalence relation on a Polish space $X$. The full group of $E$ is the group $[E]$ of all Borel automorphisms $g : X \to X$ such that $\forall x \in X \ (x Eg \cdot x)$. The aperiodic part of $E$ is given by

$$\text{Aper}(E) = \{ x \in X : |[x]_E| = \infty \}.$$

**Proposition 2.1.** Suppose that $E$ and $F$ are (not necessarily Borel) equivalence relations on Polish spaces $X$ and $Y$ and $\pi : [E] \to [F]$ is an algebraic isomorphism. Then there is a bijection $\varphi : \text{Aper}(E) \to \text{Aper}(F)$ such that

$$\forall g \in [E] \ (\pi(g)|\text{Aper}(F) = \varphi \circ (g|\text{Aper}(E)) \circ \varphi^{-1}).$$

In particular, $\varphi$ is a (not necessarily Borel) isomorphism of $E|\text{Aper}(E), F|\text{Aper}(F)$.

**Proof.** The support of $g \in [E]$ is given by $\text{supp}(g) = \{ x \in X : g \cdot x \neq x \}$, and $g$ is a transposition if its support is of cardinality 2. We use $\text{id}_X$ to denote the trivial automorphism of $X$. The order of $g \in [E]$ is given by

$$|g| = \begin{cases} n & \text{if } n \geq 1 \text{ is least such that } g^n = \text{id}_X, \\ \infty & \text{if } \forall n \geq 1 \ (g^n \neq \text{id}_X). \end{cases}$$

Let $\text{Per}_n(E) = \{ x \in X : |[x]_E| = n \}$ and $\text{Per}_{\geq n}(E) = \{ x \in X : |[x]_E| \geq n \}$.

**Lemma 2.2.** Suppose that $g \in [E]$ is of order 2. Then the following are equivalent:

1. $g|\text{Aper}(E)$ is a transposition and $\forall n \geq 3 \ (g|\text{Per}_n(E) = \text{id}_{\text{Per}_n(E)})$.
2. The following conditions are satisfied:
   a. If $h$ is a conjugate of $g$, then $|gh| \leq 3$.
   b. If $1 \leq n \leq 3$, then there is a conjugate $h$ of $g$ such that $|gh| = n$.
   c. There are infinitely many distinct conjugates of $g$.

**Proof.** It is enough to show (2) $\Rightarrow$ (1). We prove first a pair of sublemmas.

**Sublemma 2.3.** $\forall x \in X \ (|\text{supp}(g|[x]_E)| < \aleph_0)$.

**Proof.** Suppose, towards a contradiction, that there exists $S \subseteq [x]_E$ such that $g|S = \cdots \cdot (x_{-2} x_{-1})(x_0 x_1)(x_2 x_3) \cdots$, where the $x_n$ are pairwise distinct. Fix a conjugate $h$ of $g$ such that $h|S = \cdots \cdot (x_{-3} x_{-2})(x_{-1} x_0)(x_1 x_2) \cdots$, and note that $gh|S = (\cdots x_2 x_0 x_{-2} \cdots)(\cdots x_{-1} x_1 \cdots)$; thus $|gh| = \infty$, which contradicts (a). \hfill $\square$

**Sublemma 2.4.** There exists $x \in \text{Aper}(E)$ such that $\text{supp}(g) \subseteq \text{Per}_2(E) \cup [x]_E$.

**Proof.** First suppose, towards a contradiction, that

$$(1) \quad \text{supp}(g) \subseteq \text{Per}_{\leq 4}(E) \text{ and } \forall x \in \text{Per}_4(E) \ (|\text{supp}(g) \cap [x]_E| \neq 2).$$

Note that $\text{supp}(g)$ cannot intersect both $\text{Per}_3(E)$ and $\text{Per}_4(E)$, as we could then find a conjugate $h$ of $g$ such that $|gh| \geq 6$, which contradicts (a). It then follows that $\text{supp}(g)$ cannot intersect $\text{Per}_4(E)$, since then there would be no conjugate $h$ of $g$ such that $|gh| = 3$, which contradicts (b). It similarly follows that $\text{supp}(g)$ cannot intersect $\text{Per}_3(E)$, since then there would be no conjugate $h$ of $g$ such that $|gh| = 2$, which again contradicts (b). It now follows that, for every conjugate $h$ of
g, the product gh is trivial, and this final contradiction with (b) implies that (†) fails; thus there exists \( x \in \text{Per}_{\geq 4}(E) \cap \text{supp}(g) \) such that

\[
|\{x\}| = 4 \Rightarrow |\text{supp}(g)| = 2.
\]

