Infinite games

We will consider games with perfect information by two players on the integers.

<table>
<thead>
<tr>
<th>I</th>
<th>( a_0 )</th>
<th>( a_2 )</th>
<th>( a_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>II</td>
<td>( a_1 )</td>
<td>( a_3 )</td>
<td>( a_5 )</td>
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Let players I and II alternately play \( a_i \in \mathbb{N} \)

and hence a run of the game corresponds to a function \( f : \mathbb{N} \to \mathbb{N} \) given by \( f(n) = a_n \).

Given a set \( \mathcal{A} \) of atomic functions, i.e., \( a_i \in \mathbb{N} \), and a run of the game \( f \in \mathcal{A} \), we say that I wins the game if \( f \in \mathcal{A} \).

Thus, any set \( \mathcal{A} \) determines a game with two players, which we denote by \( G(\mathcal{A}) \). In this case, if \( \mathcal{A} \) is determined to be the winning condition.
A strategy $S$ for player I in this game is a function

$$S : \mathbb{N}^\omega \rightarrow \mathbb{N}$$

defined on infinite strings of numbers.

We say that $I$ plays according to this strategy if for $a_0, a_1, a_2, \ldots$, the outcome of the game we have

$$a_{2i} = S(a_0, a_1, \ldots, a_{2i-1})$$

In other words, the strategy tells $I$ what to play depending on what $II$ plays.

The notion of a strategy for $II$ is defined similarly.

Given a $S : \mathbb{N}^\omega \rightarrow \mathbb{N}$, a strategy $S$ for $I$ (resp. for $II$) is said to be winning for the game $G(S)$ if $I$ wins whenever he plays according to the strategy.
We say that a game $G(a^*)$ is determined if one of the two players have a winning strategy.

**Definition** The Axiom of Determinacy (AD) is the statement that $G(a^*)$ is determined for every $a^* \in \mathbb{N}^\mathbb{N}$.

**Theorem** Let $a \subseteq \mathbb{N}$ and let

$$a^* = \{ A \mid [A]_+ \in a \}.$$  

If $G(a^*)$ is determined, then $a$ either contains or is disjoint from a come.

**Proof** Consider first the case where $I$ has a winning strategy $s : \mathbb{N}^\mathbb{N} \to \mathbb{N}$ in the game $G(a^*)$.

Since $s$ is a sequence $(s_n, k_n)$ of pairs of finite strings and natural numbers we can recursively code $s$ as a subset of $\mathbb{N}$. Thus $s$ has a Turing degree $a$. 
We claim that $D(\geq a) \subseteq A$.

To see this, pick any $b \geq a$

and function $g = (a_1, a_3, a_5, a_7, \ldots) 

\in b$.

In the game $G(A^2)$ we let II play $a_1, a_3, a_5, \ldots$ and let I play $a_0, a_2, \ldots$ according to its strategy $\mathcal{S}$.

Clearly, $a_0, a_2, \ldots$ can be computed from $\mathcal{S}$ and $a_1, a_3, a_5, \ldots$.

Thus let $c$ be the degree of $a_0, a_2, \ldots$

and

$c \leq b \vee a = b$ (since $a \leq b$).

Also, since I wins $G(A^2)$, the sequence $a_0, a_1, a_2, a_3, \ldots$ belongs to $A$ and so its degree

$b \vee c = b$ belongs to $A$.

Therefore, $b \geq a$ and $D(\geq a) \subseteq A$.

Similarly, $D(\geq a) \cap A = \emptyset$ if II has $a$ wins.
Corollary (AD) Every set of degree either
contains or is disjoint from a come.

Theorem (AD) Every map \( \varphi : \mathcal{P} \to \mathbb{R} \)
is constant on a come.

Proof For every \( q \in \mathcal{Q} \) let
\[
\mathcal{A}_q = \{ a \in \mathcal{P} \mid \varphi(a) \geq q \}.
\]

List \( \mathcal{Q} \) as \( q_0, q_1, q_2, \ldots \).

We define inductively \( a_0 \leq a_1 \leq a_2 \leq \ldots \)
such that \( \mathcal{D}(\geq a_n) \) is either contained
in a come \( \mathcal{D}(\geq a) \) or is disjoint from \( \mathcal{A}_a \).

\( n = 0 \): \( \mathcal{A}_{q_0} \) is a come of degree, so
either contained or is disjoint from
some come \( \mathcal{D}(\geq b) \). So let
\[
a_0 = b.
\]

\( n = k + 1 \): Suppose \( \mathcal{A}_{a_k} \) is contained and check
\( k + 1 \) come \( \mathcal{D}(\geq b) \) which is either
contained in \( \mathcal{A}_{a_{k+1}} \) or disjoint from \( \mathcal{A}_{q_0} \).
Then is \[ D(\geq k) \subseteq D(\geq k), \]
\[ a_n = \dot k \forall n \text{ works}. \]

Pick \( A_n \subseteq a_n \) and let \( A = \bigcap_{n} A_n \) \[ A = [A]_\mathcal{F}. \]
Then \( a_0 \leq a, \ldots, a_n \leq a \) and thus for every \( n \),
\[ D(\geq a) \cap A_{A_n} = \emptyset \quad \text{as} \]
\[ D(\geq a) \subseteq A_{A_n}. \]

In particular, for every \( b \geq a \)
and \( q \in A \)
\[ \Theta(b) \geq q \quad \text{iff} \quad \Theta(a) \geq q, \]
whence \( \Theta(b) = \Theta(a) \).

