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ISOMORPHISM OF BOREL FULL GROUPS

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Abstract. Suppose that G and H are Polish groups which act in a Borel fashion on Polish spaces X and Y. Let E_G^X and E_H^Y denote the corresponding orbit equivalence relations, and $[G]$ and $[H]$ the corresponding Borel full groups. Modulo the obvious counterexamples, we show that $[G] \cong [H] \Leftrightarrow E_G^X \cong_B E_H^Y$.

1. Introduction

Suppose that a Polish group G acts in a Borel fashion on a Polish space X. The *orbit equivalence relation* induced by the action of G on X is given by

$$x_1 E_G^X x_2 \Leftrightarrow \exists g \in G \ (g \cdot x_1 = x_2).$$

The *(Borel) full group* associated with the action of G on X is the group $[G]$ of Borel automorphisms $f : X \to X$ such that $\forall x \in X \ (x E_G^X f(x))$.

Suppose that E and F are (not necessarily Borel) equivalence relations on Polish spaces X and Y. An *isomorphism* of E and F is a bijection $\pi : X \to Y$ such that

$$\forall x_1, x_2 \in X \ (x_1 E x_2 \Leftrightarrow \pi(x_1) F \pi(x_2)).$$

We say that E and F are *Borel isomorphic*, or $E \cong_B F$, if there is a Borel isomorphism of E and F. Here we establish the connection between Borel isomorphism of orbit equivalence relations and algebraic isomorphism of their full groups.

Theorem 1.1. *Suppose that G and H are Polish groups which act in a Borel fashion on Polish spaces X and Y, and the following conditions hold:*

1. *The actions of G and H have the same number of singleton orbits.*
2. *If the actions of G and H both have infinitely many doubleton orbits, then they have the same number of doubleton orbits.*

Then $[G] \cong [H] \Leftrightarrow E_G^X \cong_B E_H^Y$.

2. Implementing isomorphisms via point maps

Here we describe how to build isomorphisms of the aperiodic parts of equivalence relations which implement a given algebraic isomorphism of their full groups.

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Suppose that E is a (not necessarily Borel) equivalence relation on a Polish space X. The *full group* of E is the group $[E]$ of all Borel automorphisms $g : X \to X$ such that $\forall x \in X\ (xEg \cdot x)$. The *aperiodic part* of E is given by

$$\mathrm{Aper}(E) = \{x \in X : |[x]_E| = \infty\}.$$

Proposition 2.1. *Suppose that E and F are (not necessarily Borel) equivalence relations on Polish spaces X and Y and $\pi : [E] \to [F]$ is an algebraic isomorphism. Then there is a bijection $\varphi : \mathrm{Aper}(E) \to \mathrm{Aper}(F)$ such that*

$$\forall g \in [E]\ (\pi(g)|\mathrm{Aper}(F) = \varphi \circ (g|\mathrm{Aper}(E)) \circ \varphi^{-1}).$$

In particular, φ is a (not necessarily Borel) isomorphism of $E|\mathrm{Aper}(E), F|\mathrm{Aper}(F)$.

Proof. The *support* of $g \in [E]$ is given by $\mathrm{supp}(g) = \{x \in X : g \cdot x \neq x\}$, and g is a *transposition* if its support is of cardinality 2. We use id_X to denote the trivial automorphism of X. The *order* of $g \in [E]$ is given by

$$|g| = \begin{cases} n & \text{if } n \geq 1 \text{ is least such that } g^n = \mathrm{id}_X, \\ \infty & \text{if } \forall n \geq 1\ (g^n \neq \mathrm{id}_X). \end{cases}$$

Let $\mathrm{Per}_n(E) = \{x \in X : |[x]_E| = n\}$ and $\mathrm{Per}_{\geq n}(E) = \{x \in X : |[x]_E| \geq n\}$.

