COARSE EQUIVALENCE AND TOPOLOGICAL COUPLINGS OF LOCALLY COMPACT GROUPS

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ABSTRACT. M. Gromov has shown that any two finitely generated groups Γ and Λ are quasi-isometric if and only if they admit a topological coupling, i.e., a commuting pair of proper continuous cocompact actions $\Gamma \curvearrowright X \curvearrowleft \Lambda$ on a locally compact Hausdorff space. This result is extended here to all (compactly generated) locally compact second-countable groups.

In his seminal monograph on geometric group theory [1], M. Gromov formulated a topological criterion for quasi-isometry of finitely generated groups (see $0.2.C_2'$ in [1]). His idea was to replace the geometric objects, that is, the finitely generated groups, by a purely topological framework, namely, a locally compact Hausdorff space, which has no intrinsic large scale geometric structure.

As it is, Gromov's proof easily adapts to characterise coarse equivalence of arbitrary countable discrete groups, but thus far the case of locally compact groups has not been adressed and indeed Gromov's construction is insufficient to deal with these. The present paper presents a solution to this problem by establishing the following theorem.

Theorem 1. *Two locally compact second-countable groups are coarsely equivalent if and only if they admit a topological coupling.*

As coarse equivalence of locally compact, compactly generated groups is just quasi-isometry, we have the following corollary.

Corollary 2. *Two compactly generated, locally compact second-countable groups are quasi-isometric if and only if they admit a topological coupling.*

Let us recall that a *topological coupling* of two locally compact groups G and H is a pair $G \curvearrowright X \curvearrowleft H$ of commuting, proper and cocompact continuous actions on a non-empty locally compact Hausdorff space X. Here the actions are *proper* if, for every compact subset $K \subseteq X$, the sets

$$\{g \in G \mid gK \cap K \neq \emptyset\}, \quad \{h \in H \mid Kh \cap K \neq \emptyset\}$$

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are both compact. Also, the actions are cocompact if $X = G \cdot K = K \cdot H$ for some compact subset $K \subseteq X$.*

A *coarse equivalence* between two metric spaces X and Y is a map $\phi\colon X \to Y$ for which $\phi[X]$ is *cobounded* in Y, i.e., $\sup_{y\in Y} d(y, \phi[X]) < \infty$, and so that, for all sequences (x_n) and (z_n) in X, we have

$$\lim_n d(x_n, z_n) = \infty \quad\Leftrightarrow\quad \lim_n d(\phi(x_n), \phi(z_n)) = \infty.$$

Alternatively, the latter condition may be expressed by saying that there are non-decreasing functions $\kappa, \omega\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty} \kappa(t) = \infty$ so that

$$\kappa\big(d(x,z)\big) \leqslant d(\phi(x), \phi(z)) \leqslant \omega\big(d(x,z)\big)$$

for all $x, z \in X$.

On the other hand, a *quasi-isometry* is a coarse equivalence in which the bounding functions κ and ω may be taken to be affine.

We note that every locally compact second-countable group G admits a compatible left-invariant *proper* metric d, i.e., whose closed balls are compact [5]. Moreover, any two such metrics turn out to be coarsely equivalent via the identity map and hence define a unique coarse geometry on G. If, in addition, G is compactly generated, say $G = \langle K \rangle$ for some symmetric compact set K, there is a compatible left-invariant proper metric d that is quasi-isometric to the left-invariant word metric ρ_K induced by K. And as before, any two compact symmetric generating sets induce quasi-isometric word metrics and hence define a unique quasi-isometry type of G. Finally, any coarse equivalence between compactly generated, locally compact second-countable groups will in fact be a quasi-isometry.

The results of the paper here have recently been applied by J. Koivisto, D. Kyed and S. Raum to problems of measure equivalence between amenable unimodular groups [3].

Let us begin by verifying the easy direction of Theorem 1. To fix notation, we let $B_d(x, \epsilon)$ denote the open ball of d-radius ϵ centred at x and $\overline{B_d(x, \epsilon)}$ its closure, while, for any set A, we set $(A)_\epsilon = \{x \mid d(x, A) < \epsilon\}$.

