MA 430, SECOND HOMEWORK SET, DUE WEDNESDAY, SEPTEMBER 16TH.

1. Exercise

Decide whether the following argument is valid by formalising it in propositional logic and checking whether the conclusion is a tautological consequence of the hypotheses:

Either Obama fails to reform health care, or, if Blagojevich goes to prison, then Quinn stays governor. If Obama succeeds in reforming health care, then Blagojevich does not go to prison. If Quinn stays governor, then Blagojevich goes to prison. Therefore, if Blagojevich goes to prison, then Quinn stays governor.

2. Exercise

Decide whether the following hold

(1) $\{(P \leftrightarrow Q), (Q \vee \neg R)\} \models (P \to R)$,
(2) $\{(P \to (Q \to \neg P)), (R \to Q)\} \models (P \to \neg R)$.

3. Exercise

A safe has 5 locks and can be opened only when all of the 5 locks are open. Five people p_1, p_2, p_3, p_4, p_5 are to receive keys to some of the locks. Which keys depend on the person, but each key can be duplicated and given to several people. Find a distribution of the 5 keys such that the safe can be opened if and only if one of the following three situations occur:

- p_1 and p_2 are both present,
- p_1, p_3 and p_4 are all present,
- p_2, p_4 and p_5 are all present.

[Hint: Using the language $L = \{P_1, P_2, P_3, P_4, P_5\}$, the immediately preceding condition can be described by the following formula:

$$A = (P_1 \wedge P_2) \vee (P_1 \wedge P_3 \wedge P_4) \vee (P_2 \wedge P_4 \wedge P_5).$$

On the other hand, giving one key to, e.g., p_1 and p_3 corresponds to a disjunction $(P_1 \vee P_3)$ and the distribution of keys corresponds to a conjunctive normal form of the formula A.]

4. Exercise

Let L be a propositional language and let $\downarrow, \uparrow$ be new logical connectives of two variables. We define an extended set of propositional formulas of L, denoted $\mathrm{Form}^*(L)$, to be the smallest set of expressions such that

- any propositional variable belongs to $\mathrm{Form}^*(L)$,
- if $A, B \in \mathrm{Form}^*(L)$, then also $\neg A, (A \vee B), (A \wedge B), (A \to B), (A \leftrightarrow B), (A \downarrow B), (A \uparrow B) \in \mathrm{Form}^*(L)$.

So $\mathrm{Form}^*(L)$ contains all the expressions that are built up inductively using the extended set of logical connectives, $\neg, \vee, \wedge, \to, \leftrightarrow, \downarrow$ and $\uparrow$.

Moreover, if $v\colon L \to \{T, F\}$ is a valuation, we extend v to a function

$$v\colon \mathrm{Form}^*(L) \to \{T, F\}$$

by the usual rules for the connectives $\neg, \vee, \wedge, \to, \leftrightarrow$ and the additional rules

$$v\big((A \downarrow B)\big) = \begin{cases} T & \text{if } v(A) = v(B) = F, \\ F & \text{otherwise,} \end{cases}$$

and

$$v\big((A \uparrow B)\big) = \begin{cases} F & \text{if } v(A) = v(B) = T, \\ T & \text{otherwise.} \end{cases}$$

(a) Show that $(A \uparrow B) \equiv \neg(A \wedge B)$ for any $A, B \in \mathrm{Form}^*(L)$.

Recall that for any truth function $\psi\colon \mathrm{Val}(L) \to \{T, F\}$ there is a formula A in built up using only $\neg$ and $\wedge$ such that $\psi = \phi_A$.

(b) Show that for any truth function $\psi\colon \mathrm{Val}(L) \to \{T, F\}$ there is $A \in \mathrm{Form}^*(L)$ only containing the connective $\downarrow$ such that $\psi = \phi_A$.

(c) Show that for any truth function $\psi\colon \mathrm{Val}(L) \to \{T, F\}$ there is $A \in \mathrm{Form}^*(L)$ only containing the connective $\uparrow$ such that $\psi = \phi_A$.

5. Exercise

Let $L = \{P_1, P_2, \ldots, P_n\}$ be a propositional language. We define a partial ordering $\ll$ of the set $\mathrm{Val}(L)$ of all valuations of L as follows:

$$v \ll w \Leftrightarrow \text{ for all } i \leq n, \text{ if } v(P_i) = T, \text{ then also } w(P_i) = T.$$

(a) Show that indeed $\ll$ is a partial ordering of $\mathrm{Val}(L)$. Is it a total (a.k.a., linear) ordering?

(b) A formula A is called *positive* if for all valuations $v \ll w$, if $v(A) = T$, then also $w(A) = T$. Show that a formula A is positive if either

- A is a tautology,

- $\neg A$ is a tautology,
- A is tautologically equivalent to a formula containing only the connectives $\vee$ and $\wedge$.