Lemma 2.2. *Suppose that $g \in [E]$ is of order 2. Then the following are equivalent:*

1. *$g|\mathrm{Aper}(E)$ is a transposition and $\forall n \geq 3\ (g|\mathrm{Per}_n(E) = \mathrm{id}_{\mathrm{Per}_n(E)})$.*
2. *The following conditions are satisfied:*
   - (a) *If h is a conjugate of g, then $|gh| \leq 3$.*
   - (b) *If $1 \leq n \leq 3$, then there is a conjugate h of g such that $|gh| = n$.*
   - (c) *There are infinitely many distinct conjugates of g.*

Proof. It is enough to show (2) $\Rightarrow$ (1). We prove first a pair of sublemmas.

Sublemma 2.3. $\forall x \in X\ (|\mathrm{supp}(g|[x]_E)| < \aleph_0)$.

Proof. Suppose, towards a contradiction, that there exists $S \subseteq [x]_E$ such that

$$g|S = \cdots (x_{-2}\ x_{-1})(x_0\ x_1)(x_2\ x_3) \cdots,$$

where the x_n are pairwise distinct. Fix a conjugate h of g such that

$$h|S = \cdots (x_{-3}\ x_{-2})(x_{-1}\ x_0)(x_1\ x_2) \cdots,$$

and note that

$$gh|S = (\cdots\ x_2\ x_0\ x_{-2}\ \cdots)(\cdots\ x_{-1}\ x_1\ \cdots);$$

thus $|gh| = \infty$, which contradicts (a). □

Sublemma 2.4. *There exists $x \in \mathrm{Aper}(E)$ such that* $\mathrm{supp}(g) \subseteq \mathrm{Per}_2(E) \cup [x]_E$.

Proof. First suppose, towards a contradiction, that

(†) $\mathrm{supp}(g) \subseteq \mathrm{Per}_{\leq 4}(E)$ and $\forall x \in \mathrm{Per}_4(E)\ (|\mathrm{supp}(g) \cap [x]_E| \neq 2)$.

Note that $\mathrm{supp}(g)$ cannot intersect both $\mathrm{Per}_3(E)$ and $\mathrm{Per}_4(E)$, as we could then find a conjugate h of g such that $|gh| \geq 6$, which contradicts (a). It then follows that $\mathrm{supp}(g)$ cannot intersect $\mathrm{Per}_4(E)$, since then there would be no conjugate h of g such that $|gh| = 3$, which contradicts (b). It similarly follows that $\mathrm{supp}(g)$ cannot intersect $\mathrm{Per}_3(E)$, since then there would be no conjugate h of g such that $|gh| = 2$, which again contradicts (b). It now follows that, for every conjugate h of

g, the product gh is trivial, and this final contradiction with (b) implies that (†) fails; thus there exists $x \in \mathrm{Per}_{\geq 4}(E) \cap \mathrm{supp}(g)$ such that

$$|[x]_E| = 4 \Rightarrow |\mathrm{supp}(g|[x]_E)| = 2.$$

Now suppose, towards a contradiction, that there exists $y \in \mathrm{Per}_{\geq 3}(E) \cap \mathrm{supp}(g)$ which is not E-equivalent to x. If $|[y]_E| = 3$, then there is a conjugate h of g such that $|gh|[x]_E| = 2$ and $|gh|[y]_E| = 3$; thus $|gh| \geq 6$, which contradicts (a). If $|[y]_E| \geq 4$, then there is a conjugate h of g such that $|gh|[x]_E| = 3$ and $|gh|[y]_E| = 2$; thus $|gh| \geq 6$, which again contradicts (a). Thus $\mathrm{supp}(g) \subseteq \mathrm{Per}_2(E) \cup [x]_E$, and condition (c) then ensures that $x \in \mathrm{Aper}(E)$. ☐