Lemma 3. *Let $G \curvearrowright X \curvearrowleft H$ be a topological coupling of locally compact second-countable groups. Then G and H are coarsely equivalent.*

Proof. Fix compatible left-invariant proper metrics d_G and d_H on G and H respectively. Let $K \subseteq X$ be a compact subset with $X = G \cdot K = K \cdot H$ and pick a point $x \in X$. We define a map $\phi\colon G \to H$ by requiring that

$$x \in gK\phi(g)^{-1}$$

for all $g \in G$. Observe then that

$$g^{-1}fK \cap K\phi(g)^{-1}\phi(f) \neq \emptyset$$

for all $g, f \in G$.

*The G-action written on the left will be a *left*-action, while the H-action written on the right will be a *right*-action. However, both groups G and H will be equipped with their left-invariant coarse structure, which is that induced by a compatible proper left-invariant metric.

Now suppose t is a given constant. As the subset $\overline{B_{d_G}(1,t)K}$ is compact and H acts properly, there is some s so that

$$\overline{B_{d_G}(1,t)K} \cap Kh = \emptyset$$

whenever $d_H(h,1) \geqslant s$. Then, if $d_G(g,f) < t$, we have $g^{-1}fK \subseteq \overline{B_{d_G}(1,t)K}$ and so

$$\overline{B_{d_G}(1,t)K} \cap K\phi(g)^{-1}\phi(f) \neq \emptyset,$$

i.e., $d_H\big(\phi(g),\phi(f)\big) < s$.

Similarly, for every t, there is s so that

$$d_H\big(\phi(g),\phi(f)\big) < t \;\Rightarrow\; d_G(g,f) < s.$$

Thus to see that ϕ is a coarse equivalence, we must check that $\phi[G]$ is cobounded in H. So find s so that $K \cap Kh = \emptyset$ whenever $d_H(h,1) > s$. Then given $h \in H$ find some $g \in G$ so that

$$x \in gKh^{-1} \cap gK\phi(g)^{-1}$$

and thus also $K \cap K\phi(g)^{-1}h \neq \emptyset$. It follows that $d_H(h,\phi(g)) \leqslant s$, whence $\phi[G]$ is cobounded in H. □

We now proceed to the proof of Theorem 1.

Proof. Suppose G and H are locally compact second-countable groups and let d_G and d_H be proper left-invariant compatible metrics on G and H respectively. Let also μ denote left-invariant Haar measure on G scaled so that the closure of the unit ball $B = \overline{B_{d_G}(1_G,1)}$ has measure 1.

Assume that $\phi\colon H \to G$ is a coarse equivalence and let κ_ϕ, ω_ϕ be respectively the compression and the expansion moduli of ϕ,

$$\kappa_\phi(t) = \inf_{d_H(h_1,h_2)\geqslant t} d_G(\phi h_1, \phi h_2),$$

$$\omega_\phi(t) = \sup_{d_H(h_1,h_2)\leqslant t} d_G(\phi h_1, \phi h_2)$$

Choose also $s > 0$ so that $\kappa_\phi(s) \geqslant 3$.

Let $Y \subseteq H$ be a maximal s-discrete subset, i.e., so that $d_H(y,y') \geqslant s$ and hence also $d_G\big(\phi(y),\phi(y')\big) \geqslant 3$ for all $y \neq y'$ in Y. By maximality, Y is s-dense in H, i.e., for every $h \in H$ there is some $y \in Y$ with $d_H(h,y) < s$. For every $y \in Y$, define $\theta_y\colon H \to [0, s+1]$ by $\theta_y(h) = \max\{0, s+1-d_H(h,y)\}$. Note that θ_y is 1-Lipschitz and $\theta_y \geqslant 1$ on the ball of radius s centred at y, while $\theta_y = 0$ outside the ball of radius $s+1$. By properness and left-invariance of the metric d_H, it follows that

$$\Theta(h) = \sum_{y\in Y} \theta_y(h)$$

is a bounded Lipschitz function with $\Theta \geqslant 1$.