So \( \Theta \) is constant on \( D(\geq a) \). \( \square \)

Corollary (AD) There is no injective function from \( D \) to \( \mathbb{R} \).

In particular, the axiom of choice fails.
Theorem (AD) There is a countably additive non-atomic measure on $\mathcal{P}(X)$. Moreover, the measure is two-valued.

Recall first that a countably additive measure on a set $X$ is a function $\mu : \mathcal{P}(X) \to [0, 1]$ satisfying

(i) $\mu(\emptyset) = 0$, $\mu(X) = 1$

(ii) $Y \subseteq Z \subseteq X \implies \mu(Y) \leq \mu(Z)$

(iii) If $Y_n \subseteq X$ are disjoint, then

$\mu\left(\bigcup_{n \in \mathbb{N}} Y_n\right) = \sum_{n \in \mathbb{N}} \mu(Y_n)$

(iv) $\mu$ is non-atomic if $\mu(\{x, y\}) = 0$ for every $x, y \in X$. 
Proof. We simply define

$$\mu (A) = \begin{cases} 0 & \text{if } A \text{ is disjoint from } z \text{ everywhere} \\ 1 & \text{if } A \text{ contains } z \text{ everywhere} \end{cases}$$

Clearly, (i), (ii), and (iv) hold.

Now if \( A_n \subseteq D \) are disjoint,

then there are no two of them that can contain \( z \) everywhere.

For then we would have two disjoint curves, which is impossible.

So \( \sum \mu (A_n) \leq 1 \).

And if each \( A_n \) is disjoint from \( z \) nowhere \( D \) (\( \neq z_n \)), then we can find \( b > z_n \) for all \( n \), and then \( A_n \cap D (\neq b) = \emptyset \), in which case

\( U \Delta u \) is disjoint from \( z \) nowhere \( D (\neq b) \).

So \( \mu (U \Delta u) = 0 \) if \( \sum \mu (A_n) = 0 \). \( \square \)
Definition: A linear order \((X, \leq)\) is a well order if there is no infinite descending chain.

Theorem: Let \((X, \leq)\) and \((Y, \leq)\) be two well orders. Then either \((X, \leq)\) is isomorphic to an initial segment of \((Y, \leq)\) or \((Y, \leq)\) is isomorphic to an initial segment of \((X, \leq)\) or both way around.

Prop: No well order is embeddable into a proper initial segment of itself.

So well orders are linearly ordered by the relation of embeddability which coincides with the relation of being isomorphic to an initial segment.

In fact, this well orders the well orders.
Evidently there is a bijection between $\mathbb{N}$ and $\mathcal{P}(\mathbb{Q})$, so from a set-theoretic viewpoint we can identify $\mathbb{N}$ with $\mathcal{P}(\mathbb{Q})$. Therefore, we can consider $D = \mathcal{P}(\mathbb{Q})$.

Let $\mathcal{W}_0 = \{ A \in \mathcal{Q} \mid A \text{ is wellordered and } \omega \leq A \}$. Notice that if $A \in \mathcal{Q}$ is wellordered, then its ordinal type $\text{otp}(A)$ is an ordinal $< \omega$.

So $\text{otp} : \mathcal{W}_0 \rightarrow \omega$, is a surjective function since $\mathcal{Q}$ contains a copy of any well ordered ordinal.

We can therefore define a function $\Theta : \mathcal{D} \rightarrow \omega$, by

$$\Theta(A) = \sup \{ \text{otp}(A) \mid A \in \mathcal{W}_0 \text{ and } \mathcal{T}_A = \omega \}$$
Notice that if \( a \leq b \) then
\[
\Theta(a) \leq \Theta(b) \quad \text{Hence } \Theta \text{ is a monotone function from } (D, \leq) \text{ to } (\omega, \leq).
\]

Now letting \( \nu = \Theta * \mu \), i.e.,
\[
\nu(x) = \mu(\Theta^{-1}(x)) \quad \text{we get a countably additive measure on } \omega,
\]
We claim that \( \nu \) is non-atomic.

For otherwise, there would be some \( \frac{1}{2} < c \), such that \( \Theta^{-1}(\{c\}) \)
\[
= \{ a \in D \mid \Theta(a) = c \} \quad \text{has } \mu \text{-measure 1}, \quad \text{i.e., containing some } D(\geq b).
\]

But it is easy to see \( A \subseteq B \), \( \Theta(B) = \frac{n}{2} + 1 \)
and \( B \text{ has degree } \frac{n}{2} \), then
\[
\Theta(b \cup c) \geq \frac{n}{2} + 1 \quad \text{which is a contradiction. Here one uses monotonicity.}
\]
Theorem (Salavag) If $A^c$ holds

then $w_i$ is a measurable cardinal, i.e., carries a $\mathcal{S}_{0,1}$-valued $w_i$-additive measure.

In other words, there is an ultrafilter $\mathcal{U}$ on $\omega$, not containing singletons and such that $(\omega \setminus A_n \in \mathcal{U}) \Rightarrow \forall n \in \mathcal{U}$. 

\[ \forall n \in \mathcal{U} \]