Fix $x \in \mathrm{Aper}(E)$ such that $\mathrm{supp}(g) \subseteq \mathrm{Per}_2(E) \cup [x]_E$, find pairwise distinct points $x_0, x_1, \ldots, x_{2n-1} \in [x]_E$ such that

$$g|[x]_E = (x_0\ x_1)(x_2\ x_3) \cdots (x_{2n-2}\ x_{2n-1}),$$

fix $x_{2n} \in [x]_E \setminus \{x_i\}_{i<2n}$, and find a conjugate h of g such that

$$h|[x]_E = (x_1\ x_2)(x_3\ x_4) \cdots (x_{2n-1}\ x_{2n}).$$

Then $gh|[x]_E$ is a cycle of order $2n+1$; thus $n = 1$, and the lemma follows. ☐

We say that $g \in [E]$ is a *near transposition* if it satisfies the equivalent conditions of Lemma 2.2. Note that g is a near transposition $\Leftrightarrow \pi(g)$ is a near transposition.

We say that a family $\mathcal{T}$ of near transpositions is *good* if $|\mathcal{T}| \geq 4$ and $\mathcal{T}$ is maximal with the property that $\forall g, h \in \mathcal{T}\ (g \neq h \Rightarrow gh \neq hg)$. For each E-invariant set $B \subseteq X$, the *restriction* of $\mathcal{T}$ to B is given by

$$\mathcal{T}|B = \{(g|B) \cup \mathrm{id}_{X \setminus B} : g \in \mathcal{T}\}.$$

If $\mathcal{T}$ is good, then so too is $\mathcal{T}|\mathrm{Per}_{\geq 3}(E)$, so the map $\mathcal{T} \mapsto \mathcal{T}|\mathrm{Per}_{\geq 3}(E)$ associates with each good family of near transpositions a good family of transpositions. For each $x \in \mathrm{Aper}(E)$, the *good family of transpositions centered at* x is given by

$$\mathcal{T}_x = \{(x\ y) : y \in [x]_E \setminus \{x\}\}.$$

Lemma 2.5. *Suppose that $\mathcal{T}$ is a good family of near transpositions. Then there exists $x \in \mathrm{Aper}(E)$ such that $\mathcal{T}|\mathrm{Per}_{\geq 3}(E) = \mathcal{T}_x$.*

Proof. Set $\mathcal{T}' = \mathcal{T}|\mathrm{Per}_{\geq 3}(E)$, and fix distinct transpositions $(x\ y), (x\ z) \in \mathcal{T}'$. Note that $(y\ z) \notin \mathcal{T}'$, since the set $\{(x\ y), (y\ z), (x\ z)\}$ does not extend to a good family. Also observe that if $w \notin \{x, y, z\}$, then $(y\ w), (z\ w)$ are not in $\mathcal{T}'$, since they commute with $(x\ z), (x\ y)$. Thus, the only possible elements of $\mathcal{T}'$ are those of the form $(x\ w)$, where $w \in [x]_E \setminus \{x\}$, and it follows that $\mathcal{T}' = \mathcal{T}_x$. ☐

For each good family $\mathcal{T}$ of near transpositions, let $x(\mathcal{T})$ be the unique element of $\mathrm{Aper}(E)$ such that $\mathcal{T}_{x(\mathcal{T})} = \mathcal{T}|\mathrm{Per}_{\geq 3}(E)$, and define

$$\mathcal{T}_1 \sim \mathcal{T}_2 \Leftrightarrow x(\mathcal{T}_1) = x(\mathcal{T}_2).$$

Lemma 2.6. $\mathcal{T}_1 \sim \mathcal{T}_2 \Leftrightarrow \forall g_1 \in \mathcal{T}_1\ \exists! g_2 \in \mathcal{T}_2\ (g_1 g_2 = g_2 g_1)$.

Proof. To see ($\Rightarrow$), note that if $g_1 \in \mathcal{T}_1$ and $g_1|\mathrm{Per}_{\geq 3}(E) = (x\ y)$, then the unique $g_2 \in \mathcal{T}_2$ such that $g_2|\mathrm{Per}_{\geq 3}(E) = (x\ y)$ is also the unique element of $\mathcal{T}_2$ which commutes with g_1.