Let now $Z = \phi[Y]$ and, for $z = \phi(y)$, define $\alpha_z = \frac{\theta_y}{\Theta}$. Then the family $\{\alpha_z\}_{z\in Z}$ is a partition of unity of H consisting of N-Lipschitz functions $\alpha_z\colon H \to [0,1]$ for some $N > 0$, so that

$$\mathrm{supp}(\alpha_{\phi(y)}) \subseteq B_{d_H}(y, s+1),$$

for each $y \in Y$. Thus, for all $h \in H$, we have

$$Z_h = \{z \in Z \mid \alpha_z(h) > 0\} \subseteq B_{d_G}\big(\phi(h), \omega_\phi(s+1)\big).$$

So Z_h is a 3-discrete subset of G of diameter at most $2\omega_\phi(s+1)$. We let M be the maximal size of a 3-discrete subset of diameter at most $2\omega_\phi(s+1)$ in G.

Let $\lambda\colon G \curvearrowright L^1(G,\mu)$ denote the left-regular representation and define a function $\psi\colon H \to L^1(G,\mu)$ by

$$\psi_h = \sum_{z\in Z} \alpha_z(h)\cdot \chi_{zB} = \sum_{z\in Z_h} \alpha_z(h)\cdot \chi_{zB} \in L^1(G,\mu).$$

Thus each ψ_h is the convex combination of at most M disjointly supported left translates $\lambda(z)\chi_B = \chi_{zB}$ of the characteristic function χ_B. Therefore, for $h, f \in H$,

$$\begin{aligned}\|\psi_h - \psi_f\|_{L^1} &= \Big\| \sum_{z\in Z_h\cup Z_f} (\alpha_z(h) - \alpha_z(f))\cdot \chi_{zB}\Big\|_{L^1} \\ &\leqslant \sum_{z\in Z_h\cup Z_f} |\alpha_z(h)-\alpha_z(f)|\cdot \|\chi_{zB}\|_{L^1} \\ &\leqslant \sum_{z\in Z_h\cup Z_f} N\cdot d_H(h,f) \\ &\leqslant 2MN\cdot d_H(h,f).\end{aligned}$$

I.e., ψ is $2MN$-Lipschitz. Also, $\|\psi_h\|_{L^1} = 1$, while the essential support of ψ_h, denoted $\mathrm{supp}(\psi_h)$, satisfies

$$\mathrm{supp}(\psi_h) \subseteq \overline{B_{d_G}(\phi(h), \omega_\phi(s+1)+1)}.$$

Set

$$X = \{\xi \in L^1(G,\mu) \mid \quad \xi = \sum_{i=1}^m \alpha_i \chi_{g_iB} \text{ where } \{g_1,\dots,g_m\} \text{ ranges over 3-discrete}$$
$$\text{subsets of } G \text{ with diameter} \leqslant 2\omega_\phi(s+1) \text{ and } \alpha_i \geqslant 0 \text{ with } \sum_{i=1}^m \alpha_i = 1\}.$$

Observe that X is invariant under the left-regular presentation $\lambda\colon G \curvearrowright L^1(G,\mu)$.

We denote the duality pairing between $L^1(G)$ and $L^\infty(G)$ by $\langle\cdot\mid\cdot\rangle$, i.e.,

$$\langle \xi \mid \zeta\rangle = \int \xi(g)\zeta(g)d\mu(g)$$

for $\xi \in L^1(G)$ and $\zeta \in L^\infty(G)$.

Claim 4. *X is locally compact in the norm topology on $L^1(G,\mu)$. In fact,*

$$[K,\epsilon] = \{\xi \in X \mid \langle \xi \mid \chi_K\rangle \geqslant \epsilon\}$$

is norm compact for every compact subset $K \subseteq G$ and $\epsilon > 0$. Conversely, every compact subset of X is contained in some $[K,\epsilon]$.