Now suppose, towards a contradiction, that there exists \( y \in \text{Per}_{\geq 3}(E) \cap \text{supp}(g) \) which is not \( E \)-equivalent to \( x \). If \( |\{y\}| = 3 \), then there is a conjugate \( h \) of \( g \) such that \( |\{gh\}| = 2 \) and \( |\{gh\}| = 3 \); thus \( |gh| \geq 6 \), which contradicts (a). If \( |\{y\}| \geq 4 \), then there is a conjugate \( h \) of \( g \) such that \( |\{gh\}| = 3 \) and \( |\{gh\}| = 2 \); thus \( |gh| \geq 6 \), which again contradicts (a). Thus \( \text{supp}(g) \subseteq \text{Per}_{\geq 2}(E) \cup \{x\} \), and condition (c) then ensures that \( x \in \text{Aper}(E) \). \( \square \)

Fix \( x \in \text{Aper}(E) \) such that \( \text{supp}(g) \subseteq \text{Per}_{\geq 2}(E) \cup \{x\} \), find pairwise distinct points \( x_0, x_1, \ldots, x_{2n-1} \in \{x\} \) such that

\[
g|\{x\} = (x_0 x_1)(x_2 x_3) \cdots (x_{2n-2} x_{2n-1}),
\]

fix \( x_{2n} \in \{x\} \), and find a conjugate \( h \) of \( g \) such that

\[
h|\{x\} = (x_1 x_2)(x_3 x_4) \cdots (x_{2n-1} x_{2n}).
\]

Then \( gh|\{x\} \) is a cycle of order \( 2n + 1 \); thus \( n = 1 \), and the lemma follows. \( \square \)

We say that \( g \in \{x\} \) is a near transposition if it satisfies the equivalent conditions of Lemma 2.2. Note that \( g \) is a near transposition \( \Leftrightarrow \pi(g) \) \( g \) is a near transposition.

We say that a family \( T \) of near transpositions is good if \( |T| \geq 4 \) and \( T \) is maximal with the property that \( \forall q, h \in T \) \( g \neq h \Rightarrow gh \neq hg \).

Each \( E \)-invariant set \( B \subseteq X \), the restriction of \( T \) to \( B \) is given by

\[
T|B = \{(g|B) \cup id_{X \setminus B} : g \in T\}.
\]

If \( T \) is good, then so too is \( T|\text{Per}_{\geq 3}(E) \), so the map \( T \mapsto T|\text{Per}_{\geq 3}(E) \) associates with each good family of near transpositions a good family of transpositions. For each \( x \in \text{Aper}(E) \), the good family of transpositions centered at \( x \) is given by

\[
T_x = \{(x y) : y \in \{x\}\}.
\]

Lemma 2.5. Suppose that \( T \) is a good family of near transpositions. Then there exists \( x \in \text{Aper}(E) \) such that \( T|\text{Per}_{\geq 3}(E) = T_x \).

Proof. Set \( T' = T|\text{Per}_{\geq 3}(E) \), and fix distinct transpositions \((x y), (x z) \in T' \). Note that \((y z) \notin T' \), since the set \( \{(x y), (y z), (x z)\} \) does not extend to a good family. Also observe that if \( w \notin \{x, y, z\} \), then \((y w), (z w) \) are not in \( T' \), since they commute with \((x z), (x y) \). Thus, the only possible elements of \( T' \) are those of the form \((x w) \), where \( w \in \{x\} \), and it follows that \( T' = T_x \). \( \square \)

For each good family \( T \) of near transpositions, let \( x(T) \) be the unique element of \( \text{Aper}(E) \) such that \( T_x(T) = T|\text{Per}_{\geq 3}(E) \), and define

\[
T_1 \sim T_2 \iff x(T_1) = x(T_2).
\]

Lemma 2.6. \( T_1 \sim T_2 \iff \forall g_1 \in T_1 \exists g_2 \in T_2 \) \( (g_1 g_2 = g_2 g_1) \).

Proof. To see \((\Rightarrow)\), note that if \( g_1 \in T_1 \) and \( g_1|\text{Per}_{\geq 3}(E) = (x y) \), then the unique \( g_2 \in T_2 \) such that \( g_2|\text{Per}_{\geq 3}(E) = (x y) \) is also the unique element of \( T_2 \) which commutes with \( g_1 \).