To see $\mathcal{T}_1 \not\sim \mathcal{T}_2 \Rightarrow \exists g_1 \in \mathcal{T}_1\ (\neg \exists! g_2 \in \mathcal{T}_2\ (g_1 g_2 = g_2 g_1))$, note that if $\mathcal{T}_1 \not\sim \mathcal{T}_2$, then $x(\mathcal{T}_1) \neq x(\mathcal{T}_2)$, in which case we can easily find an element of $\mathcal{T}_1$ which commutes with infinitely many elements of $\mathcal{T}_2$. ☐

Now let $\varphi : \mathrm{Aper}(E) \to \mathrm{Aper}(F)$ be the unique map such that

$$\forall x \in \mathrm{Aper}(E)\ (\pi(\mathcal{T}_x) \sim \mathcal{T}_{\varphi(x)}),$$

and suppose that $x, y \in \mathrm{Aper}(E)$ are E-equivalent. As $(x\ y)$ is the unique element of $\mathcal{T}_x \cap \mathcal{T}_y$, it follows that $\pi[(x\ y)]$ is the unique element of $\pi(\mathcal{T}_x) \cap \pi(\mathcal{T}_y)$; thus

$$\pi[(x\ y)]|\mathrm{Per}_{\geq 3}(E) = (\varphi(x)\ \varphi(y)).$$

For each $g \in [E]$, we now have that

$$\begin{aligned} \pi(g)[\{\varphi(x), \varphi(y)\}] &= \pi(g)[\mathrm{supp}[(\varphi(x)\ \varphi(y))]] \\ &= \mathrm{Per}_{\geq 3}(F) \cap \pi(g)[\mathrm{supp}(\pi[(x\ y)])] \\ &= \mathrm{Per}_{\geq 3}(F) \cap \mathrm{supp}(\pi(g) \circ \pi[(x\ y)] \circ \pi(g)^{-1}) \\ &= \mathrm{Per}_{\geq 3}(F) \cap \mathrm{supp}(\pi(g \circ (x\ y) \circ g^{-1})) \\ &= \mathrm{Per}_{\geq 3}(F) \cap \mathrm{supp}(\pi[(g \cdot x\ g \cdot y)]) \\ &= \{\varphi(g \cdot x), \varphi(g \cdot y)\}, \end{aligned}$$

and it follows that $\pi(g) \cdot \varphi(x) = \varphi(g \cdot x)$, which completes the proof. □

3. Orbit equivalence relations

Here we describe a technical condition under which the map φ of Proposition 2.1 is automatically Borel. We then use this to draw out our main theorem regarding the connection between Borel isomorphism of orbit equivalence relations and algebraic isomorphism of their full groups.

Suppose that E is a (not necessarily Borel) equivalence relation on a Polish space X. We say that E is *countable* if each of its equivalence classes is countable, and E is *good* if it admits a countable Borel subequivalence relation $F \subseteq E$ such that

$$\forall x \in X\ (|[x]_E| \geq 3 \Rightarrow |[x]_F| \geq 3).$$

Our interest in such equivalence relations stems from the following connection between their full groups and the underlying σ-algebra of Borel sets.

Proposition 3.1. *Suppose that E is an equivalence relation on a Polish space X. Then the following are equivalent:*

1. *E is good.*
2. *The σ-algebra generated by $\mathcal{A} = \{\mathrm{supp}(g) : g \in [E]\}$ contains every set of the form $A \cap B$, where $A = \mathrm{Per}_{\geq 3}(E)$ and $B \subseteq X$ is Borel.*

Proof. To see (1) ⇒ (2), fix a countable Borel equivalence relation $F \subseteq E$ with

$$\forall x \in X\ (|[x]_E| \geq 3 \Rightarrow |[x]_F| \geq 3),$$

and suppose that $B \subseteq X$ is Borel. As $A = \mathrm{Per}_{\geq 3}(F)$ and the latter set is Borel, we can write $A \cap B = B_1 \cup B_2$, where B_1 is a Borel set which intersects every equivalence class of F in at most one point, and B_2 is a Borel set which intersects every equivalence class of F in an even or infinite number of points. It is not difficult to find involutions $g_1, g_2 \in [F]$ such that $B_1 = \mathrm{supp}(g_1) \cap \mathrm{supp}(g_2)$, and Proposition 7.4 of Kechris-Miller [2] ensures the existence of an involution $g \in [F]$ such that $\mathrm{supp}(g) = B_2$. As $B \subseteq X$ was arbitrary, condition (2) follows.