Proof. Observe that, for every compact subset $K \subseteq G$ and $\epsilon > 0$, the set $[K,\epsilon]$ is weakly closed and thus also norm closed in X. Moreover, for every $\xi \in X$, there is a compact subset $K \subseteq G$ so that $\langle \xi \mid \chi_K\rangle > 0$, whereby, for $\epsilon = \frac{1}{2}\langle \xi \mid \chi_K\rangle$, the set $[K,\epsilon]$ is a norm neighbourhood of ξ in X.

To see that $[K,\epsilon]$ is compact and hence X is locally compact, it suffices to show that every sequence (ξ_n) in $[K,\epsilon]$ has a convergent subsequence. By passing to

a subsequence, there is some $m \leqslant M$ so that each ξ_n can be written as a convex combination
$$\xi_n = \sum_{i=1}^{m} \alpha_{i,n}\chi_{g_{i,n}B}$$
of some 3-discrete subset $\{g_{1,n}, \dots, g_{m,n}\} \subseteq G$ with diameter at most $2\omega_\phi(s+1)$. Note then that, as $\langle \xi_n \mid \chi_K \rangle > 0$, we have $g_{i,n}B \cap K \neq \emptyset$ for some $i \leqslant m$ and thus that $d_G(g_{j,n}, K) \leqslant 2\omega_\phi(s+1)+1$ for all j. Therefore, by passing to a further subsequence, we may assume that $g_i = \lim_n g_{i,n}$ and $\alpha_i = \lim_n \alpha_{i,n}$ exist for all $i \leqslant m$. It follows that $\{g_1, \dots, g_m\}$ is a 3-discrete subset of G with radius at most $2\omega_\phi(s+1)$, that $\alpha_i \geqslant 0$ with $\sum_i \alpha_i = 1$ and that
$$\xi = \sum_{i=1}^{m} \alpha_i \chi_{g_i B} \in X$$
is the norm limit of the ξ_n.

Suppose now instead that $C \subseteq X$ is compact. Then C is covered by open subsets of the form
$$(K, \epsilon) = \{\xi \in X \mid \langle \xi \mid \chi_K \rangle > \epsilon\}$$
for K compact and $\epsilon > 0$ and hence may be covered by finitely many of these, $C \subseteq \bigcup_{i=1}^{p} (K_i, \epsilon_i)$. It thus follows that $C \subseteq [\bigcup_{i=1}^{p} K_i, \min_{i=1}^{p} \epsilon_i]$. □

Consider now the space of maps X^H and note that $\psi \in X^H$. Endow X^H with the product topology and commuting left and right actions $G \curvearrowright X^H \curvearrowleft H$ by homeomorphisms given by
$$(g \cdot \xi)_h = \lambda(g)\xi_h, \quad (\xi \cdot h)_f = \xi_{hf}$$
for $g \in G$, $h, f \in H$ and $\xi \in X^H$. We set $\Omega = \overline{G \cdot \psi \cdot H}$. Note that Ω is $G \times H$-invariant. We will see that Ω is a topological coupling of G and H.

