The jump operator

Definition
(i) $A \equiv^* B$ if $A \leq^* B \equiv^* A$.

(ii) The Turing degree of $A$ (or the degree of unsolvability) is the set
$$\text{deg}(A) = \{ B \mid B \equiv^* A \}.$$ 

(iii) $\text{deg}(A) \cup \text{deg}(B) = \text{deg}(A \oplus B)$

(iv) Degrees, i.e., $\equiv^*$ -equivalence classes are denoted by smaller case letters $a, b, c$ in boldface and $\mathcal{D}$ denotes the class of degrees.

(v) $(\mathcal{D}, \leq)$ is defined by $a \leq b$ iff
$$\exists A \in a \exists B \in b \ A \equiv^* B$$

(vi) $a \in \mathcal{D}$ is recursively enumerable if there is an r.e. set $A \in a$

(vii) $a \equiv a$ r.e. c.e. in b. Let $\exists A \in a \exists B \in b$
$$A \equiv a \ \text{r.e. in} \ B.$$
Definition \[ KA = \{ x \mid \phi^A_x(x) \uparrow \} \]

This set is called the jump of \( A \) and is also denoted by \( A' \).

By the Myhill isomorphism theorem one can see that \[ KA \cong K_A^A = \{ <x,y> \mid \phi^A_y(x) \uparrow \} \]

It is enough to notice that \( K^A_0 \equiv_1 KA \)

Theorem

(i) \( A' \) is r.e. in \( A \)

(ii) \( A' \not\equiv_1 A \)

(iii) If \( A \) is r.e. in \( B \) and \( B \leq_1 C \), then \( A \) is r.e. in \( C \).

(iv) \( B \) is r.e. in \( A \) iff \( B \leq_1 A' \)

(v) \( B \leq_1 A \iff B' \leq_1 A' \)

(vi) \( B \equiv_1 A \rightarrow B' \equiv_1 A' \rightarrow B' \equiv A' \rightarrow B' \equiv_1 A' \)

(vii) \( A \) is r.e. in \( B \) \( \iff \) \( A \) is r.e. in \( \overline{B} \).

(viii) \( A \leq_1 A' \) and \( \overline{A} \leq_1 A' \).
Proof. (i) It is clear, as $A^* = KA = S^*$.

(ii) This follows from relativising the proof at $K \not\vdash \emptyset$. To see what this latter says, notice that $K \leq_T \emptyset$ means that $\chi_K$ belongs to the recursive closure of $\exists \chi_\emptyset$, which is just the set of partial recursive functions.

But $\chi_K$ is not total recursive, so $K \not\vdash \emptyset$. Working in the class of partial $A$-recursive functions, one sees that $K^* \not\vdash A$.

(iii) This is also the relativised version of "$B$ is r.e. iff $B \leq_T K$".

(iv) If $A \neq \emptyset$ is r.e. in $\emptyset$, then $A = \text{rg}(f)$ for some total $\emptyset$-recursive function $f$.

Moreover, if $B \leq_T C$ then $f$ belongs to the recursive closure of $\exists \chi_B$ and $\chi_B$ belongs to the recursive closure of $\exists \chi_\emptyset$, whence $f$ belongs to the recursive closure of $\exists \chi_C$, witnessing that $A$ is r.e. in $C$. 

\[ \exists x \left( \forall y, (\exists z < (x, y, x, y) \land z \in A) \right) \]
(iv) If \( B \leq_A A \) then \( B' \) is re. in \( B \) (by (i)) and hence \( B' \) is re. in \( A \) (by (iii)) and \( B' \leq_A A' \) (by (iv)).

Commonly, if \( B' \leq_A A' \), then by (viii)
\[ B, CB \leq_A B' \leq_A A' \text{ and re by (iv)} \]
\[ B, CB \text{ are re. in } A \] Therefore, \( B \leq_A A \).

(vi) \[ B \equiv_A A \Rightarrow B' \equiv_A A' \Rightarrow B' \equiv_A A' \text{ are known.} \]

So if \( h \) is a recursive permutation of \( N \) such that \( h(B') = A' \), notice that \( X_B = X_A \circ h \Rightarrow X_A = X_B \circ h^{-1} \).

Thus, \( B' \equiv_A A' \Rightarrow A \equiv_A B' \).

(vii) Notice that \( B \equiv_A CB \) (by \( X_B = 1 - X_{CB} \)) so the result follows from (iii).

(viii) Relativise the proof above that \( K \) is 1-complete, i.e., that any re. set \( C \), \( C \leq_1 K \). Now, both \( A \) and \( CA \) are re. in \( A \), so re. in \( A \) and hence \( A, CA \leq_1 KA \equiv_A A' \).
Definition \[ A' = \deg (A') \] for any some \( A \).

This is well-defined by \((vi)\) above.

Moreover, put \( \varphi = \deg (\varphi) \), so \( \varphi' = \deg (K) \).

To see this one can use the following

Proposition If \( A \equiv B \), then there is a recursive function \( h : \mathbb{N} \to \mathbb{N} \) such that
\[ \phi^A_e = \phi^B_{h(e)} \] for all \( e \in \mathbb{N} \).

Proof Suppose \( \chi_A = \phi^B_{h(e)} \). Then \( h \) will be the function that to the program with index \( e \) calculates \( \chi_A \) of the program such that:

- calculate \( \phi^A_e \) but such that whenever a value \( \chi_A (n) \) is needed, calculate instead \( \phi^B_{h(e)} \).

Notice that \( h \) does not depend on \( A, B \). \( \square \)

By padding one can moreover suppose that \( h \) is injective. So as for acceptable systems of indices, one can show that if \( A \equiv B \) then there is a recursive bijection \( \sigma \) such that
\[ \phi^A_{\sigma(e)} = \phi^B_e \] for.