First order theories

A first order theory of a language $L$ is just a set $T$ of $L$-formulas. Sometimes we call formulas in $T$ the axioms.

Examples:

Group theory. The language of group theory has one constant $e$ and one binary function symbol $\cdot$. The theory of group theory has axioms:

\[ \forall x \exists y \ (x \cdot y = e \land y \cdot x = e) \]
\[ \forall x \forall y \forall z \ (x \cdot (y \cdot z) = (x \cdot y) \cdot z) \]
\[ \forall x \ (x \cdot e = x \land e \cdot x = x) \]

The theory of abelian groups moreover has the following axiom:

\[ \forall x \forall y \ (x \cdot y = y \cdot x) \]

Here and in what follows we shall use variables $x, y, z, \ldots$ instead of $x_0, x_1, x_2, \ldots$ in order to simplify notation.
The theory of linearly ordered sets is a theory of first language with a single binary relation symbol \(<\) and with axioms

\[ \forall x \neg (x < x) \]
\[ \forall x \forall y \forall z \ ((x < y \land y < z) \rightarrow x < z) \]
\[ \forall x \forall y \ (x = y \lor x < y \lor y < x) . \]

The expressive power of first order logic is greater than that of propositional logic.

Example. Formalize the following in first order logic:

"If every ancestor of an ancestor of an individual is also an ancestor of the same individual and no individual is his/her own ancestor, then there is someone without any ancestors."

Here \( L = \exists R E \), where \( R \) is a binary relation symbol:

\[ Rxy \iff x \text{ is an ancestor of } y . \]
Formalised we have:

\[
\left[ \left( \forall x \forall y \forall z \left( (R_{xy} \land R_{yz}) \rightarrow R_{xz} \right) \right) \land \forall x \neg R_{xx} \right]
\]

\rightarrow \exists x \forall y \exists z R_{yx}

Example