Note first that, for $g \in G$ and $h, f_1, f_2 \in H$,
$$\|(g\psi h)_{f_1} - (g\psi h)_{f_2}\|_{L^1} = \|\lambda(g)\big(\psi_{hf_1} - \psi_{hf_2}\big)\|_{L^1} \leqslant 2MN \cdot d_H(f_1, f_2)$$
and conclude that for all $\xi \in \Omega$ and $f_1, f_2 \in H$,
$$\|\xi_{f_1} - \xi_{f_2}\|_{L^1} \leqslant 2MN \cdot d_H(f_1, f_2).$$
We also have
$$\mathrm{supp}\big((g\psi h)_f\big) = \mathrm{supp}\big(\lambda(g)\psi_{hf}\big) = g \cdot \mathrm{supp}\big(\psi_{hf}\big) \subseteq \overline{B_{d_G}(g\phi(hf), \omega_\phi(s+1)+1)}.$$
Therefore,
$$\begin{aligned}
\kappa_\phi\big(d_H(f_1, f_2)\big) - 2\omega_\phi(s+1) - 2 &\leqslant d_G(g\phi(hf_1), g\phi(hf_2)) - 2\omega_\phi(s+1) - 2 \\
&\leqslant d_G\Big(\mathrm{supp}\big((g\psi h)_{f_1}\big), \mathrm{supp}\big((g\psi h)_{f_2}\big)\Big) \\
&\leqslant d_G(g\phi(hf_1), g\phi(hf_2)) + 2\omega_\phi(s+1) + 2 \\
&\leqslant \omega_\phi\big(d_H(f_1, f_2)\big) + 2\omega_\phi(s+1) + 2.
\end{aligned}$$
By consequence, we have that for all $\xi \in \Omega$ and $f_1, f_2 \in H$,
$$\begin{aligned}
\kappa_\phi\big(d_H(f_1, f_2)\big) - 2\omega_\phi(s+1) - 2 &\leqslant d_G\Big(\mathrm{supp}\big(\xi_{f_1}\big), \mathrm{supp}\big(\xi_{f_2}\big)\Big) \\
&\leqslant \omega_\phi\big(d_H(f_1, f_2)\big) + 2\omega_\phi(s+1) + 2.
\end{aligned}$$

Claim 5. *The space $\Omega = \overline{G \cdot \psi \cdot H}$ is locally compact.*

Proof. Indeed, given $\xi \in \Omega$, let $K = \mathrm{supp}(\xi_1)$ and consider the neighbourhood

$$\{\eta \in \Omega \mid \eta_1 \in [K, 1/2]\}$$

of ξ. Then, if $\zeta \in \{\eta \in \Omega \mid \eta_1 \in [K, 1/2]\}$, we have $K \cap \mathrm{supp}(\zeta_1) \neq \emptyset$ and thus

$$\mathrm{supp}(\zeta_f) \subseteq (K)_{\omega_\phi(d_H(f,1))+6\omega_\phi(s+1)+6}.$$

By consequence

$$\{\zeta \in \Omega \mid \zeta_1 \in [K, 1/2]\} \subseteq \prod_{f \in H} [(K)_{\omega_\phi(d_H(f,1))+6\omega_\phi(s+1)+6}, 1],$$

where the latter product is compact. $\square$

Claim 6. *The action $\Omega \curvearrowleft H$ is continuous and proper.*

Proof. To show continuity at $(\xi, h) \in \Omega \times H$, we must show that, for all $f \in H$ and $\epsilon > 0$, there are neighbourhoods V and W of ξ and h respectively so that $\|(\xi \cdot h)_f - (\zeta \cdot k)_f\|_{L^1} < \epsilon$ whenever $\zeta \in V$ and $k \in W$.

So assume $f \in H$ and $\epsilon > 0$ are given and define

$$V = \{\zeta \in \Omega \mid \|\xi_{hf} - \zeta_{hf}\|_{L^1} < \frac{\epsilon}{2}\}$$

and

$$W = \{k \in H \mid d_H(hf, kf) < \frac{\epsilon}{4MN}\}.$$

Then, for all $\zeta \in V$ and $k \in W$, we have

$$\begin{aligned}\|(\xi \cdot h)_f - (\zeta \cdot k)_f\|_{L^1} &\leqslant \|\xi_{hf} - \zeta_{hf}\|_{L^1} + \|\zeta_{hf} - \zeta_{kf}\|_{L^1} \\ &< \frac{\epsilon}{2} + 2MN \cdot d_H(hf, kf) \\ &< \epsilon\end{aligned}$$

as required.

To see that the action is proper, observe that, for any compact subset of Ω, the projection on the coordinate $1 \in H$ is also compact, hence the subset is contained in a set of the form

$$\{\xi \in \Omega \mid \xi_1 \in [K, \epsilon]\},$$

for some compact $K \subseteq G$ and $\epsilon > 0$.