To see \((\Leftarrow)\), note that if \( T_1 \sim T_2 \), then \( x(T_1) = x(T_2) \), in which case we can easily find an element of \( T_1 \) which commutes with infinitely many elements of \( T_2 \). \( \square \)
Now let \( \varphi : \text{Aper}(E) \to \text{Aper}(F) \) be the unique map such that
\[
\forall x \in \text{Aper}(E) \ (\pi(T_x) \sim T_{\varphi(x)}),
\]
and suppose that \( x, y \in \text{Aper}(E) \) are \( E \)-equivalent. As \( (x, y) \) is the unique element of \( T_x \cap T_y \), it follows that \( \pi([x, y]) \) is the unique element of \( \pi(T_x) \cap \pi(T_y) \); thus
\[
\pi([x, y])|_{\text{Per}_{\geq 3}(E)} = (\varphi(x), \varphi(y)).
\]
For each \( g \in [E] \), we now have that
\[
\pi(g)[\{\varphi(x), \varphi(y)\}] = \pi(g)[\text{supp}(\varphi(x) \cdot \varphi(y))]
\]
\[
= \text{Per}_{\geq 3}(F) \cap \pi(g)[\text{supp}(\pi([x, y]))]
\]
\[
= \text{Per}_{\geq 3}(F) \cap \text{supp}(\pi(g) \circ \pi([x, y]) \circ \pi(g)^{-1})
\]
\[
= \text{Per}_{\geq 3}(F) \cap \text{supp}(\pi(g \circ (x, y) \circ g^{-1}))
\]
\[
= \text{Per}_{\geq 3}(F) \cap \text{supp}(\pi([g \cdot x, g \cdot y]))
\]
\[
= \{\varphi(g \cdot x), \varphi(g \cdot y)\},
\]
and it follows that \( \pi(g) \cdot \varphi(x) = \varphi(g \cdot x) \), which completes the proof. \( \square \)

3. Orbit equivalence relations

Here we describe a technical condition under which the map \( \varphi \) of Proposition 2.1 is automatically Borel. We then use this to draw out our main theorem regarding the connection between Borel isomorphism of orbit equivalence relations and algebraic isomorphism of their full groups.

Suppose that \( E \) is a (not necessarily Borel) equivalence relation on a Polish space \( X \). We say that \( E \) is countable if each of its equivalence classes is countable, and \( E \) is good if it admits a countable Borel subequivalence relation \( F \subseteq E \) such that
\[
\forall x \in X \ (|[x]_E| \geq 3 \Rightarrow |[x]_F| \geq 3).
\]
Our interest in such equivalence relations stems from the following connection between their full groups and the underlying \( \sigma \)-algebra of Borel sets.

**Proposition 3.1.** Suppose that \( E \) is an equivalence relation on a Polish space \( X \). Then the following are equivalent:

1. \( E \) is good.
2. The \( \sigma \)-algebra generated by \( A = \{\text{supp}(g) : g \in [E]\} \) contains every set of the form \( A \cap B \), where \( A = \text{Per}_{\geq 3}(E) \) and \( B \subseteq X \) is Borel.

**Proof.** To see (1) \( \Rightarrow \) (2), fix a countable Borel equivalence relation \( F \subseteq E \) with
\[
\forall x \in X \ (|[x]_E| \geq 3 \Rightarrow |[x]_F| \geq 3),
\]
and suppose that \( B \subseteq X \) is Borel. As \( A = \text{Per}_{\geq 3}(F) \) and the latter set is Borel, we can write \( A \cap B = B_1 \cup B_2 \), where \( B_1 \) is a Borel set which intersects every equivalence class of \( F \) in at most one point, and \( B_2 \) is a Borel set which intersects every equivalence class of \( F \) in an even or infinite number of points. It is not difficult to find involutions \( g_1, g_2 \in [F] \) such that \( B_1 = \text{supp}(g_1) \cap \text{supp}(g_2) \), and Proposition 7.4 of Kechris-Miller \( [2] \) ensures the existence of an involution \( g \in [F] \) such that \( \text{supp}(g) = B_2 \). As \( B \subseteq X \) was arbitrary, condition (2) follows.

To see (2) \( \Rightarrow \) (1), suppose that the \( \sigma \)-algebra generated by \( A \) contains every Borel set of the form \( A \cap B \), with \( B \subseteq X \) Borel, fix a countable family of Borel automorphisms \( g_0, g_1, \ldots \) in \([E]\) such that the corresponding family of Borel sets
A_n = \text{supp}(g_n)\) separates points of \(A\), let \(G\) be the group generated by these automorphisms, and define \(B \subseteq X\) by

\[
B = \{x \in X : |[x]_{E_G^Y}| \leq 2\}.
\]

Note that if \(x \in A \cap B\), then \(|[x]_{E_G^Y}| = 1\), since otherwise there exists \(y \neq x\) in \([x]_{E_G^Y}\), and we can then find \(g \in G\) such that exactly one of \(x, y\) lie in \(\text{supp}(g)\); thus \(\{x, y, g \cdot x, g \cdot y\} \subseteq [x]_{E_G^Y}\) consists of 3 points. It follows that

\[
A \cap B = \{x \in A : \forall g \in G (x \notin \text{supp}(g))\},
\]

and therefore \(A \cap B\) consists of at most one point. If \(A \cap B = \emptyset\), we set \(F = E_G^X\). If \(A \cap B = \{x\}\), we fix \(y \in [x]_E \setminus \{x\}\) and define

\[
x_1F_1x_2 \Leftrightarrow x_1E_G^X x_2 \text{ or } x_1, x_2 \in \{x\} \cup [y]_{E_G^X}.
\]