To see (2) ⇒ (1), suppose that the σ-algebra generated by $\mathcal{A}$ contains every Borel set of the form $A \cap B$, with $B \subseteq X$ Borel, fix a countable family of Borel automorphisms $g_0, g_1, \ldots$ in $[E]$ such that the corresponding family of Borel sets

$A_n = \mathrm{supp}(g_n)$ separates points of A, let G be the group generated by these automorphisms, and define $B \subseteq X$ by

$$B = \{x \in X : |[x]_{E_G^X}| \leq 2\}.$$

Note that if $x \in A \cap B$, then $|[x]_{E_G^X}| = 1$, since otherwise there exists $y \neq x$ in $[x]_{E_G^X}$, and we can then find $g \in G$ such that exactly one of x, y lie in $\mathrm{supp}(g)$; thus $\{x, y, g \cdot x, g \cdot y\} \subseteq [x]_{E_G^X}$ consists of 3 points. It follows that

$$A \cap B = \{x \in A : \forall g \in G \ (x \notin \mathrm{supp}(g))\},$$

and therefore $A \cap B$ consists of at most one point. If $A \cap B = \emptyset$, we set $F = E_G^X$. If $A \cap B = \{x\}$, we fix $y \in [x]_E \setminus \{x\}$ and define

$$x_1 F x_2 \Leftrightarrow x_1 E_G^X x_2 \text{ or } x_1, x_2 \in \{x\} \cup [y]_{E_G^X}.$$

In either case, we have that $|[x]_E| \geq 3 \Rightarrow |[x]_F| \geq 3$; hence E is good. □

Next, we have our main technical result:

Theorem 3.2. *Suppose that E and F are good equivalence relations on Polish spaces X and Y and $\pi : [E] \to [F]$ is an algebraic isomorphism. Then there is a Borel isomorphism φ of $E|\mathrm{Aper}(E)$ and $F|\mathrm{Aper}(F)$ such that*

$$\forall g \in [E] \ (\pi(g)|\mathrm{Aper}(F) = \varphi \circ (g|\mathrm{Aper}(E)) \circ \varphi^{-1}).$$

Proof. By Proposition 2.1 there is a bijection $\varphi : \mathrm{Aper}(E) \to \mathrm{Aper}(F)$ such that

$$\forall g \in [E] \ (\pi(g)|\mathrm{Aper}(F) = \varphi \circ (g|\mathrm{Aper}(E)) \circ \varphi^{-1}).$$

Now, for each $g \in [E]$, we have that

$$\begin{aligned} \varphi(\mathrm{supp}(g) \cap \mathrm{Aper}(E)) &= \varphi(\mathrm{supp}(g|\mathrm{Aper}(E))) \\ &= \mathrm{supp}(\varphi \circ (g|\mathrm{Aper}(E)) \circ \varphi^{-1}) \\ &= \mathrm{supp}(\pi(g)|\mathrm{Aper}(F)) \\ &= \mathrm{supp}(\pi(g)) \cap \mathrm{Aper}(F). \end{aligned}$$

As E and F are good, the sets $\mathrm{Per}_{\geq 3}(E)$ and $\mathrm{Per}_{\geq 3}(F)$ are Borel, and Proposition 3.1 ensures that the Borel subsets of $\mathrm{Per}_{\geq 3}(E)$ are generated by the sets of the form $\mathrm{supp}(g)$, where $g \in [E]$. Similarly, the Borel subsets of $\mathrm{Per}_{\geq 3}(F)$ are generated by the sets of the form $\mathrm{supp}(g)$, where $g \in [F]$, and it easily follows that φ is a Borel isomorphism of $E|\mathrm{Aper}(E)$ and $F|\mathrm{Aper}(F)$. □