So suppose that $\xi, \zeta \in \Omega$ satisfy $\xi_1, \zeta_1 \in [K, \epsilon]$, while $h \in H$ satisfies

$$\kappa_\phi(d_H(h,1)) > \mathrm{diam}_{d_G}(K) + 2\omega_\phi(s+1) + 2.$$

Then $\mathrm{supp}(\xi_1)$ and $\mathrm{supp}(\zeta_1)$ must both intersect K. Therefore,

$$\begin{aligned}\mathrm{diam}_{d_G}(K) &< \kappa_\phi(d_H(h,1)) - 2\omega_\phi(s+1) - 2 \\ &\leqslant d_G\Big(\mathrm{supp}(\zeta_h), \mathrm{supp}(\zeta_1)\Big)\end{aligned}$$

and so $\mathrm{supp}(\zeta_h)$ must be disjoint from K, whence $\mathrm{supp}(\zeta_h) \neq \mathrm{supp}(\xi_1)$. In particular, $(\zeta \cdot h)_1 = \zeta_h \neq \xi_1$.

By consequence,

$$\{\xi \in \Omega \mid \xi_1 \in [K, \epsilon]\} \cap \{\xi \in \Omega \mid \xi_1 \in [K, \epsilon]\} \cdot h = \emptyset$$

for all $h \in H$ with $\kappa_\phi\big(d_H(h,1)\big) > \mathrm{diam}_{d_G}(K) + 2\omega_\phi(s+1) + 2$, witnessing properness of the action. $\square$

Claim 7. *The action $\Omega \curvearrowleft H$ is cocompact.*

Proof. Fix $R > \sup_{g\in G} d_G(g, \phi[H])$ and let $K \subseteq G$ be the closed ball of radius $R + \omega_\phi(s+1) + 1$ centred at 1_G. We will show that

$$\Omega = \{\xi \in \Omega \mid \xi_1 \in [K, 1/2]\} \cdot H.$$

To see this, fix $\zeta \in \Omega$ and set $C = \mathrm{supp}\big(\zeta_1\big)$. Now, suppose that $\mathrm{supp}\big((g\psi h)_1\big)$ intersects C for some $g \in G$ and $h \in H$, whence $d_G(C, g\phi(h)) \leqslant \omega_\phi(s+1) + 1$. Pick $f \in H$ so that $d_G(g\phi(hf), 1_G) = d_G(\phi(hf), g^{-1}) < R$ and observe that then

$$\begin{aligned}\kappa_\phi\big(d_H(f,1)\big) &\leqslant d_G(g\phi(h), g\phi(hf))\\ &\leqslant d_G(g\phi(h), C) + \mathrm{diam}_{d_G}(C) + d_G(C, 1_G) + d_G(1_G, g\phi(hf))\\ &\leqslant \omega_\phi(s+1) + 1 + \mathrm{diam}_{d_G}(C) + d_G(C, 1_G) + R.\end{aligned}$$

Letting

$$r = \sup\big\{t \mid \kappa_\phi(t) \leqslant \omega_\phi(s+1) + 1 + \mathrm{diam}_{d_G}(C) + d_G(C, 1_G) + R\big\},$$

we find that for all $g \in G$ and $h \in H$ for which $\mathrm{supp}\big((g\psi h)_1\big)$ intersects C, there is some $f \in B_{d_H}(1_H, r)$ with $d_G(g\phi(hf), 1_G) < R$, whence also

$$\mathrm{supp}\big((g\psi h)_f\big) \subseteq K.$$

and $\langle (g\psi h)_f \mid \chi_K\rangle = 1$. It follows that ζ is a limit of points $g\psi h \in \Omega$ for which there are $f \in B_{d_H}(1_H, r)$ with $\langle (g\psi h)_f \mid \chi_K\rangle = 1$.