In either case, we have that \(|[x]_E| \geq 3 \Rightarrow |[x]_F| \geq 3\); hence \(E\) is good. \(\Box\)

Next, we have our main technical result:

**Theorem 3.2.** Suppose that \(E\) and \(F\) are good equivalence relations on Polish spaces \(X\) and \(Y\) and \(\pi : [E] \to [F]\) is an algebraic isomorphism. Then there is a Borel isomorphism \(\varphi\) of \(E|\text{Aper}(E)\) and \(F|\text{Aper}(F)\) such that

\[
\forall g \in [E] \ (\pi(g)|\text{Aper}(F) = \varphi \circ (g|\text{Aper}(E)) \circ \varphi^{-1}).
\]

**Proof.** By Proposition 2.1 there is a bijection \(\varphi : \text{Aper}(E) \to \text{Aper}(F)\) such that

\[
\forall g \in [E] \ (\pi(g)|\text{Aper}(F) = \varphi \circ (g|\text{Aper}(E)) \circ \varphi^{-1}).
\]

Now, for each \(g \in [E]\), we have that

\[
\varphi(\text{supp}(g) \cap \text{Aper}(E)) = \varphi(\text{supp}(g|\text{Aper}(E)))
= \text{supp}(\varphi \circ (g|\text{Aper}(E)) \circ \varphi^{-1})
= \text{supp}(\pi(g)|\text{Aper}(F))
= \text{supp}(\pi(g)) \cap \text{Aper}(F).
\]

As \(E\) and \(F\) are good, the sets \(\text{Per}_{\geq 3}(E)\) and \(\text{Per}_{\geq 3}(F)\) are Borel, and Proposition 2.1 ensures that the Borel subsets of \(\text{Per}_{\geq 3}(E)\) are generated by the sets of the form \(\text{supp}(g)\), where \(g \in [E]\). Similarly, the Borel subsets of \(\text{Per}_{\geq 3}(F)\) are generated by the sets of the form \(\text{supp}(g)\), where \(g \in [F]\), and it easily follows that \(\varphi\) is a Borel isomorphism of \(E|\text{Aper}(E)\) and \(F|\text{Aper}(F)\). \(\Box\)

We say that an equivalence relation \(E\) is very good if there is a countable Borel subequivalence relation \(F \subseteq E\) such that

\[
\forall x \in X \forall n \in \mathbb{N} \ (|[x]_E| \geq n \Rightarrow |[x]_F| \geq n).
\]

**Theorem 3.3.** Suppose that \(E\) and \(F\) are very good equivalence relations on Polish spaces \(X\) and \(Y\), and the following conditions hold:

(1) \(E\) and \(F\) have the same number of singleton equivalence classes.

(2) If \(E\) and \(F\) both have infinitely many doubleton equivalence classes, then they have the same number of doubleton equivalence classes.

Then \([E] \cong [F] \Leftrightarrow E \cong_B F\).
Proposition 3.4. Suppose that $\kappa = \min(2^{\aleph_0}, 2^{\|\operatorname{Per}_n(E)\|}) = \min(2^{\aleph_0}, 2^{\|\operatorname{Per}_2(F)\|})$, and condition (2) then ensures that $|\operatorname{Per}_2(E)| = |\operatorname{Per}_2(F)|$.

For $n = 4$, note that the minimal normal subgroups of $[E]$ which are isomorphic to $A_n$, the alternating group on $n$ elements, are exactly those of the form

$$N = \{\{g \in [E] : \supp(g) \subseteq [x]_E \text{ and } g \text{ is of even cycle type}\},$$

where $x \in \operatorname{Per}_n(E)$. Letting $\kappa$ denote the number of such subgroups, it follows that $\kappa = |\operatorname{Per}_n(E)| = |\operatorname{Per}_n(F)|$. $\Box$

Theorem 1.1 is now a consequence of the following fact:

**Proposition 3.4.** Suppose that $G$ is a Polish group which acts in a Borel fashion on a Polish space $X$. Then $E^X_G$ is very good.

**Proof.** By Theorem 2.6.6 of Becker-Kechris [1], we can assume that the action of $G$ on $X$ is continuous. Fix a countable dense subgroup $H \leq G$, and note that if $g_1 \cdot x, g_2 \cdot x, \ldots, g_n \cdot x$ are distinct then, by choosing $h_1$ sufficiently close to $g_i$, we can ensure that $h_1 \cdot x, h_2 \cdot x, \ldots, h_n \cdot x$ are also distinct; thus the countable Borel equivalence relation $F = E^X_H$ witnesses that $E^X_G$ is very good. $\Box$

**References**


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