We say that an equivalence relation E is *very good* if there is a countable Borel subequivalence relation $F \subseteq E$ such that

$$\forall x \in X \, \forall n \in \mathbb{N} \ (|[x]_E| \geq n \Rightarrow |[x]_F| \geq n).$$

Theorem 3.3. *Suppose that E and F are very good equivalence relations on Polish spaces X and Y, and the following conditions hold:*

1. *E and F have the same number of singleton equivalence classes.*
2. *If E and F both have infinitely many doubleton equivalence classes, then they have the same number of doubleton equivalence classes.*

Then $[E] \cong [F] \Leftrightarrow E \cong_B F$.

Proof. It is enough to show ($\Rightarrow$). In light of Theorem 3.2, it only remains to show that for all $n \geq 1$, the equivalence relations $E|\mathrm{Per}_n(E)$ and $F|\mathrm{Per}_n(F)$ are Borel isomorphic. As E and F are very good, it follows that the sets $\mathrm{Per}_n(E)$ and $\mathrm{Per}_n(F)$ are Borel, so it is enough to show that $|\mathrm{Per}_n(E)| = |\mathrm{Per}_n(F)|$. Condition (1) ensures that this is the case when $n = 1$.

For $n = 2$, note that the normal subgroups of $[E]$ of cardinality 2 are exactly those of the form $\{1, g\}$, where $\mathrm{supp}(g) \subseteq \mathrm{Per}_2(E)$. Letting κ denote the number of such subgroups, it follows that

$$\kappa = \min(2^{\aleph_0}, 2^{|\mathrm{Per}_2(E)|}) = \min(2^{\aleph_0}, 2^{|\mathrm{Per}_2(F)|}),$$

and condition (2) then ensures that $|\mathrm{Per}_2(E)| = |\mathrm{Per}_2(F)|$.

For $n = 4$, note that the minimal normal subgroups of $[E]$ of cardinality 4 are exactly those of the form

$$N = \{\mathrm{id}_X, (x_1\ x_2)(x_3\ x_4), (x_1\ x_3)(x_2\ x_4), (x_1\ x_4)(x_2\ x_3)\},$$

where x_1, x_2, x_3, x_4 make up a single equivalence class of E. Letting κ denote the number of such subgroups, it follows that $\kappa = |\mathrm{Per}_4(E)| = |\mathrm{Per}_4(F)|$.

For the remaining n, the minimal normal subgroups of $[E]$ which are isomorphic to A_n, the alternating group on n elements, are exactly those of the form

$$N = \{g \in [E] : \mathrm{supp}(g) \subseteq [x]_E \text{ and } g \text{ is of even cycle type}\},$$

where $x \in \mathrm{Per}_n(E)$. Letting κ denote the number of such subgroups, it follows that $\kappa = |\mathrm{Per}_n(E)| = |\mathrm{Per}_n(F)|$. ☐

Theorem 1.1 is now a consequence of the following fact:

Proposition 3.4. *Suppose that G is a Polish group which acts in a Borel fashion on a Polish space X. Then E_G^X is very good.*

Proof. By Theorem 2.6.6 of Becker-Kechris [1], we can assume that the action of G on X is continuous. Fix a countable dense subgroup $H \leq G$, and note that if $g_1 \cdot x, g_2 \cdot x, \ldots, g_n \cdot x$ are distinct then, by choosing h_i sufficiently close to g_i, we can ensure that $h_1 \cdot x, h_2 \cdot x, \ldots, h_n \cdot x$ are also distinct; thus the countable Borel equivalence relation $F = E_H^X$ witnesses that E_G^X is very good. ☐

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