Assume for a contradiction that $(\zeta \cdot f)_1 = \zeta_f \notin [K, 1/2]$ for all $f \in H$ and let $f_1, \ldots, f_n$ be $\frac{1}{5MN}$-dense in $B_{d_H}(1_H, r)$. Choose $g\psi h \in \Omega$ close enough to ζ so that $(g\psi h)_{f_i} \notin [K, 1/2]$ for all $i \leqslant n$, while, on the other hand, $\langle (g\psi h)_f \mid \chi_K\rangle = 1$ for some $f \in B_{d_H}(1_H, r)$. Pick then i with $d_H(f_i, f) \leqslant \frac{1}{5MN}$, whereby

$$\begin{aligned}1/2 &\leqslant \big|\langle (g\psi h)_{f_i} \mid \chi_K\rangle - \langle (g\psi h)_f \mid \chi_K\rangle\big|\\ &\leqslant \|(g\psi h)_{f_i} - (g\psi h)_f\|_{L^1}\\ &\leqslant 2MN \cdot d_H(f_i, f)\\ &< 1/2,\end{aligned}$$

which is absurd.

Thus, $\zeta \in \{\xi \in \Omega \mid \xi_1 \in [K, 1/2]\} \cdot f$ for some $f \in H$. Since $\zeta \in \Omega$ was arbitrary and $\{\xi \in \Omega \mid \xi_1 \in [K, 1/2]\}$ relatively compact in Ω, this proves cocompactness of the action. $\square$

Claim 8. *The action $G \curvearrowright \Omega$ is continuous, proper and cocompact.*

Proof. Continuity is trivial since already the action of G on X and hence on X^H is continuous. For properness, it suffices to see that, for every $\epsilon > 0$ and compact subset $1 \in K \subseteq G$, the set of $g \in G$ for which

$$\{\xi \in \Omega \mid \xi_1 \in [K, \epsilon]\} \cap g \cdot \{\xi \in \Omega \mid \xi_1 \in [K, \epsilon]\} \neq \emptyset$$

is relatively compact in G. But, if $\xi_1 \in [K, \epsilon]$ and $\lambda(g)\xi_1 = (g \cdot \xi)_1 \in [K, \epsilon]$, then $\mathrm{supp}(\xi_1)$ intersects both K and $g^{-1}K$, whereby

$$d_G(K, g^{-1}K) \leqslant \mathrm{diam}_{d_G}\big(\mathrm{supp}(\xi_1)\big) \leqslant 2\omega_\phi(s+1) + 2$$

and thus $d_G(g, 1) \leqslant 2\omega_\phi(s+1) + 2 + 2\,\mathrm{diam}_{d_G}(K)$.

Finally, for cocompactness, let K be the ball of radius $4\omega_\phi(s+1) + 4$ centred at 1_G. Then every subset of G of diameter at most $2\omega_\phi(s+1) + 2$ can be translated via an element of G into K. But this means that, if $\xi \in \Omega$, then

$$\mathrm{supp}\big((g \cdot \xi)_1\big) = g \cdot \mathrm{supp}(\xi_1) \subseteq K$$

for some $g \in G$ and thus $g \cdot \xi \in \{\zeta \in \Omega \mid \zeta_1 \in [K, 1]\}$. In other words,

$$\Omega = G \cdot \{\zeta \in \Omega \mid \zeta_1 \in [K, 1]\},$$

showing cocompactness. □

As both of the actions are proper, cocompact and continuous, we have a topological coupling. □

As stated, Theorem 1 only applies to locally compact second-countable groups. However, every locally compact group G comes equipped with a canonical coarse structure induced by the entourages

$$E_C = \{(g, f) \in G \times G \mid g^{-1}f \in C\},$$

where C ranges over compact subsets of G, and thus one may wonder if Theorem 1 still applies to them. While we have not investigated the general case, it is easy to extend our result to locally compact σ-compact groups, since, by the Kakutani–Kodaira Theorem [2], every such G contains arbitrarily small compact normal subgroups K so that G/K is locally compact second-countable. As G is then coarsely equivalent to G/K via the quotient map, a coarse equivalence between two such groups G and H would give rise to a topological coupling of their respective quotients and hence of G and H themselves.

In this connection, it is perhaps more interesting to note that, with appropriate adjustments, one may formulate and prove a generalisation of Theorem 1 for a larger class of *Polish* groups, i.e., separable and completely metrisable topological groups, namely those with bounded geometry. This forms part of the forthcoming monograph [4].

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